2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Application of pseudospectral method in stochastic optimal control of nonlinear structural systems

Wei Song and Shirley J. Dyke

Abstract—**This paper presents the results of numerical study on the optimal control strategy of nonlinear stochastic systems. The systems under investigation are mechanical oscillators and a damping device. The numerical approach to obtain the optimal control strategy involves solving a nonlinear partial differential equation --- the Hamilton-Jacobi-Bellman equation. Since civil engineering structural systems usually exhibit nonlinear hysteretic behavior under extreme loading conditions, the potential application of the obtained control strategy could provide an optimal feedback control law to reduce the system response under the random excitations (such as earthquakes, wind load and sea waves). Several numerical examples are presented to verify optimality and demonstrate the efficacy of the proposed optimal control solution. First, a linear oscillator is used to verify that the obtained solution is indeed the optimal solution by comparing it to the closed form solution. Then the proposed method is applied to several nonlinear systems. In each case, optimality is demonstrated by comparing the system responses and costs under optimal control with those obtained using linearized optimal control.**

I. Introduction

In structural control design, a control law that can effectively integrate the structure system with the energy dissapation devices is needed. Therefore, to realize better control effects, a desirable control strategy has to consider the fact that the expendable damping devices (passive friction devices, base isolation devices, magneto-rheological dampers, etc.), along with the structure itself, will exhibit nonlinear hysteretic behavior in the energy dissipation process. Meanwhile, the external input (traffic, wind and earthquake) often takes on random characteristics. As a result, the study of the control of nonlinear or hysteretic civil engineering structures has attracted considerable attention. In the past three decades, several techniques have been developed and successfully applied to analyze the stochastic dynamics of certain hysteretic models [11]. The linear dynamic system assumption is widely used in the present practical controller design for civil engineering structures. Linear quadratic Gaussian [3] and H_∞ controllers [13] have shown positive results in terms of their performance. To better accommodate the nonlinearity in the hysteretic system, researchers have also applied several comprehensive nonlinear control methods [12,14], and combined these with stochastic averaging method [15,16], all reported positive results.

For civil structures subjected to hazardous loading events, the external excitations are random in nature. A feedback control law is more desirable in this case. Theoretically, the optimal feedback control strategy can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) resulting from dynamic programming. In most cases, the obtained HJB equation can be treated as initial value problem, and time stepping schemes coupled with spatial discretizations are often applied. Both finite element and finite difference methods have been applied to solve the nonlinear HJB equation [1,9,10]. In recent years, an emerging numerical technique--- pseudospectral (PS) method has been successfully applied in solving nonlinear partial differential equations [2]. In this paper, PS is applied in solving the HJB equation in nonlinear stochastic system. The PS method is a type of collocation methods with collocation nodes carefully chosen to reduce the approximation error. It is worth pointing out that, PS method has been extensively applied in solving optimal control problems resulting from Pontryagin's maximum principle, which generates open-loop controls [4,5].

In this paper, the nonlinear stochastic system is assumed to be described by an Ito sense stochastic differential equation (SDE), and the theoretical background on obtaining the formulation will be briefly introduced in section II. The technical difficulty and certain characteristics of HJB will be briefly outlined in section III. And section IV describes the PS method used in this study and the associated properties of the method. In section V, the proposed numerical method is verified and applied to several nonlinear dynamic systems.

II. Stochastic Control Problem Formulation

Consider the following system described by a Ito-sense SDE

$$\mathrm{d}\mathbf{x} = \left[\mathbf{f}\big(\mathbf{x}(t)\big) + \mathbf{g}\big(\mathbf{x}(t)\big)\cdot\mathbf{u}(t)\right]\mathrm{d}t + \boldsymbol{\sigma}\big(\mathbf{x}(t)\big)\mathrm{d}\mathbf{w}(t) \qquad (1)$$

where $\mathbf{f} + \mathbf{g}\cdot\mathbf{u}$ is the drift term, and $\boldsymbol{\sigma}$ is the diffusion term. This paper assumes that the origin is a stable fixed point of the system. $\mathbf{x} \in \mathbb{R}^n$ represents the state vector, function $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ characterizes the system dynamics, function $\mathbf{g}: \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the coefficient matrix for control,

Manuscript received Sep 26, 2010. This work was supported from the National Science Foundation (Grant no. CMMI-1002641).

