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Nonlinear Observer Design for Lipschitz Nonlinear Systems

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Abstract— **This paper presents a nonlinear observer design methodology for a class of Lipschitz nonlinear systems via convex optimization. A sufficient condition for the existence of an observer gain matrix to stabilize the estimation error dynamics is given in term of a quadratic stability margin. In addition, the observer gain matrix is optimally designed by minimizing the magnitude of elements of the observer gain matrix to reduce the amplification of sensor measurement noise. Furthermore, when disturbances considered as unknown deterministic inputs are imposed on the error dynamics in an additive form, the observer gain matrix is redesigned to minimize an induced $\mathcal{L}_2$ gain between the disturbance to the estimation error as well as the effect of measurement noise. Finally a systematic design algorithm is applied to a flexible joint robot system.**

I. INTRODUCTION

Consider the class of Lipschitz nonlinear systems for observer design as follows:

$$\begin{aligned} \dot{x} &= Ax + B_u u + f(x) \\ y &= Cx \end{aligned} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}^m$, the measurement $y \in \mathbb{R}^p$ where $p < n$, f is Lipschitz with a Lipschitz constant γ, *i.e.*,

$$\|f(x) - f(\hat{x})\| \leq \gamma \|x - \hat{x}\|, \quad \forall (x, \hat{x}) \in \mathcal{D}, \tag{2}$$

and the pair (A, C) is observable. If $\mathcal{D} = \mathbb{R}^n$, f is globally Lipschitz. Otherwise, it is locally Lipschitz.

A "Luenberger-like" nonlinear observer for (1) was proposed originally in [1] as follows:

$$\dot{\hat{x}} = A\hat{x} + B_u u + f(\hat{x}) + L(y - C\hat{x}) \tag{3}$$

After defining the estimation error by $e := x - \hat{x}$ and using the inequality constraint (2), the estimation error dynamics are written as follows:

$$\begin{aligned} \dot{e} &= (A - LC)e + \{f(x) - f(\hat{x})\} := A_{ob}e + \phi \\ \|\phi\| &\leq \gamma \|e\| \end{aligned} \tag{4}$$

where $A_{ob} = A - LC$ and $\phi = f(x) - f(\hat{x})$.

There have been a series of results for the observer design in the literature [1]-[8] and their contributions are summarized in sequential order as follows:

- Stability analysis for the estimation error dynamics given in (4) (refer to [1], [2])
- Sufficient conditions for existence of the observer gain matrix L to gurantee asymptotic stability of the error dynamics (refer to [3]-[6])
- Optimal design of the observer gain matrix with consideration of disturbances (refer to [7], [8])

Since all results of the stability analysis can be used to check the stability of the estimation error dynamics only after the gain matrix L is assigned, this motivates a problem for the existence of the observer gain matrix to guarantee stability. Then, under the sufficient condition for the existence of the gain matrix, the optimal design problem is considered from an input-output stability point of view via a linear matrix inequality (LMI) approach.

The contribution of this paper is to show equivalences or analogies among the previous results of nonlinear observer design in the literature and formulate them in the form of LMI. Then, it will be shown that the existence of the observer gain matrix to stabilize the estimation error dynamics can be guaranteed by checking whether the quadratic stability margin of the error dynamics is greater than the Lipschitz constant. Finally, the observer gain matrix can be designed optimally in the sense that amplification of sensor measurement noise is reduced and an induced $\mathcal{L}_2$ gain between the disturbance to the estimation error is minimized.

II. QUADRATIC STABILITY

The error dynamics in (4) can be considered as a linear system subject to a vanishing perturbation in the sense that $\phi(t, x, \hat{x}) \to 0$ as $e(t) \to 0$. The corresponding result for stability analysis was addressed originally by Thau [1] and may be modified by use of the result of [9] as follows: If there is a gain matrix L such that

$$\gamma < \frac{1}{2\lambda_{max}(P)} \tag{5}$$

where P is a symmetric positive definite matrix satisfying

$$A_{ob}^T P + P A_{ob} = -I,$$

the origin in (4) is exponentially stable.

If the estimation error dynamics in (4) are regarded as a norm-bound linear differential inclusion (NLDI) [10]. Using the result in [10], the stability analysis problem can be addressed in the form of LMI and the same result is also found in [7].

Theorem 1: If there exist $P > 0$ and $\sigma \geq 0$ such that

$$\begin{bmatrix} A_{ob}^T P + P A_{ob} + \sigma\gamma^2 I & P \\ P & -\sigma I \end{bmatrix} < 0 \tag{6}$$

the origin in (4) is exponentially stable for the given L.

