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# Robust Stabilization of Sampled-Data Distributed Processes Using a Dynamic Sensor-Controller Communication Logic

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*Abstract*— **This paper presents a methodology for the robust stabilization of spatially distributed processes with sampled sensor measurements that are transmitted to the actuators over a resource-constrained communication medium. Initially, a finite-dimensional system that captures the slow process dynamics is derived and used to design a Lyapunov-based controller that enforces closed-loop stability in the absence of communication suspensions. An explicit characterization of the stability properties of the closed-loop system under discrete measurement sampling is obtained and then used to devise a dynamic communication logic which can adaptively adjust the rate of information transfer from the sensors to the controller. The key idea is to monitor the evolution of the Lyapunov function at the sampling times and suspend communication for periods when the prescribed stability threshold is satisfied. During such periods, the controller switches to a finite-dimensional model that provides estimates of the slow states to compute the control action. At times when the sampled state begins to breach the expected stability threshold, communication is restored and the controller switches back to the sampled measurements. The results are illustrated through an application to a representative diffusion-reaction process.**

## I. Introduction

With the advent of wireless sensor networks (WSNs) in recent years and the push by major industrial organizations, such as WINA, ZigBee and ISA, for adoption of wireless technology by the process industries, there has been a growing realization that WSNs can play a major role in expanding the capabilities of existing process control systems beyond what is currently feasible with wired devices and point-to-point architectures alone [1]–[3]. The low cost, flexibility and ease of installation and maintenance of wireless sensors mean that more devices could be deployed and more process variables could be monitored and controlled than is cost-effective with solely wired networks. This provides opportunities for improving the existing control quality (e.g., through high-density sensing and deployments in unsafe areas) and for pursuing capabilities that cannot be realized otherwise, including proactive fault-tolerance and real-time plant reconfiguration to accommodate projected market demand changes. Realizing this potential, however, requires handling the challenges that WSNs introduce from a control point of view, including resource constraints (e.g., limited battery power and communication capabilities) and the occasional unreliability due to interference in the field or environmental impact [4], which can significantly limit the control benefits of WSNs if not handled properly in the controller design framework. While a substantial and growing body of research work on networked control systems already exists (e.g., [5]–[9]), the majority of these studies deal with lumped parameter systems modeled by ordinary differential (or difference) equations. Many important engineering applications, however, are characterized by spatial variations and are naturally modeled by partial differential equations (PDEs) such as transport-reaction processes.

Compared with the extensive literature on control of distributed parameter systems in process control (e.g., [10]–[16]), results on networked control of spatially distributed processes are limited at present. Efforts to address this problem were initiated in [17], [18] where resource-aware networked control and scheduling strategies were developed on the basis of appropriate finite-dimensional approximations of the infinite-dimensional system. A key idea – inspired by the results in [19] – was to utilize model-based networked control techniques to enforce closed-loop stability with minimal sensor-controller communication requirements. These results were subsequently extended in [20] where a state-dependent communication logic was introduced to further reduce the sensor-controller communication and allow the control system to respond quickly and adaptively to changes in operating conditions. The implementation of these approaches, however, requires the availability of process measurements at all times. In practice, process measurements are typically available from the sensors at discrete times instances and not continuously. The limitations on the frequency of measurement availability imposes restrictions on the implementation of the feedback controller and can erode the closed-loop stability properties if not explicitly accounted for at the design stage. Furthermore, the lack of continuous measurements limits our ability to accurately monitor the evolution of the state which is a necessary pre-requisite for the correct implementation of the sensor-controller communication logic.

Motivated by these considerations, we focus on the problem of controlling spatially distributed processes with sampled sensor measurements over a resource-constrained wireless sensor network. A model-based networked control structure with a dynamic sensor-controller communication policy is developed to robustly stabilize the process with minimal sensor-controller communication. The rest of the

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paper is organized as follows. In Section II, we consider distributed processes modeled by highly-dissipative nonlinear PDEs and use model reduction techniques to obtain a finite-dimensional system that captures the dominant dynamics of the PDE. This system is then used in Section III to design a robust Lyapunov-based controller and characterize the closed-loop stability properties under discrete measurements. This characterization is then used in Section IV to derive the communication logic which uses the Lyapunov stability bound as a threshold for adaptively adjusting the rate of information transfer from the sensors to the controller. Finally, the results are illustrated through simulations in Section V.

## II. Preliminaries

### A. Class of systems

We consider spatially-distributed processes modeled by nonlinear parabolic PDEs of the form:

$$\frac{\partial \bar{x}(z,t)}{\partial t} = \alpha\frac{\partial^2 \bar{x}}{\partial z^2} + \beta\bar{x} + f(\bar{x}) + \omega\sum_{i=1}^{m} b_i(z)u_i(t) + \sum_{j=1}^{n} w_j(\bar{x})d_j(z)\theta_j(t),\ |\theta_j(t)| \le \theta_b^j \tag{1}$$

subject to the boundary and initial conditions:

$$\bar{x}(0,t) = \bar{x}(\pi,t) = 0,\ \bar{x}(z,0) = \bar{x}_0(z) \tag{2}$$

where $\bar{x}(z,t) \in \mathbb{R}$ denotes the process state variable, $z \in [0,\pi]$ is the spatial coordinate, $t \in [0,\infty)$ is time, $u_i$ denotes the $i$-th manipulated input, $b_i(\cdot)$ is a known function that describes how the control action is distributed in $[0,\pi]$, $\theta_j$ is an uncertain variable, which may represent uncertain process parameters or exogenous disturbances, $\theta_b^j$ is a positive real number that captures the maximum size of the uncertain variable, $f(\cdot)$ and $w_i(\cdot)$ are sufficiently smooth nonlinear functions, $d_j(\cdot)$ is a known function that specifies the positions of action of the uncertain variable $\theta_j$, the parameters $\alpha > 0, \beta$ are constants, and $\bar{x}_0(z)$ is a smooth function of $z$. Throughout the paper, the norm notations $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_2$ will be used to denote the standard Euclidean norm, the $L_2$ norm associated with a finite-dimensional Hilbert space, and the $L_2$ norm associated with an infinite-dimensional Hilbert space, respectively. Furthermore, the notation $x(t_k^-)$ will be used to denote the limit $\lim_{t\to t_k^-} x(t)$.

### B. Formulation of the infinite-dimensional system

Using standard techniques from operator theory [21], the PDE of (1)-(2) can be formulated as an infinite-dimensional system of the following form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + f(x) + \mathcal{W}(x)\theta,\ x(0) = x_0 \tag{3}$$

where $x(t) = \bar{x}(z,t)$, $t \ge 0$, $0 < z < \pi$, is the state function defined on the Hilbert space $\mathcal{H} = L_2(0,\pi)$ endowed with inner product and norm:

$$\langle \omega_1, \omega_2\rangle = \int_0^\pi \omega_1(z)\omega_2(z)dz, \|\omega_1\|_2 = \langle \omega_1,\omega_1\rangle^{\frac{1}{2}} \tag{4}$$

where $\omega_1, \omega_2$ are two elements of $L_2(0,\pi)$, $\mathcal{A}$ is the differential operator defined by $\mathcal{A}\phi = \alpha\frac{d^2\phi}{dz^2} + \beta\phi,\ 0 < z < \pi$, where $\phi(\cdot)$ is a smooth function on $(0,\pi)$ with $\phi(0) = \phi(\pi) = 0$, $\mathcal{B}$ is the input operator defined by $\mathcal{B}u = \omega\sum_{i=1}^{m} b_i(\cdot)u_i$, $u = [u_1 \cdots u_m]^T$, $\mathcal{W}$ is the uncertainty operator with $\mathcal{W}(x)\theta = \sum_{j=1}^{n} w_j(\bar{x})d_j(z)\theta_j(t)$, $\theta = [\theta_1 \cdots \theta_n]^T$ and $x_0 = \bar{x}_0(z)$.

For $\mathcal{A}$, the eigenvalue problem is given by $\mathcal{A}\phi_j = \lambda_j\phi_j,\ \ j = 1,\ldots,\infty$, where $\lambda_j$ denotes an eigenvalue and $\phi_j$ denotes an eigenfunction. The solution to this eigenvalue problem is given by $\lambda_j = \beta - \alpha j^2$, $\phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz),\ \ j = 1,\ldots,\infty$. It can be seen that all the eigenvalues of $\mathcal{A}$ are real and ordered. Also, for a given $\alpha$, only a finite number of unstable eigenvalues exists, and the distance between two consecutive eigenvalues increases as $j$ increases. Furthermore, the spectrum of $\mathcal{A}$ can be partitioned as $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \bigcup \sigma_2(\mathcal{A})$, where $\sigma_1(\mathcal{A}) = \{\lambda_1,\ldots,\lambda_m\}$ contains the first $m$ (with $m$ finite) "slow" eigenvalues and $\sigma_2(\mathcal{A}) = \{\lambda_{m+1},\lambda_{m+2},\ldots\}$ contains the remaining "fast" stable eigenvalues where $|\lambda_m|/|\lambda_{m+1}| = O(\epsilon)$ and $\epsilon < 1$ is a small positive number that characterizes the extent of separation between the slow and fast eigenvalues of $\mathcal{A}$. This implies that the dominant dynamics of the PDE can be described by a finite-dimensional system.

### C. Modal decomposition

Defining the orthogonal projection operators, $\mathcal{P}_s$ and $\mathcal{P}_f$, such that $x_s = \mathcal{P}_s x \in \mathcal{H}_s := \mathrm{span}\{\phi_1,\ldots,\phi_m\}$, $x_f = \mathcal{P}_f x \in \mathcal{H}_f := \mathrm{span}\{\phi_{m+1},\phi_{m+2},\ldots\}$, the state of the system of (3) can be decomposed as $x = x_s + x_f$. Applying $\mathcal{P}_s$ and $\mathcal{P}_f$ to the system of (3) and using the decomposition of $x$, the system of (3) can be decomposed as:

$$\dot{x}_s = \mathcal{F}_s(x_s,x_f) + \mathcal{B}_s u + \mathcal{W}_s(x_s,x_f)\theta,\ x_s(0) = \mathcal{P}_s x_0 \tag{5}$$

$$\dot{x}_f = \mathcal{F}_f(x_s,x_f) + \mathcal{B}_f u + \mathcal{W}_f(x_s,x_f)\theta,\ x_f(0) = \mathcal{P}_f x_0 \tag{6}$$