Wei Song is currently a Ph.D. candidate of School of Civil Engineering, Purdue University, West Lafayette, IN 47907 USA. (corresponding author, phone: 765-494-9303; fax: 765-494-9886; e-mail: songwei@purdue.edu).

Shirley J. Dyke is currently a Professor of Mechanical and Civil Engineering, Purdue University, Lafayette, IN 47907 USA. (e-mail: sdyke@purdue.edu).

$\mathbf{u} \in \mathbb{R}^m$ is the control dynamics to be designed, and the external excitation is given as n-dimensional Brownian motion $\mathbf{w}$. Before discussing the optimal control formulation, it is noted that this paper assumes state feedback, i.e. all state variables are available for the controller to be designed. This can be implemented by using certain nonlinear observers, which will not be discussed in this paper.

The optimal control problem is defined by minimizing the following bounded control cost functional

$$J = \mathbb{E}\int_0^\infty e^{-\beta t} L(\mathbf{x}(t), \mathbf{u}(t)) \cdot dt < \infty \tag{2}$$

where $\mathbb{E}(\cdot)$ is the expectation operator, $\beta > 0$ is the exponential discount factor, which discount the future costs and ensure the cost functional is bounded. Cost rate $L: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a non-negative function of the Markov diffusion process $\mathbf{x}$ that solves (1), and the control $\mathbf{u}$. In this paper, the cost rate function L is defined as

$$L(\mathbf{x}, \mathbf{u}) = \|\mathbf{x}\|_{\mathbf{Q}}^2 + \|\mathbf{u}\|_{\mathbf{R}}^2 = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} \tag{3}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric positive definite matrices.

Note that the cost functional in (3) is defined on infinite time horizon, which results in a time invariant formulation as shown later in (5). The associated cost-to-go function (also called value function, meaning the minimum cost by starting at any $t \in [0, \infty)$ with any state $\mathbf{x}(t)$) is given as

$$V(\mathbf{x}(t), t) = \min_{\mathbf{u}} \left\{\mathbb{E}\int_t^\infty e^{-\beta s} L(\mathbf{x}(s), \mathbf{u}(s)) \cdot ds\right\} \tag{4}$$

By the principle of optimality and Ito calculus, the following HJB equation can be derived (interested readers can refer to [6] for a detailed derivation)

$$\beta V^*(\mathbf{x}) = \min_{\mathbf{u}} \left\{V_{\mathbf{x}}^{*T}(\mathbf{f} + \mathbf{g} \cdot \mathbf{u}) + L(\mathbf{x}, \mathbf{u}) + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}^T V_{\mathbf{xx}}^*)\right\} \tag{5}$$

where $\mathrm{Tr}(\cdot)$ is the trace operator, V^* is the value function corresponding to the optimal control, $V_{\mathbf{x}}^*$ and $V_{\mathbf{xx}}^*$ are the first and second order partial derivatives of V^* with respect to $\mathbf{x}$. In general, the optimal control u^* can be obtained by the following equation

$$u^*(\mathbf{x}) = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{g}^T V_{\mathbf{x}}^* \tag{6}$$

Substituting (6) into (5), the minimized HJB equation is expressed as

$$\beta V^*(\mathbf{x}) = V_{\mathbf{x}}^{*T}\mathbf{f} + \|\mathbf{x}\|_{\mathbf{Q}}^2 + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}^T V_{\mathbf{xx}}^*) - \frac{1}{4}V_{\mathbf{x}}^{*T}\mathbf{g}\mathbf{R}^{-1}\mathbf{g}^T V_{\mathbf{x}}^* \tag{7}$$

Equation (7) needs to be solved to obtain the optimal value function V^*, and subsequently the optimal control $\mathbf{u}^*$ can be calculated from (6).