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Instead of (5) and (6), other sufficient conditions for stability have been proposed by Rajamani and Cho [4] and Aboky *et al* [6] as follows:

Theorem 2: The estimation error dynamics in (4) are exponentially stable if one of the following equivalent statements is satisfied for the given gain matrix L:

1. there exists a positive definite matrix P such that
$$(A-LC)^TP+P(A-LC)+\gamma^2PP+I<0 \quad (7)$$
2. there exists a positive definite matrix P such that
$$\begin{bmatrix} A_{ob}^TP+PA_{ob}+\gamma^2I & P \\ P & -I \end{bmatrix}<0 \quad (8)$$
3. there exist a positive definite matrix P and $\sigma>0$ satisfying LMI (6)

Proof: 1. Consider the Lyapunov function candidate $V=e^TPe$ where $P>0$. Its derivative of V along the trajectory of (4) is
$$\dot{V}=e^T[A_{ob}^TP+PA_{ob}]e+2e^TP[f(x)-f(\hat{x})] \quad (9)$$
Using the inequality constraint (2),
$$\begin{aligned} 2e^TP[f(x)-f(\hat{x})] &\le 2\gamma\|Pe\|\|e\| \\ &\le \gamma^2e^TP^TPe+e^Te \end{aligned} \quad (10)$$
If the inequality (10) is used, (9) is bounded as
$$\dot{V}\le e^T[(A-LC)^TP+P(A-LC)+\gamma^2PP+I]e$$
Therefore, if the matrix inequality condition (7) is satisfied, $\dot{V}<0$, thus the error dynamics (4) becomes asymptotically stable.

$1\leftrightarrow 2$: The inequality condition (10) can be also written as
$$2e^TP[f(x)-f(\hat{x})]\le e^TP^TPe+\gamma^2e^Te. \quad (11)$$
Similarly, if the inequality (11) is used, the LMI condition for $\dot{V}<0$ for all nonzero e is
$$(A-LC)^TP+P(A-LC)+PP+\gamma^2I<0. \quad (12)$$
The Schur complement of the above condition is LMI (8).

$2\leftrightarrow 3$: After defining $P=\tilde{P}/\sigma$ where $\sigma>0$, LMI (12) becomes
$$(A-LC)^T\tilde{P}+\tilde{P}(A-LC)+\frac{1}{\sigma}\tilde{P}\tilde{P}+\sigma\gamma^2I<0$$
Using the Schur complement of the above linear matrix inequality, this is equivalent to LMI (6). ■

As pointed out in [3], [4], all results in (5) and Theorem 2 can be used to check the stability of the estimation error dynamics only after the gain matrix L is assigned and the Lipschitz constant γ is known. However, if the stability is discussed in the form of LMI as suggested in Theorem 2, a region of attraction of the estimation error dynamics (4) for locally Lipschitz nonlinear systems can be estimated. In other words, instead of checking the feasibility of P in Theorem 2, we can calculate the maximum value of γ which is computed in the framework of convex optimization. The *quadratic stability margin* which is the largest nonnegative α for which the origin in (4) satisfying $\phi^T\phi\le\alpha^2e^Te$ is exponentially stable is considered [10]. That is, we can maximize γ in (7) and (8) and the corresponding quadratic stability margin can be computed as follows:

Corollary 1: The estimation error dynamics in (4) are exponentially stable if there exists a solution to the following convex optimization problem:
$$\begin{aligned} &\textit{maximize} && \alpha^2 \\ &\textit{subject to} && P>0, \\ &&& \begin{bmatrix} A_{ob}^TP+PA_{ob}+\alpha^2I & P \\ P & -I \end{bmatrix}<0 \end{aligned} \quad (13)$$
for the given L and the computed quadratic stability margin α is greater than or equal to γ.

Remark 1: It is remarked that the Schur complement of (7) is
$$\begin{bmatrix} A_{ob}^TP+PA_{ob}+I & P \\ P & -\frac{1}{\gamma^2}I \end{bmatrix}<0. \quad (14)$$
After defining $\beta=1/\gamma^2$, the quadratic stability margin can also be computed as follows:
$$\begin{aligned} &\textit{minimize} && \beta \\ &\textit{subject to} && P>0,\quad \beta\ge 0, \\ &&& \begin{bmatrix} A_{ob}^TP+PA_{ob}+I & P \\ P & -\beta I \end{bmatrix}<0 \end{aligned} \quad (15)$$
Then, the stability margin α can be calculated by $1/\sqrt{\beta}$.