where $\mathcal{F}_s(x_s,x_f) = \mathcal{A}_s x_s + f_s(x_s,x_f)$, $\mathcal{A}_s = \mathcal{P}_s\mathcal{A}$ is an $m\times m$ diagonal matrix of the form $\mathcal{A}_s = \mathrm{diag}\{\lambda_j\}$, $\mathcal{B}_s = \mathcal{P}_s\mathcal{B}$, $f_s = \mathcal{P}_s f$, $\mathcal{W}_s = \mathcal{P}_s\mathcal{W}$, $\mathcal{F}_f(x_s,x_f) = \mathcal{A}_f x_s + f_f(x_s,x_f)$, $\mathcal{A}_f = \mathcal{P}_f\mathcal{A}$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s$ and $\mathcal{H}_f$), $\mathcal{B}_f = \mathcal{P}_f\mathcal{B}$, $f_f = \mathcal{P}_f f$, and $\mathcal{W}_f = \mathcal{P}_f\mathcal{W}$. In what follows, the $x_s$- and $x_f$-subsystems will be referred to as the slow and fast subsystems, respectively. Neglecting the fast and stable $x_f$-subsystem of (6), the following approximate, $m$-dimensional slow system can be obtained:

$$\dot{\bar{x}}_s = \mathcal{F}_s(\bar{x}_s,0) + \mathcal{B}_s u + \mathcal{W}_s(\bar{x}_s,0)\theta \tag{7}$$

where the bar symbol denotes that the variable is associated with a finite-dimensional system.

## III. Robust stabilization using sampled sensor measurements

To realize the desired robust distributed networked control structure, the first step is to synthesize a nonlinear feedback controller that enforces robust closed-loop stability with an arbitrary degree of asymptotic attenuation of the effect of the uncertainty on the closed-loop system in the absence of communication suspension. As an example, we consider the following robust nonlinear controller [22]:

$$u = k(\bar{x}_s,\theta_b,\rho,\chi,\phi) = -\left(\frac{L^*_{\mathcal{F}_s}V + \sqrt{(L^{**}_{\mathcal{F}_s}V)^2 + |(L_{\mathcal{B}}V)^T|^4}}{|(L_{\mathcal{B}}V)^T|^2}\right)(L_{\mathcal{B}}V)^T \tag{8}$$

when $|(L_{\mathcal{B}}V)^T| \neq 0$, and $u = 0$ when $|(L_{\mathcal{B}}V)^T| = 0$, where

$$L^{**}_{\mathcal{F}_s}V = L_{\mathcal{F}_s}V + \rho\|\,\bar{x}_s\,\| + \chi|(L_{\mathcal{W}_s}V)^T|\theta_b \tag{9}$$

$$L^{*}_{\mathcal{F}_s}V = L_{\mathcal{F}_s}V + \left(L^{**}_{\mathcal{F}_s}V - L_{\mathcal{F}_s}V\right)\left(\frac{\|\,\bar{x}_s\,\|}{\|\,\bar{x}_s\,\| + \phi'}\right) \tag{10}$$

$V : \mathcal{H}_s \to \mathbb{R} \geq 0$ is a robust control Lyapunov function [23], [24] for the system of (7), $L_{\mathcal{F}_s}V$, $L_{\mathcal{B}}V$ and $L_{\mathcal{W}_s}V$ are Lie derivatives of $V$, $\theta_b$ is an upper bound on $|\theta(t)|$, and $\rho$, $\chi$, $\phi'$ are tunable parameters that satisfy $\rho > 0$, $\chi > 1$ and $\phi' > 0$.

Consider the system of (7) under the control law of (8)-(10) and continuous communication. Evaluating $\dot{V}$ along the trajectories of the closed-loop system, it can be verified that there exists a positive real number $\phi^*$ such that if $\phi := \phi'(\chi - 1)^{-1} \in (0, \phi^*]$, $\dot{V}$ satisfies the following bound:

$$\dot{V}(\bar{x}_s) \quad \leq \quad -\varphi V(\bar{x}_s) + \gamma_1(\phi) \tag{11}$$

for some positive constant $\varphi$ and a class $\mathcal{K}$ function $\gamma_1(\cdot)$, which implies that the state of the approximate closed-loop system remains bounded and converges in finite-time to a terminal neighborhood around the origin whose size can be made arbitrarily small by appropriate selection of the controller tuning parameters.

To analyze the stability properties of the closed-loop system when the state measurements are sampled and transmitted to the controller at discrete times, we consider the typical case when the sampling period is constant and the same for all the sensors. The following proposition establishes the fact that the robust controller of (8)-(10) possesses a robustness property that preserves closed-loop stability when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time.

*Proposition 1: Consider the system of (7) under the control law of (8)-(10) with* $\phi \in (0, \phi^*]$. *Let* $u(t) = k(\bar{x}_s(t_j))$ *for all* $t \in [t_j, t_{j+1})$, $j = 0, 1, 2, \cdots$, *where* $t_{j+1} - t_j = \Delta$. *Then given any positive real numbers* $\delta_b$ *and* $\delta_d$, *where* $\delta_b > \delta_d + 2\varphi^{-1}\gamma_1(\phi)$, *there exists a positive real number* $\Delta^*$ *such that if* $\bar{x}_s(0) \in \Omega_b := \{\bar{x}_s \in \mathcal{H}_s : V(\bar{x}_s) \leq \delta_b\}$ *and* $\Delta \in (0, \Delta^*]$, *we have* $\limsup_{t\to\infty} V(\bar{x}_s(t)) \leq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$.