III. TECHNICAL CHALLENGES

There is no general closed form solution to the HJB equation (7), except for linear case. The HJB equation is a second order nonlinear PDE, which is difficult to solve in general. Actually the HJB can be considered as a convection-diffusion equation. Conventional numerical methods, such as finite element (FE) or finite volume (FV) methods, will yield inaccurate or unstable solutions if the convection term $V_{\mathbf{x}}^{*T}\mathbf{f}$ is dominant. To overcome these difficulties, researchers have proposed several numerical techniques to offer special treatment in terms of solving the HJB equation [1,9,10].

Most of the numerical methods are grid based, and therefore suffers from the so called "curse-of-dimensionality", i.e. the computational effort will grow exponentially with the dimension of the state space n. The PS method used in this paper is also a grid based method, but with the grid points chosen at certain locations, higher accuracy can be achieved with smaller number of grid points. This is one major reason to choose PS method to solve HJB equation.

IV. PSEUDOSPECTRAL METHOD

The PS method is a powerful computational tool in solving PDEs. It has been applied extensively in computational fluid dynamics [2]. It also has been successfully applied to obtain optimal control solutions for open loop control problems [4,5]. The PS method used in this study is based on interpolation functions collocated on Chebyshev nodes, which are distributed over the interval $[-1, 1]$. To accommodate arbitrary computational domain $[\tau_i, \tau_f]$, the following affine transformation is applied,

$$\tau(x) = \frac{(\tau_f - \tau_i)x + (\tau_f + \tau_i)}{2}, x \in [-1, 1] \tag{8}$$

The solution V^* to the HJB equation is approximated by a truncated expansion V_N with $N + 1$ interpolation nodes (for succinctness, only one dimensional case is shown here)

$$V_N(x) = \sum_{i=0}^{N} v_i \varphi_i(x) \tag{9}$$

where φ_i is a set of chosen polynomial basis functions, and v_i is the corresponding coefficient. In this paper, Lagrange polynomial interpolation functions are selected as the basis functions, and the interpolation nodes are chosen at the Chebyshev nodes $x_k = \cos([(k-1)\pi]/(N-1)), k = 1,2, \dots, N$, for which the approximation error $V^*(x) - V_N(x)$ is particularly small in terms of ∞-norm [7]. To approximate the partial derivatives, (9) is differentiated to give

$$V_N^{(\ell)}(x) = \sum_{i=0}^{N} v_i \varphi_i^{(\ell)}(x), \ \ell = 1,2, \dots \tag{10}$$

where the superscript $(\cdot)$ indicates the order of the derivative, and the same notation is used hereinafter. With the interpolation basis functions and nodes are determined, the differentiation matrix can be defined as [2]

$$\mathbf{D}_{i,j}^{(1)} \triangleq \varphi_j^{(1)}(x_i) = \begin{cases} \frac{c_i(-1)^{i+j}}{c_j(x_i-x_j)} & i \neq j \\ -\frac{1}{2}\frac{x_i}{(1-x_i^2)} & i = j \neq 1, N \\ \frac{2(N-1)^2+1}{6} & i = j = 1 \\ -\frac{2(N-1)^2+1}{6} & i = j = N \end{cases} \quad (11)$$

and

$$\mathbf{D}^{(\ell)} = \left(\mathbf{D}^{(1)}\right)^{\ell},\ \ell = 1,2,\ldots \quad (12)$$

By substituting (9), (10), (11) and (12) into the original HJB equation (7), the nonlinear PDE is discretized over the computational domain, and converted into a nonlinear equation of v_i. Now the HJB equation can be solved by invoking any desirable nonlinear equation solver to solve for v_i, and subsequently the discretized control values u_i can be obtained by substituting v_i into (6). However, to circumvent the time-consuming process of solving the nonlinear equation, a successive approximation technique [8] is applied in this study.

The PS method has many attractive features, especially its efficiency in gaining sufficient accuracy requirements with relatively coarse grids. However, when the problem has irregular domain, the PS method is generally not the best choice. There are many alternatives [1,9,10] to solve the HJB equation in this case as mentioned in section III.