III. Existence of Observer Gain Matrix

All results in the previous section provide a method to check the stability and to compute the stability margin for the given gain matrix L. However, they do not tell us how to design L to satisfy the stability condition. This observer design problem has been considered by several researchers in the literature. Algebraic Riccati equations (ARE) or linear matrix inequalities (LMI) for guaranteeing the quadratic stability have been proposed by Raghavan and Hedrick [2], Rajamani and Cho [4], and Aboky *et al* [6]. The results can be summarized as:

Theorem 3: The following statements are equivalent:

1. [2] For some small ϵ, if there exists a positive definite P such that
$$AP+PA^T+P(\gamma^2I-\frac{1}{\epsilon}C^TC)P+I+\epsilon I=0 \quad (16)$$
then the error dynamics (4) can be stabilized by $L=PC^T/2\epsilon$.
2. [4] If there exists a positive definite P such that
$$A^TP+PA+\gamma^2PP+I-\frac{1}{\gamma^2}C^TC<0 \quad (17)$$
then the error dynamics (4) can be stabilized by $L=P^{-1}C^T/2\gamma^2$.
3. [6] If there exists a positive definite P such that
$$A^TP+PA+\gamma^2I-C^TC+PP<0 \quad (18)$$

then the error dynamics (4) can be stabilized by $L = P^{-1}C^T/2$.

4. If there exist $P > 0$ and $\epsilon > 0$ such that

$$\begin{bmatrix} A^TP + PA + \gamma^2 I - \frac{1}{\epsilon}C^TC & P \\ P & -I \end{bmatrix} < 0 \quad (19)$$

then the error dynamics (4) can be stabilized by

$$L = \frac{P^{-1}C^T}{2\epsilon}. \quad (20)$$

Proof: Since the proof of the statement 1 is in [2], we will only show its equivalence to the statement 2 , 3, and 4.

$1 \leftrightarrow 2$: ARE (16) can be written as the following matrix inequality without loss of generality

$$AP + PA^T + P(\gamma^2 I - \frac{1}{\epsilon}C^TC)P + I < 0 \quad (21)$$

Let $P = P_1^{-1}/\gamma^2$ where $P_1 > 0$. Then

$$A\frac{P_1^{-1}}{\gamma^2} + \frac{P_1^{-1}}{\gamma^2}A^T + \frac{P_1^{-1}}{\gamma^2}(\gamma^2 I - \frac{1}{\epsilon}C^TC)\frac{P_1^{-1}}{\gamma^2} + I < 0$$

Multiplying the above inequality on the left and right by $(\gamma P_1)^T$ and γP_1 respectively, we get

$$A^TP_1 + P_1A + I - \frac{1}{\epsilon\gamma^2}C^TC + \gamma^2 P_1P_1 < 0 \quad (22)$$

Since the gain matrix L is

$$L = \frac{PC^T}{2\epsilon} = \frac{P_1^{-1}C^T}{2\epsilon\gamma^2},$$

substituting L in (22) gives that

$$(A - LC)^TP_1 + P_1(A - LC) + \gamma^2 P_1P_1 + I < 0$$

which is equivalent to (7). Using the result of Theorem 2, the error dynamics (4) are quadratically stabilized by the observer (1). Finally, when $P_1 = P$ and $\epsilon = 1$, the matrix inequality (22) is the same as (17).

$2 \leftrightarrow 3$: Let $P_2 = \gamma^2 P_1$. After substituting P_1 with P_2 in (22),

$$A^TP_2 + P_2^TA + \gamma^2 I - \frac{1}{\epsilon}C^TC + P_2P_2 < 0 \quad (23)$$

Since the gain matrix L is

$$L = \frac{P_1^{-1}C^T}{2\gamma^2\epsilon} = \frac{P_2^{-1}C^T}{2\epsilon},$$

substituting L in (23) gives that

$$(A - LC)^TP_2 + P_2(A - LC) + P_2P_2 + \gamma^2 I < 0$$

which is equivalent to (8) in Theorem 2. When $P_2 = P$ and $\epsilon = 1$, the inequality condition (23) is equal to (18).