*Proof:* Let the control action be computed for some $\bar{x}_s(t_j) \in \Omega$ and held constant for some time $\Delta > 0$. Then, $\forall t \in [t_j, t_{j+1})$, where $t_{j+1} = t_j + \Delta$, we have:

$$\begin{aligned} \dot{V}(t) \;&\leq\; -\varphi V(\bar{x}_s(t)) + \gamma_1(\phi) \\ &+\; L_{\mathcal{B}_s}V(\bar{x}_s(t))[k(\bar{x}_s(t_j)) - k(\bar{x}_s(t))] \end{aligned} \tag{12}$$

where (11) was used to establish the inequality in (12). Since $\Omega_b$ is compact, one can find, for all $\bar{x}_s(t_j) \in \Omega_b$ and a given $\Delta$, a positive real number $K_1$ such that $\|\bar{x}_s(t_j) - \bar{x}_s(t)\| \leq K_1\Delta$ , $\forall t \in [t_j, t_{j+1})$. Then due to the continuous properties of $\bar{x}(t)$, the function $k(\cdot)$ and $L_{\mathcal{B}_s}V(\cdot)$, one can easily find positive real numbers, $K_1$ and $M$, such that $\|k(\bar{x}_s(t_j)) - k(\bar{x}_s(t))\| \leq K_1\Delta$ and $\|L_{\mathcal{B}_s}V(\bar{x}_s(t))\| \leq M\Delta$. Substituting these inequalities in (12) yields for all $t \in [t_j, t_{j+1})$:

$$\dot{V}(\bar{x}_s(t)) \leq -\varphi V(\bar{x}_s(t)) + \gamma_1(\phi) + MK_1\Delta^2 \tag{13}$$

which implies that:

$$\begin{aligned} &\dot{V}(\bar{x}_s(t)) \leq -\tfrac{1}{2}\varphi V(\bar{x}_s(t)) \\ &\forall\, V(\bar{x}_s(t)) \geq \delta_r + 2\varphi^{-1},\ \delta_r = 2\varphi^{-1}MK_1\Delta^2 > 0 \end{aligned} \tag{14}$$

The above analysis implies that, given $\delta_d$, we can choose $\delta_r < \delta_d$ and find a corresponding value of $\Delta \leq \Delta^{\star\star} := \sqrt{\delta_d\varphi}/\sqrt{2MK_1}$, such that if the control action is computed for any $\bar{x}_s(t_j) \in \Omega_b$ and the hold time is less than $\Delta^{\star\star}$, we get that $\dot{V}(\bar{x}_s(t))$ remains negative for $V(\bar{x}_s(t)) \geq \delta_r + 2\varphi^{-1}\gamma_1(\phi)$, and $\bar{x}_s(t)$ remains inside $\Omega_b$ and converges in finite time to the level set $\Omega_r := \{\bar{x}_s \in \mathcal{H}_s : V(\bar{x}_s) \leq \delta_r + 2\varphi^{-1}\gamma_1(\phi)\}$.Then we consider $\Delta'$ such that:

$$\delta_d + 2\varphi^{-1}\gamma_1(\phi) = \max_{\bar{x}_s(t_j)\in\Omega_r, t\in[t_j, t_j+\Delta']} V(\bar{x}_s(t)) \tag{15}$$

Since $\Omega_r$ is compact, $V(\cdot)$ is a continuous function of $\bar{x}_s$, and $\bar{x}_s$ evolves continuously in time, then one can always choose a sufficiently small $\Delta'$ such that (15) holds. Let $\Delta^\star = \min\{\Delta^{\star\star}, \Delta'\}$ and Choosing $\Delta \in (0, \Delta^*]$, then for all $\bar{x}(t_j) \in \Omega_d \bigcap \Omega_r$, by definition $\bar{x}_s(t) \in \Omega_d$ for $t \in [t_j, t_{j+1})$. For all $\bar{x}_s(t_j) \in \Omega_d \backslash \Omega_r$, $\dot{V}(\bar{x}_s(t)) < 0$ for $t \in [t_j, t_{j+1})$. Since $\Omega_d$ is a level set of $V$, then $\bar{x}_s(t) \in \Omega_d$, for $t \in [t_j, t_{j+1})$. Either way, for all initial conditions in $\Omega_d$, $\bar{x}_s(t) \in \Omega_d$ for all future times. Note that for $\bar{x}_s(t_j) \in \Omega_b \backslash \Omega_d$, negative definiteness of $\dot{V}$ is guaranteed for $\Delta \leq \Delta^\star$. Hence, all trajectories originating in $\Omega_b$ converge to $\Omega_d$, which has been shown to be invariant under the control law with a hold time $\Delta \leq \Delta^*$. This implies that, for all $\bar{x}_s(t_0) \in \Omega_b$, $\limsup_{t\to\infty} V(\bar{x}_s(t)) \leq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$. ■