V. NUMERICAL EXAMPLES

Four numerical examples are presented in this section. The first one is a stochastic linear regulator, which is used to verify the control generated by the proposed method is indeed the solution to the linear problem. Then the proposed method is applied to several nonlinear systems including Van der Pol and Duffing oscillators and a Bouc-Wen hysteretic system. For each example, a linearized control is obtained from the available linear optimal control solution of the corresponding linearized system (at the origin). And the response as well as the control cost yield from optimal control are compared with the ones from linearized control.

To evaluate the control performance, the following performance indices are introduced. The first index is the standard deviation ratio of state variable (displacement and velocity) under controlled and uncontrolled condition

$$\mathbb{J}_1 = \sigma_x^c/\sigma_x^u \quad (13)$$

where σ_x^c is the standard deviation for controlled case, and σ_x^u is the standard deviation for uncontrolled case. This index measures the energy dissipation capacity of the controller.

The second index is the maximum displacement of the controlled case relative to the uncontrolled case, i.e.

$$\mathbb{J}_2 = \max_t(|x_t^c|)/\max_t(|x_t^u|) \quad (14)$$

where x_t^c and x_t^u represent the controlled and uncontrolled displacement evaluated at each time instant t. The third index is the overall measurement of control effectiveness, i.e.

$$\mathbb{J}_3 = \textstyle\sum_{\text{all } t}|x_t^c| \cdot dt \quad (15)$$

where dt is the fixed time step during simulation, if not particularly mentioned, $dt = 0.01s$. The fourth index is the cost of the control system, i.e.

$$\mathbb{J}_4 = \textstyle\sum_{\text{all } t} L_t \cdot dt \quad (16)$$

in which L_t is the value of function L in (3) evaluated at each time instant t. This index is used to evaluate the optimality of the corresponding control. The lower the values of indices $\mathbb{J}_1$, $\mathbb{J}_2$, $\mathbb{J}_3$ and $\mathbb{J}_4$, the better the performance.

A. Linear system

This case is used to verify the obtained control is indeed optimal, by evaluating the error between the control obtained from the proposed method and the one from the available closed form solution for linear stochastic system. With the following linear stochastic oscillator

$$\ddot{x}(t) + 0.1\dot{x}(t) + x(t) = u(t) + dw(t)/dt \quad (17)$$

Then the functions $\mathbf{f}$ and $\mathbf{g}$ in (1) can be expressed as $\mathbf{f}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{g}(\mathbf{x}(t)) = [0 \quad 1]^T$. The diffusion term is given as $\boldsymbol{\sigma}(\mathbf{x}(t)) = [0 \quad 1]^T$, the weighting matrices $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = 1$, and discount factor $\beta = 0.01$. In this case, the optimal solution can be obtained analytically by solving the Riccati equation resulted from the HJB equation (7), which is given as

$$\mathbf{u}_L^*(\mathbf{x}) = -\mathbf{R}^{-1}\mathbf{g}^T\mathbf{P}\mathbf{x} \quad (18)$$

where $\mathbf{P}$ is the solution to the following Riccati equation

$$\mathbf{P}(\mathbf{A} - \beta\mathbf{I}/2) + (\mathbf{A}^T - \beta\mathbf{I}/2)\mathbf{P} - \mathbf{P}\mathbf{g}\mathbf{R}^{-1}\mathbf{g}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \quad (19)$$

The error between $\mathbf{u}_L^*$ and $\mathbf{u}^*$ is evaluate as

$$\mathbb{J}_5 = \|\mathbf{u}_L^*(\mathbf{x}_k) - \mathbf{u}^*(\mathbf{x}_k)\|_{Fro} \quad (20)$$

where $\|\cdot\|_{Fro}$ is the Frobenius norm, and $\mathbf{x}_k$ indicates the grid nodes.