$3 \leftrightarrow 4$: The Schur complement of (23) is LMI (19). ■

For the design of the gain matrix L, the next question is under what conditions the positive definite matrix P and/or positive constant ϵ exist to satisfy one of (16) - (19) in Theorem 3. To address the problem, the idea of the distance to unobservability was originally proposed in [4] and was modified with an additional constraint by Aboky *et al* [6]. While a bisection algorithm is used for computing δ numerically to guarantee the existence of a quadratically stabilizing observer gain matrix [6], [11], the existence condition of a gain matrix L can be formulated in the framework of convex optimization as follows:

Theorem 4: If there exists a solution of the following convex optimization problem (COP): For a fixed $\epsilon^* > 0$,

$$\begin{aligned} &\text{maximize} \quad \alpha^2 \\ &\text{subject to} \quad P > 0, \\ &\qquad \begin{bmatrix} A^TP + PA + \alpha^2 I - \frac{1}{\epsilon^*}C^TC & P \\ P & -I \end{bmatrix} < 0 \end{aligned} \quad (24)$$

and the computed quadratic stability margin $\alpha \geq \gamma$, it guarantees the existence of P for all $\epsilon \in (0, \epsilon^*]$ in LMI (19).

Proof: The Schur complement of the above LMI in COP (24) is

$$A^TP_1 + P_1A + \alpha_1^2 I - \frac{1}{\epsilon^*}C^TC + P_1P_1 < 0.$$

where P_1 and α_1 are a solution of COP (24). This LMI can be written as

$$A^TP_1 + P_1A + \gamma^2 I - \frac{1}{\epsilon^*}C^TC + P_1P_1 < -(\alpha_1^2 - \gamma^2)I.$$

Since $\alpha_1 \geq \gamma$, a pair (P_1, ϵ^*) is also a solution of LMI (19). Furthermore, it is shown in [2] that all $\epsilon \in (0, \epsilon^*)$ solve ARE (16) if there exists a solution of ARE (16) for a certain ϵ^*. By use of the equivalence of Theorem 2, there exists a solution of LMI (19) for all $\epsilon \in (0, \epsilon^*)$. ■

The coordinate transformation has been proposed to reduce the Lipschitz constant and increase the distance to unobservability in the new coordinates [2], [4]. Similarly, this can be used to increase the quadratic stability margin α in Theorem 4. Suppose $z = Tx$ and $\hat{z} = T\hat{x}$ where T is invertible and called a transformation matrix. Then, the errors in the new coordinates are defined as $\tilde{e} = Te$. Equation (4) in the new coordinates is given by

$$\begin{aligned} \dot{\tilde{e}} &= T(A - LC)T^{-1}\tilde{e} + T\{f(T^{-1}z) - f(T^{-1}\hat{z})\} \\ &= \tilde{A}_{ob}\tilde{e} + \tilde{\phi} \end{aligned} \quad (25)$$

where

$$\tilde{A}_{ob} = T(A - LC)T^{-1},$$
$$\tilde{\phi} = T\{f(T^{-1}z) - f(T^{-1}\hat{z})\}.$$

Since A_{ob} and $\tilde{A}_{ob}$ are similar, they have the same eigenvalues, but result in different values of $\lambda_{max}(P)$ in (5) or α in COP (24). However, there is no clear relation between T and either $\lambda_{max}(P)$ or the stability margin α.

IV. Optimal Design

Either the solutions of AREs proposed in [2], [4], [6] or the results of Theorem 2 and 4 do not provide a specific level of performance for the observer [12]. Therefore, the last question in this paper is how to optimize the observer's performance as well as guarantee the existence of a quadratically stabilizing observer gain matrix.

To optimize the performance of an observer, we need to specify a desirable property for the observer. Among many desirable properties such as the magnitude of the elements of the gain matrix, decay rate, and $\mathcal{L}_2$ gain [10], [12], the magnitude of the elements of the gain matrix is considered to reduce the amplification of sensor measurement noise. Since the observer gain matrix depends upon the inverse of P in Theorem 3, the minimum eigenvalue of P should be maximized to reduce the maximum singular value of the observer gain matrix elements. Therefore, it can be formulated as follows: For the given γ,

$$\begin{aligned} &\text{maximize} \quad \lambda_{min}(P) \\ &\text{subject to} \quad P > 0,\ \text{LMI (19)} \end{aligned} \tag{26}$$

where $\epsilon \in (0, \epsilon^*)$ is given and ϵ^* is a solution of COP (24). Then, the resulting observer gain matrix is given in (20).