## IV. Networked Control Using Dynamic Sensor-Controller Communication

### A. Model-based networked controller design

The implementation of the controller of (8)-(10) requires that the state measurements be transmitted at each sampling time. Since the sensor-controller communication link is resource-constrained, it is desired to reduce the communication rate below the sampling rate without jeopardizing closed–loop stability. To accomplish this goal, a finite-dimensional model of the system of (7) is included in the control system to provide the controller with an estimate of the state when the sensor-controller communication is suspended. The state of the model is then updated based on the actual state measurements when communication is restored. Under this model-based control strategy, the controller of (8)-(10) is implemented as follows:

$$\begin{aligned} u(t) \;&=\; k(\widehat{x}_s(t)),\ t \in (t_j, t_k) \\ \dot{\widehat{x}}_s(t) \;&=\; \nu(t)[\widehat{\mathcal{F}}_s(\widehat{x}_s(t)) + \widehat{\mathcal{B}}_s u(t)],\ \nu(t) \in \{0, 1\} \\ \widehat{x}_s(t_j) \;&=\; \bar{x}_s(t_j),\ j, k = 0, 1, 2, \cdots,\ j < k \end{aligned} \tag{16}$$

where $\widehat{x}_s$ is the state of the model that generates estimates of $\bar{x}_s$, $\widehat{\mathcal{F}}_s(\cdot)$ and $\widehat{\mathcal{B}}_s$ are bounded operators that represent models of $\mathcal{F}_s(\cdot)$ and $\mathcal{B}_s$, respectively (notice that in general $\widehat{\mathcal{F}}_s(\cdot) \neq \mathcal{F}_s(\cdot)$, $\widehat{\mathcal{B}}_s \neq \mathcal{B}_s$). The times $t_j = j\Delta$ and $t_k = k\Delta$ are the $j$-th and $k$-th sampling times, respectively, where $t_k$ is the earliest transmission time after $t_j$, $\nu(t)$ is a model switching signal which is a binary variable that takes a value of $\nu = 1$ when the approximate model is used by the controller to compute the control action, and a value of $\nu = 0$ when the sample-and-hold scheme is used instead (i.e., when the controller uses the sampled measurements).

### B. An adaptive sensor-controller communication policy

Consider the finite-dimensional system of (7) subject to the model-based network controller of (16). Using the result of Proposition 1 and evaluating the time-derivative of $V$ along the trajectories of the networked closed-loop system, starting from any $\bar{x}_s(t_j) \in \Omega$, for $t \in [t_j, t_{j+1})$ yields:

$$\begin{aligned} \dot{V}(\bar{x}_s(t)) \;&\leq\; -\varphi V(\bar{x}_s(t)) + \gamma_1(\phi) + \gamma_2(\Delta) \\ &+\; L_{\mathcal{B}_s}V\bar{x}_s(t)[k(\widehat{x}_s(t)) - k(\bar{x}_s(t_j))] \end{aligned} \tag{17}$$

where we have used the bound in (13) with $\gamma_2(\Delta) = MK_1\Delta^2$ to derive the inequality in (17). Examining this inequality and comparing it with (13) reveals explicitly the perturbation effect of communication suspension on closed-loop stability. Specifically, the discrepancy between $k(\widehat{x}_s(t))$ and $k(\bar{x}_s(t_j))$ alters the rate at which the Lyapunov function decays. As the model estimation error grows, the error in the implemented control action grows as well, and may become large enough to cause the growth of the Lyapunov function and render the closed-loop system potentially unstable. When this happens, the sensors must be allowed to send their measurements to update the state of the model at next sampling time, and the controller must switch back to the sample-and-hold scheme to compensate for the increase of the Lyapunov function to avert instability. This communication policy is formalized in the following theorem.

*Theorem 1: Consider the closed-loop system of (7) and (8)-(10), for which the Lyapunov function, $V$, satisfies (13) for $\phi \in (0, \phi^*]$ and $\Delta \in (0, \Delta^*]$ when state measurements are transmitted at every sampling time. Consider also the networked closed-loop system of (7) and (16) for any $\bar{x}_s(t_0) \in \Omega_b$ such that $V(\bar{x}_s(t_0)) > \delta_d + 2\varphi^{-1}\gamma_1(\phi)$, and set $\widehat{x}_s(t_0) = \bar{x}_s(t_0)$. Let $\nu(t) = 1$, $\forall\, t \in [t_0, t_{j^\star})$, where $t_{j^\star}$ is the earliest time such that:*

$$V(\bar{x}(t_{j^\star}^-)) \geq V(\bar{x}(t_{j-1})) \tag{18}$$

*and $V(\bar{x}_s(t_{j^\star}^-)) > \delta_d + 2\varphi^{-1}\gamma_1(\phi)$. Then the update law given by $\widehat{x}_s(t_k) = \bar{x}_s(t_k)$, $\nu(t) = 0$ for $t \in [t_k, t_{k+1})$, $\forall\, k \geq j^\star$, guarantees that $\limsup_{t\to\infty} V(\bar{x}_s(t)) \leq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$.*