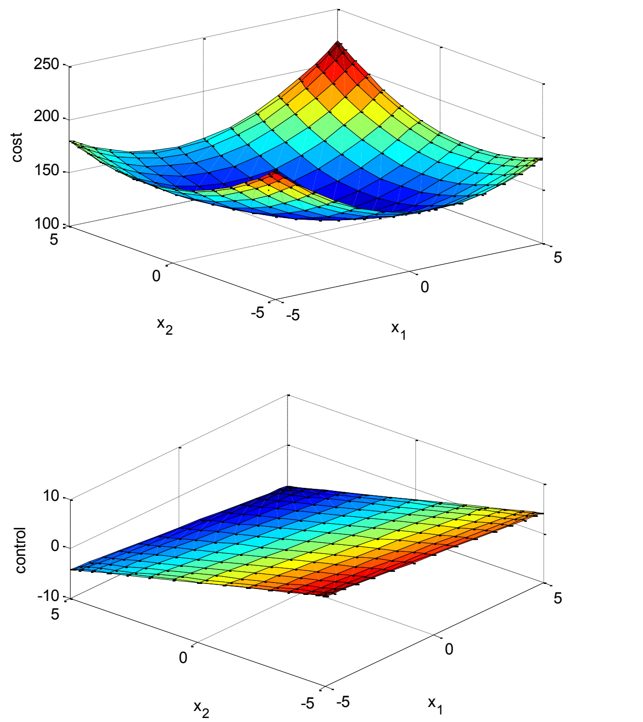

Fig. 1. Optimal value function and optimal control manifold for linear stochastic system

The optimal value function V^* and the optimal control manifold $\mathbf{u}^*$ obtained from (18) are discretized by a 17×17 grid on domain $[-5,5] \times [-5,5]$, and plotted in Fig. 1. Apparently, for linear stochastic system, the optimal control manifold is a linear surface, which indicates a linear function of the states. The numerical optimal value function and control solution are obtained also by the same discretization scheme using the proposed method, and the error index $\mathbb{J}_5 = 3.3e^{-10} \approx 0$, which implies the numerically obtained control is indeed the optimal control.

B. Van der Pol oscillator (nonlinear damping)

Consider the following stochastic Van der Pol oscillator

$$\ddot{x}(t) + 0.1\big(1 - 0.5x^2(t)\big)\dot{x}(t) + 0.1x(t) = u(t) + 10\,dw(t)/dt \qquad (21)$$

The functions $\mathbf{f}$ and $\mathbf{g}$ in (1) can be expressed as $\mathbf{f}\big(\mathbf{x}(t)\big) = \begin{bmatrix} x_2 \\ -0.1x_1 - 0.1(1 - 0.5x_1^2)x_2 \end{bmatrix}$ and $\mathbf{g}\big(\mathbf{x}(t)\big) = [0 \quad 1]^T$. Other parameters are given as $\boldsymbol{\sigma}\big(\mathbf{x}(t)\big) = [0 \quad 10]^T$, $\mathbf{Q} = 10 \cdot \mathbf{I}$, $\mathbf{R} = 1$, and discount factor $\beta = 0.05$. The corresponding linearized optimal control $\mathbf{u}_L^*$ is obtained by replacing $\mathbf{A}$ in the previous section with $[\partial \mathbf{f}/\partial \mathbf{x}]_{\mathbf{x}=\mathbf{0}}$, and the same way is used in obtaining linearized optimal control for the rest of the numerical examples. The PS method is applied with a 17×17 grid discretization scheme on computational domain $[-10,10] \times [-10,10]$ (see Fig. 2). The optimal value functions and control manifold are obtained and plotted in Fig. 2.

Actually, the oscillator used in this example is a "negative" Van der Pol oscillator --- the damping term has a different sign since a stable fixed point is required at the origin, as most of the realistic stable structural systems. To validate the effectiveness and efficiency of the obtained optimal control, both linearized control and the obtained control are applied to the oscillator to check the performance. During the numerical simulation, the uncontrolled case has a diverged solution, and therefore $\mathbb{J}_1$ index is replaced with σ_x^c and $\mathbb{J}_2$ index is replaced with $\max_t(|x_t^c|)$. All performance indices are summarized in Table I.