Alternatively, since the magnitude of elements of $L = P^{-1}C^T/2\epsilon$ is dependent on $1/\epsilon$ as well as the inverse of P, we can minimize both of them and this is called a multi-objective optimization problem. The following multi-objective optimization problem can be considered by using scalarization for finding Pareto optimal points [12], [13]: For the given $\delta \in [0, 1]$,

$$\begin{aligned} &\text{maximize} \quad \delta\lambda_{min}(P) - (1-\delta)\sigma \\ &\text{subject to} \quad P > 0,\ \sigma - \sigma^* \geq 0,\ \text{LMI (19)} \end{aligned} \tag{27}$$

where $\sigma = 1/\epsilon$, $\sigma^* = 1/\epsilon^*$, and δ represents a relative weight between two objective functions. The resulting observer gain matrix is

$$L = \sigma \frac{P^{-1}C^T}{2}.$$

If an unknown exogenous input d is considered, a class of nonlinear systems considered are written as

$$\dot{x} = Ax + B_u u + f(x) + B_d d. \tag{28}$$

The corresponding estimation error dynamics are

$$\begin{aligned} \dot{e} &= (A - LC)e + f(x) - f(\hat{x}) + B_d d \\ &= A_{ob}e + \phi + B_d d \end{aligned} \tag{29}$$

Then, a desirable property of an observer is to make the state estimates insensitive to d representing disturbances and uncertainties. To consider this property, the induced $\mathcal{L}_2$ gain between the exogenous input d and the estimation error e, signified as $\|H_{d\to e}\|_\infty$, are minimized by redesigning the observer gain matrix L.

Theorem 5: For the given nonlinear system in (28) and nonlinear observer in (3), the observer error dynamics in (29) has $\|H_{d\to e}\|_\infty \leq \kappa$ if there exist $P > 0$, $\epsilon > 0$, and $\kappa \geq 0$ such that

$$\begin{bmatrix} \begin{pmatrix} A^TP + P^TA + \\ (1+\gamma^2)I - \frac{1}{\epsilon}C^TC \end{pmatrix} & P & PB_d \\ P & -I & 0 \\ B_d^TP & 0 & -\kappa^2 I \end{bmatrix} < 0 \tag{30}$$

and the resulting observer gain matrix is $L = \frac{P^{-1}C^T}{2\epsilon}$.

Proof: Suppose there exist $V(e) = e^TPe$, $P > 0$, and $\kappa \geq 0$ such that

$$\dot{V} + e^Te - \kappa^2 d^Td \leq 0. \tag{31}$$

After integrating the left side of (31) from 0 to T with the assumption that $e(0) = 0$,

$$V(T) + \int_0^T (e^Te - \kappa^2 d^Td)dt \leq 0$$

Since $V(T) \geq 0$, this implies that $\|H_{d\to e}\|_\infty \leq \kappa$ by the definition [10]

$$\kappa^2 = \|H_{d\to e}\|_\infty^2 = \sup_{\|d\|_2 \neq 0} \frac{\|e\|^2}{\|d\|^2}.$$

The inequality (31) is equivalent to

$$e^T(A_{ob}^TP + PA_{ob} + I)e + 2e^TP(\phi + B_d d) - \kappa^2 d^Td \leq 0 \tag{32}$$

for all (e, ϕ, d) satisfying $\|\phi\| \leq \gamma\|e\|$. Using the inequality condition in (11), the inequality condition in (32) holds if

$$e^T\{A_{ob}^TP + PA_{ob} + PP + (1+\gamma^2)I\}e + 2e^TPB_d d \\ -\kappa^2 d^Td < 0$$

If $L = \frac{P^{-1}C^T}{2\epsilon}$ is used, the above inequality becomes

$$e^T\left\{A^TP + PA + PP + (1+\gamma^2)I - \frac{1}{\epsilon}C^TC\right\}e \\ +2e^TPB_d d - \kappa^2 d^Td < 0$$

This is written in the matrix form as follows:

$$\begin{bmatrix} A^TP + PA + PP + (1+\gamma^2)I - \frac{1}{\epsilon}C^TC & PB_d \\ B_d^TP & -\kappa^2 I \end{bmatrix} < 0$$

Finally, using the Schur complement, the above inequality condition is equivalent to LMI (30). ■

Finally, the design procedure for the observer gain matrix is summarized as:

Algorithm 1: Observer design procedure

Step 1. *Solve COP (24) iteratively with logarithmic spacing of* $\epsilon_k \in [10^{-n}, 10^n]$. *If there exists an integer* k^* *such that* $\alpha_{k^*} \geq \gamma$ *, go to Step 3 with* $T = I$. *Otherwise, go to Step 2 .*

Step 2. *Use a coordinate transformation as suggested in (25) to reduce the Lipschitz constant. Go to Step 1 with* $\tilde{A} = TAT^{-1}$, $\tilde{C} = CT^{-1}$, *and a new Lipschitz constant* $\tilde{\gamma}$.