*Proof:* Note that at any time the state of the model is re-set as $\widehat{x}_s(t_k) = \bar{x}_s(t_k)$ and $\nu(t) = 0$ for all $t \in [t_k, t_{k+1})$, we have that $\widehat{x}_s(t) = \bar{x}_s(t_k)$ and, therefore, $k(\widehat{x}_s(t)) - k(\bar{x}_s(t_k)) = 0$, which when substituted into (17) yields $\dot{V}(\bar{x}_s(t)) \leq -\varphi V(\bar{x}_s(t)) + \gamma_1(\phi) + \gamma_2(\Delta)$, for all $t \in [t_k, t_{k+1})$. This is the same bound obtained when the controller is implemented using the sample-and-hold scheme, which implies that $\dot{V}(\bar{x}_s(t)) < -\frac{1}{2}\varphi V(\bar{x}_s(t)) < 0$ for all $V(\bar{x}_s(t)) \geq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$. Since the control action is computed using the sample-and-hold scheme for all $k \geq j^\star$ (i.e., for all future times), then it follows from the result of Proposition 1 that $\limsup_{t\to\infty} V(\bar{x}_s(t)) \leq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$. ■

*Remark 1:* The implementation of the adaptive communication strategy proposed in Theorem 1 requires monitoring the evolution of the state at each sampling time to determine if (18) holds. As long as this condition is not satisfied, the approximate model is used to compute the control action and sensor-controller communication is terminated. When this condition is satisfied, however, the model state is updated and the controller reverts to the sample-and-hold strategy to calculate the control action and ensure practical stability and ultimate boundedness of the closed-loop state. In this manner, the sensor-controller communication rate is always kept below the sampling rate (as long as $j^\star > 0$), and is increased only when necessary to maintain closed-loop stability. Depending on the quality of the model chosen, the resulting communication rate can be as low as zero (in cases where a perfect model is used), and as high as the sampling rate (in case where poor models are used).

*Remark 2:* Note that the communication logic presented in Theorem 1 tolerates an increase in $V$ for at most one sampling period. This could be restrictive in cases where a temporary finite increase in $V$ does not necessarily lead to continued growth in $V$ in the future or subsequent instability. In such cases, a less restrictive policy that can be used would be not to restore communication at the first sampling time that $V$ increases, and keep relying on the model until a pre-specified threshold is breached. For example, one can suspend communication and request an update (i.e., switch to the sample-and-hold strategy) only when $V(\bar{x}_s(t_j)) \geq V(\bar{x}_s(t_0))$. This logic can lead to further reduction in sensor-controller communication without loss of stability.

*Remark 3:* Another aspect of the result of Theorem 1 is that once the sampled state (under model-based control) begins to breach the specified stability threshold, the controller is forced to switch to using the sampled measurements for all future sampling times. While this is sufficient to guarantee closed-loop stability, it is possible to relax this requirement by implementing the sample-and-hold strategy only until it forces the closed-loop state to converge closer to the desired terminal set (this is guaranteed from Proposition 1), and then switching back at this time to the model-based control strategy. This can lead to additional savings in sensor-controller communication, especially when the model-based controller is stabilizing only in some small neighborhood of the origin and the initial condition lies outside this region. The following algorithm formalizes this idea and represents a generalization of the communication logic of Theorem 1:

- Starting from any initial condition $\bar{x}_s(t_0) \in \Omega$, set $\widehat{x}_s(t_0) = \bar{x}_s(t_0)$, $\nu(t) = 1$ and implement the model-based controller for $t \in [t_0, t_j)$, where $t_j$ is the earliest time such that $V(\bar{x}_s(t_j)) \geq V(\bar{x}_s(t_{j-1}))$.
- Switch to and implement the sample-and-hold strategy for all $t \in [t_j, t_k)$, i.e., set $\widehat{x}_s(t_i) = \bar{x}_s(t_i)$ and $\nu(t) = 0$ for all $j \leq i \leq k$ and $t \in [t_j, t_k)$, where $t_k$ is the earliest time such that $V(\bar{x}_s(t_k)) < V(\bar{x}_s(t_{j-1}))$
- Set $\widehat{x}_s(t_k) = \bar{x}_s(t_k)$ and $\nu(t) = 1$ for all $t \in [t_k, t_m)$ (i.e., switch back to model-based control), where $t_m$ is the earliest time such that $V(\bar{x}_s(t_m)) \geq V(\bar{x}_s(t_{m-1}))$.
- Repeat steps 2-3 until $V(\bar{x}_s(t)) \leq \delta_d + 2\varphi^{-1}\gamma_1(\phi)$.

Finally, it should be noted that while the above algorithm attempts to keep the sensor-controller information transfer to a minimum without loss of stability, some performance deterioration may occur if too many back-and-forth switchings between the model-based and sample-and-hold control strategies are allowed. Ultimately, the number of switchings should be chosen in a way that balances the fundamental tradeoff between the extent of network utilization and the achievable closed-loop performance.

*Remark 4:* In addition to modifying the switching threshold and the frequency of switching between the model-based and sample-and-hold control strategies, an additional measure that can be taken to enhance the flexibility of the adaptive communication policy is to consider several Lyapunov function candidates instead of a single one. The idea here would be to monitor those functions simultaneously and request an update only if all the functions begin to breach their respective thresholds. As long as at least one candidate continues to decrease at every sampling time, no updates from the sensors are necessary and the model-based control scheme can be implemented.