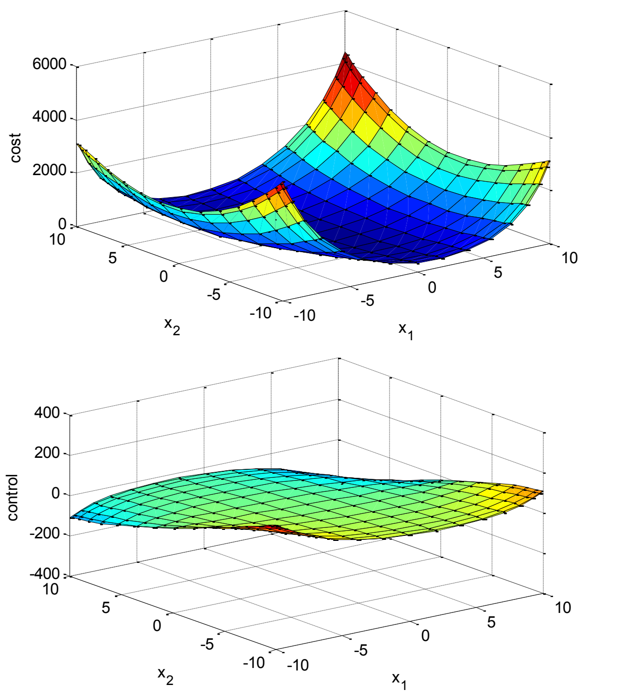

Fig. 2. Optimal value function and optimal control manifold for Van der Pol oscillator

From the performance index results in Table I, the optimal control provide a better performance than the linearized optimal control.

TABLE I
CONTROL PERFORMANCE INDEX FOR VAN DER POL OSCILLATOR

index	$\mathbb{J}_1(\mathbf{x})$	$\mathbb{J}_1(\dot{\mathbf{x}})$	$\mathbb{J}_2$	$\mathbb{J}_3$	$\mathbb{J}_4$
$\mathbf{u}^*$	1.8	3.5	5.7	149.6	8255.7
$\mathbf{u}_L^*$	2.2	3.7	7.8	174.6	8425.9

C. Duffing oscillator (nonlinear stiffness)

A stochastic Duffing oscillator is given as

$$\ddot{x}(t) + 0.1\dot{x}(t) + x(t) - 0.003x^3(t) = u(t) + 10\,dw(t)/dt \qquad (22)$$

The functions $\mathbf{f}$ and $\mathbf{g}$ in (1) can be expressed as $\mathbf{f}\big(\mathbf{x}(t)\big) = \begin{bmatrix} x_2 \\ -x_1 + 0.003x_1^3 - 0.1x_2 \end{bmatrix}$ and $\mathbf{g}\big(\mathbf{x}(t)\big) = [0 \quad 1]^T$. Other parameters are given as $\boldsymbol{\sigma}\big(\mathbf{x}(t)\big) = [0 \quad 10]^T$, $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = 1$, and discount factor $\beta = 0.01$. The

PS method is applied with a 17×17 grid discretization scheme on computational domain $[-20, 20] \times [-20, 20]$ (see Fig. 3).

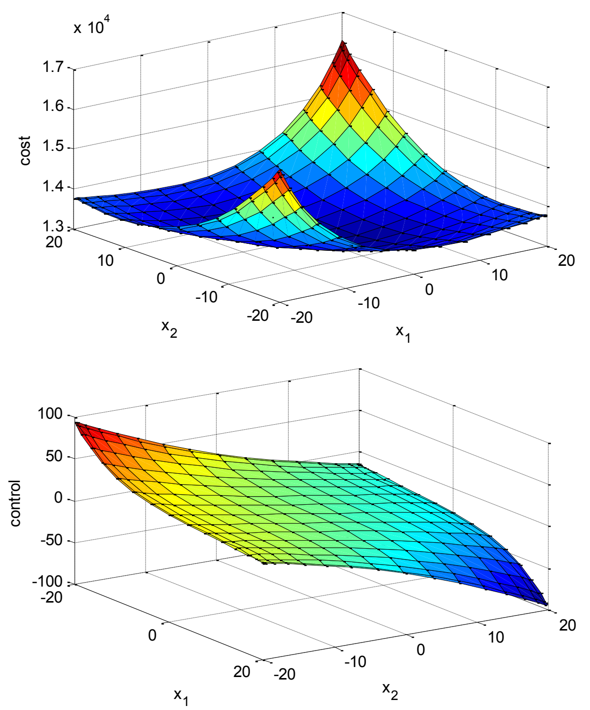

Fig. 3. Optimal value function and optimal control manifold for Duffing oscillator

The uncontrolled response in this case is also diverged, and therefore, for the same reason as the Van der Pol oscillator case, the performance indices $\mathbb{J}_1$ and $\mathbb{J}_2$ are modified in the same manner.