*Remark 5:* It can be shown that the networked controller and adaptive communication logic that stabilizes the approximate finite-dimensional system continuous to stabilize the infinite-dimensional system provided that the separation between slow and fast eigenvalues of the differential operator is sufficiently large. This argument can be justified using singular perturbation techniques and is omitted for brevity.

## V. SIMULATION STUDY: APPLICATION TO A DIFFUSION-REACTION PROCESS

To illustrate the design and implementation of the networked control structure under adaptive communication policy with sampled measurements, we consider a long, thin catalytic rod in a reactor where a zeroth-order exothermic reaction takes place. A cooling medium in contact with the rod is used to provide/remove heat from the rod. Under standard modeling assumptions, the spatiotemporal evolution of the rod temperature is described by:

$$\begin{aligned}\frac{\partial \bar{x}}{\partial t} &= \frac{\partial^2 \bar{x}}{\partial z^2} - \beta_U \bar{x} + (\beta_T + \theta_1)[e^{-\gamma/(1+\bar{x})} - e^{-\gamma}] \\ &+ \beta_U b(z) u(t)\end{aligned} \tag{19}$$

where $\bar{x}$ denotes the dimensionless temperature, $\beta_T = 50.0$, $\beta_U = 4.0$, $\gamma = 2.0$ denote dimensionless heat of reaction, heat transfer coefficient, activation energy, respectively. $u(t)$ denotes the dimensionless temperature of the cooling medium, $\theta_1$ is a parametric uncertainty in the heat of reaction, which for simulation purposes is chosen as $\theta_1 = 2.0$. It can be verified that the operating steady state $\bar{x}(z,t) = 0$ (with $u = \theta_1 = 0$) is unstable. The control objective is to stabilize the temperature profile at this unstable steady state by manipulating the temperature of the cooling medium.

The solution of the eigenvalue problem for the differential operator yields $\lambda_j = -j^2 - \beta_U$, $\phi_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz)$, $j = 1, 2, \cdots, \infty$. The first eigenvalue is considered to be the dominant one and Galerkins method is applied to derive an ODE that describes the approximate temporal evolution of the amplitude of the first eigenmode:

$$\dot{\bar{a}}_1 = \lambda_1 \bar{a}_1 + f(\bar{a}_1) + g(z_a)u + w(\bar{a}_1)\theta_1 \tag{20}$$

where $\bar{x}(z,t) = \sum_{i=1}^{\infty} a_i(t)\phi_i(z)$, $f(\bar{a}_1) = \beta_U\langle \phi_1(z), h(\bar{a}_1)\rangle$, $g(z_a) = \beta_U\langle\phi_1(z), b(z)\rangle$, $w(\bar{a}_1) = \beta_U\langle\phi_1(z), h(\bar{a}_1)\rangle$, $h(\bar{a}_1) = e^{-\gamma/(1+\bar{a}_1\phi_1(z))} - e^{-\gamma}$ and a single point actuator (with finite support) is used for stabilization. This ODE is used to design the networked controller and derive the communication logic which are then implemented on a 30-th order Galerkin discretization of the PDE (higher-order discretizations led to identical results). Specifically, a Lyapunov-based controller of the form of (8)-(10) is designed by using a quadratic Lyapunov function of the form $V(\bar{a}_1) = \bar{a}_1^2$ and setting the controller tuning parameters as $\chi = 1.1$, $\rho = 0.0001$, $\phi = 0.0001$ to ensure that the closed-loop state converges in finite time to a small neighborhood of the desired steady state.

In order to further reduce the sensor-controller communication rate, a model is considered as $\dot{\widehat{a}}_1(t) = \nu(t)(\lambda_1\widehat{a}_1(t) + \widehat{f}(\widehat{a}_1(t)) + \widehat{g}(z_a)u(t))$ where, for simplicity, $\widehat{f} = f$ and $\widehat{g} = g$. The control law is implemented as in (16) where the model state is used by the controller so long as no measurements are transmitted, but is updated using the true measurement whenever it is transmitted. To determine the appropriate update times and the value of $\nu$, the evolution of $V(a_1(t_j))$ is monitored, and the model state is updated and $\nu$ is re-set to zero (such that controller switches to using the sampled measurements) when either: (1) $V(a_1(t_j)) \geq V(a_1(t_{j_1}))$ while $a_1(t_j)$ is outside the terminal set $\{V(a_1) \leq \delta_d = 0.0001\}$, or (2) the state $a_1(t_j)$ is on the verge of escaping the terminal set while previously inside. The controller reverts to relying on the finite-dimensional model by re-setting $\nu = 1$ whenever $V(a_1(t_k))$ becomes less than $V(a_1(t_{j-1}))$ for $k > j$ (see the algorithm described in Remark 3).

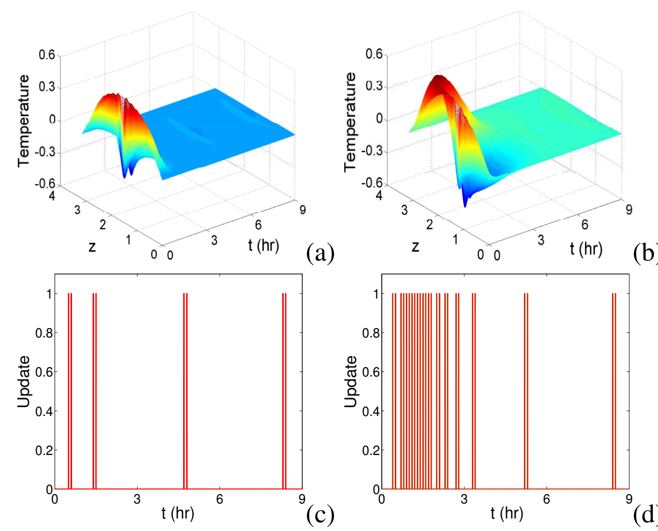


Fig. 1. Plots (a)-(b): Closed-loop state profiles under the dynamic communication policy with sampled sensor measurements and $z_a = 0.5\pi$ and $z_a = 0.4\pi$. Plots (c)-(d): Update times of the model under the dynamic communication policy for $z_a = 0.5\pi$ and $z_a = 0.4\pi$, respectively.

Figs.1 (a) depicts the evolution of the closed-loop state profiles when the process is operated using the dynamic communication policy with $\Delta = 0.1$ hr and $z_a = 0.5\pi$. It can be seen that the closed-loop system is successfully stabilized near the desired steady-state. Fig.1 (c) shows when the model embedded in the controller is updated. The variable "Update" can either be 1 or 0 representing on/off sensor-controller communication. It can be seen that communication is restored twice (and the measurement held for one sampling period each time) before $a_1$ converges inside the desired terminal set. Following this, communication is suspended until the state attempts to escape the terminal set at $t = 4.6$ $s$. At this sampling time, the sensors are prompted to send their measurements to update the model and the sample-and-hold scheme is implemented until the corrective action forces the state back into the terminal set. While not shown in the plot, examination of the behavior of the "Update" variable over a longer time period shows that, once the state is inside the terminal set, the sensor-controller communication is re-established in a nearly periodic fashion (roughly every 4 hr), which is needed to compensate for the approximation error resulting from the discrepancy between $a_1$ and $\bar{a}_1$. It should be noted that the adaptive communication policy is able to enforce closed-loop stability and maintain a closed-loop performance that is quite similar to the one obtained under the static policy with the higher communication rate.

From the stability analysis in Section IV, it can be seen that, for a given model, the actuator location is one of the key factors that influences the communication rate. To show this dependence, we implemented the dynamic communication policy described above on the process for the case when the actuator is placed at $z_a = 0.4\pi$ and the sampling period is maintained at $\Delta = 0.1$. Figs.1(b) show the evolution of the closed-loop state profiles in this case. It can be seen that the performance of closed-loop system deteriorates after changing the actuator location and that the frequency of sensor-controller communication is increased substantially to ensure the convergence of the closed-loop state close to the desired steady state. This increase can be observed from Fig.1(d) which shows the update times of the model.

In addition to network utilization considerations, we also investigated the disturbance-handling capabilities of both static and dynamic communication policies in order to compare their robustness with respect to unanticipated disturbances during process operation. To this end, a point disturbance in the jacket temperature at $z_d = \pi/8$ with an amplitude $\theta_b = 0.5$ is introduced at $t = 3$ hr and lasts until $t = 5$ hr. Figs.2(a)-(c), which depict the closed-loop state and manipulated input profiles subject to the unexpected external disturbance, show that both communication policies can successfully overcome the influence of the disturbance and force the temperature to converge near the desired steady-state. Fig.2(d) shows the update times when the dynamic communication policy is implemented. It can be seen that the dynamic communication policy can quickly detect the disturbance and respond by increasing the sensor-controller communication rate to its maximum value (i.e., the sampling rate) which helps the closed-loop state return near the desired steady-state once the disturbance has disappeared. Given the comparable closed-loop performance in both cases, the dynamic policy leads to a large saving in information transfer.

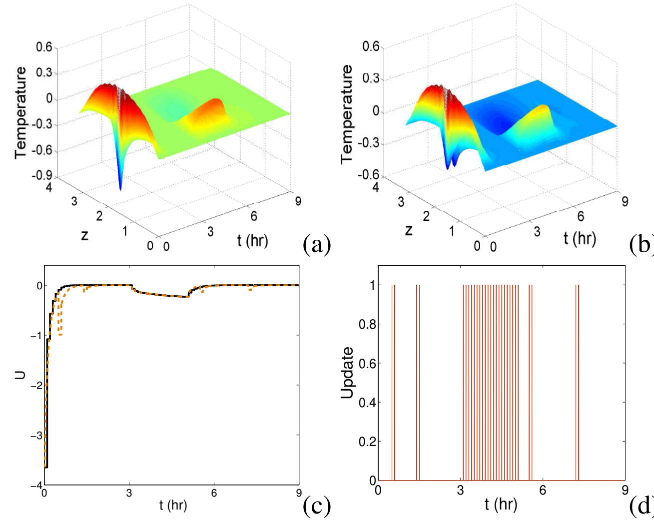


Fig. 2. Plots (a)-(b): Closed-loop state profiles under static and dynamic communication policies when the closed-loop system is subject to an unexpected external disturbance in the jacket temperature. Plot (c): Manipulated input profiles under the static (solid) and dynamic (dashed) communication policies. Plot (d): Update times of the model under the dynamic communication policy.

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