The performance index values are summarized in Table II. Again, the optimal controller outperforms the linearized optimal controller according to all indices.

TABLE II
CONTROL PERFORMANCE INDEX FOR DUFFING OSCILLATOR

index	$\mathbb{J}_1(\mathbf{x})$	$\mathbb{J}_1(\dot{\mathbf{x}})$	$\mathbb{J}_2$	$\mathbb{J}_3$	$\mathbb{J}_4$
$\mathbf{u}^*$	4.6	5.6	11.8	371.0	7043.3
$\mathbf{u}_L^*$	5.2	5.8	14.2	415.8	7056.3

D. Bouc-Wen hysteretic system

The nonlinear system in this case is the Bouc-Wen hysteresis system that contains three state variables, and therefore demands more computational power. The stochastic Bouc-Wen system is given as

$$\ddot{x}(t) + \sqrt{2}/200\,\dot{x}(t) + 0.5x(t) + 0.05z(t) = u(t) + 8\,dw(t)/dt \quad (23)$$

where $z(t)$ is the Bouc-Wen hysteretic component that can be described by

$$\dot{z} = 0.5\dot{x} - 2\dot{x}|z|^2 - 3|\dot{x}|z|z| \quad (24)$$

The functions $\mathbf{f}$ and $\mathbf{g}$ in (1) can be expressed as $\mathbf{f}(\mathbf{x}(t)) = \begin{bmatrix} x_2 \\ -0.5x_1 - \sqrt{2}/200\,x_2 - 0.05x_3 \\ 0.5x_2 - 2x_2|x_3|^2 - 3|x_2|x_3|x_3| \end{bmatrix}$ and $\mathbf{g}(\mathbf{x}(t)) = [0 \quad 1 \quad 0]^T$. Other parameters are given as $\boldsymbol{\sigma}(\mathbf{x}(t)) = [0 \quad 8 \quad 0]^T$, $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = 10$, and discount factor $\beta = 0.01$. The PS method is applied with a 7×7 grid discretization scheme on computational domain $[-20, 20] \times [-20, 20]$ to accommodate the computational demand. In this case, the time step is chosen as $dt = 0.001s$.

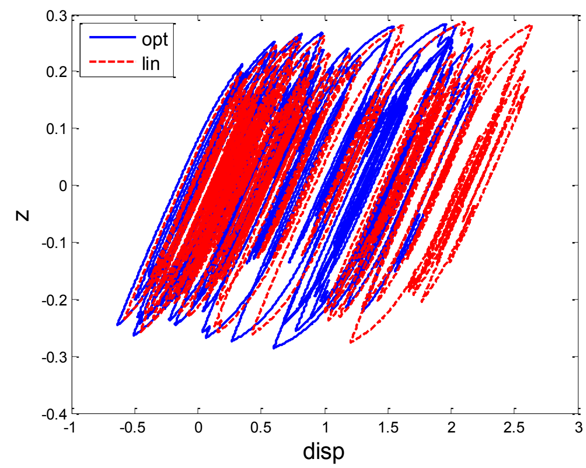

Fig. 4. Bouc-Wen hysteretic component vs. displacement

In Fig. 4, it is shown that the Bouc-Wen hysteretic component z varies nonlinearly with displacement. The performance index values are summarized in Table III. It is noted that (the highlighted cell in Table III), in this case, the index value of $\mathbb{J}_1(\dot{\mathbf{x}})$ shows that the linearized optimal control has a slightly better performance in terms of velocity variance suppression. However, the calculated optimal control shows better performance in terms of the other indices, especially $\mathbb{J}_4$, which is the control cost to be minimized.

TABLE III
CONTROL PERFORMANCE INDEX FOR BOUC-WEN HYSTERETIC SYSTEM

index	$\mathbb{J}_1(\mathbf{x})$	$\mathbb{J}_1(\dot{\mathbf{x}})$	$\mathbb{J}_2$	$\mathbb{J}_3$	$\mathbb{J}_4$
$\mathbf{u}^*$	0.014	0.057	0.024	74.6	283543.4
$\mathbf{u}_L^*$	0.017	0.055	0.028	88.3	297248.6

VI. SUMMARY

This paper applied PS method to obtain the optimal control solution to a verity of nonlinear systems. After validating the solution with linear case, the performance of obtained optimal controller is compared with linearized optimal control solution. In the future, the possibility of applying parallel computing technique in combination with PS method will be investigated.

ACKNOWLEDGEMENTS

The financial support from the National Science Foundation (Grant no. CMMI-1002641) is gratefully acknowledged.

REFERENCES

[1] M. Boulbrachene and M. Haiour, "The finite element approximation of Hamilton-Jacobi-Bellman equations," *Computers and Mathematics with Applications*, vol. 41(7-8), pp. 993-1007, 2001.

[2] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, Germany, 1988.

[3] S. J. Dyke, B. F. Spencer, Jr., M. K. Sain, and J. D. Carlson, "Modeling and control of magnetorheological dampers for seismic response reduction," *Smart Materials and Structures*, vol. 5, pp. 565-575, 1996.

[4] J. Elnagar, M. A. Kazemi, and M. Razzaghi, "The Pseudospectral Legendre Method for Discretizing Optimal Control Problems," *IEEE Transactions on Automatic Control*, vol. 40(10), pp. 1793–1796, 1995.

[5] F. Fahroo, and I. M. Ross, "Costate Estimation by a Legendre Pseudospectral Method", *Journal of Guidance, Control and Dynamics*, vol. 24(2), pp. 270–277, 2001.

[6] W. H. Fleming, and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd Edition, Springer US, 2008.

[7] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, 1995.

[8] G. N. Saridis, and C. G. Lee, "An approximation theory of optimal control for trainable manipulators," *IEEE Transactions on System, Man and Cybernetics*, vol. 9(3), pp. 152-159, 1979.

[9] J. Wang, and P. A. Forsyth, "Numerical solution of the Hamilton–Jacobi–Bellman formulation for continuous time mean variance asset allocation," *Journal of Economic Dynamics and Control*, vol. 34(2), pp. 207-230, 2010.

[10] S. Wang, L. S. Jennings, and K. L. Teo, "Numerical Solution of Hamilton-Jacobi-Bellman Equations by an Upwind Finite Volume Method," *Journal of Global Optimization*, vol. 27(2-3), pp. 177 - 192, 2003.

[11] Y. K. Wen, "Method for random vibration of hysteretic systems," *Journal of Engineering Mechanics Division, ASCE*, vol. 102(2), pp. 249-263, 1976.

[12] J. N. Yang, A. K. Agrawal, and S. Chen, "Optimal polynomial control for seismically excited non-linear and hysteretic structures," *Earthquake Engineering and Structural Dynamics*, vol. 25, pp. 1211-1230, 1996.

[13] J. N. Yang, S. Lin, and F. Jabbari, "H∞-based control strategies for civil engineering structures," *Structural Control and Health Monitoring*, vol. 11(3), pp. 223-237, 2004.

[14] J. N. Yang, J. C. Wu, and A. K. Agrawal, "Sliding mode control for nonlinear and hysteretic structures," *Journal of Engineering Mechanics, ASCE*, vol. 121(12), pp. 1330-1339, 1995.

[15] Z. G. Ying, Y. Q. Ni, and J. M. Ko, "Semi-active optimal control of linearized systems with multi-degree of freedom and application," *Journal of Sound and Vibration*, vol. 279(1-2), pp. 373-388, 2005.

[16] W. Q. Zhu, Z. G. Ying, Y. Q. Ni, and J. M. Ko, "Optimal Nonlinear Stochastic Control of Hysteretic Systems," *Journal of Engineering Mechanics, ASCE*, vol. 126(10), pp. 1027-1032, 2000.