Step 3. *Solve COP (26) for the given* $\epsilon \in [10^{-n}, \epsilon_{k^*}]$. *With the solution, the observer gain matrix* L *is*

$$L = T^{-1}\frac{P^{-1}C^T}{2\epsilon}. \tag{33}$$

Step 4. *If exogenous unknown inputs are considered,* P *in (33) is computed by solving the following convex optimization problem: for a fixed* $\delta \in [0\ 1]$ *and* $\epsilon \in [10^{-n}, \epsilon_{k^*}]$,

$$\begin{aligned} &\textit{minimize} \quad \delta\kappa^2 - (1-\delta)\lambda_{min}(P) \\ &\textit{subject to} \quad P > 0, \quad \textit{LMI } (30) \end{aligned} \tag{34}$$

If the induced $\mathcal{L}_2$ gain κ in LMI (30) is minimized, the magnitude of elements of the observer gain matrix in general becomes large. On the other hand, it can be minimized if the objective function is defined as in COP (26). Therefore, the gain matrix can be chosen by performing a trade-off between two objectives as stated in COP (34).

V. Illustrative Example

A fourth-order nonlinear model which represents a flexible joint robotic arm is considered to illustrate the proposed observer design technique [2], [4]. The system model can be described by the following equation:

$$\begin{aligned} \dot{x} &= Ax + B_u u + f(x) + B_d d \\ y &= Cx \end{aligned} \tag{35}$$

where $x = [\theta_m \ \omega_m \ \theta_l \ \omega_l]^T \in \mathbb{R}^4$, and the matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \ C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_u = [0 \ 21.6 \ 0 \ 0]^T, \ f(x) = [0 \ 0 \ 0 \ -3.33 \sin(\theta_l)]^T,$$

and B_d and d are the additional terms considered as an exogenous known input and will be defined in Step 4. Furthermore, the pair (A, C) is controllable and there exists a positive constant γ such that

$$\|f(x) - f(\hat{x})\| \le \gamma \|e\|.$$

That is, we can let $\gamma = 3.33$ in this example.

a) Step 1. Calculation of stability margin: After solving COP (24) iteratively with logarithmic spacing of $\epsilon_k \in [10^{-1} \ 10]$ for the given γ, the quadratic stability margin α_k with respect to ϵ_k for $k = 1, \ldots, 20$ is shown in Fig. 1. It is noted that the results are shown up to $k = 17$ where the solution of COP (24) is calculated numerically via cvx [14]. Since $\alpha_k < \gamma$ for all k, we need to go to Step 2 to reduce the Lipschitz constant via coordinate transformation.

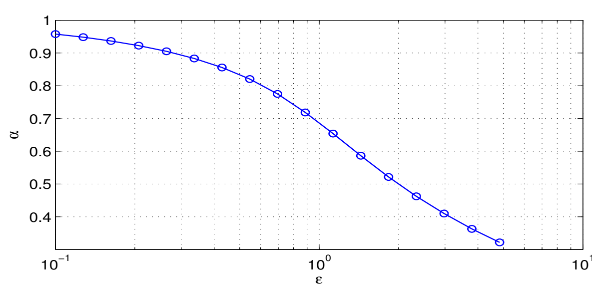

Fig. 1. Quadratic stability margin with respect to ϵ_k for the given $\gamma = 3.33$

b) Step 2. Coordinate transformation: Suppose $T = diag(1, 1, 1, 0.1)$ which is a 4×4 diagonal matrix with the vector $[1 \ 1 \ 1 \ 0.1]$ forming the diagonal and used in [4]. With this coordinate transformation $z = Tx$, the system (35) becomes

$$\begin{aligned} \dot{z} &= TAT^{-1}z + TB_u u + Tf(T^{-1}z) + TB_d d \\ y &= CT^{-1}z \end{aligned} \tag{36}$$

with the inequality constraint

$$\|Tf(T^{-1}z) - Tf(T^{-1}\hat{z})\| \le 0.1\gamma \|\tilde{e}\| = \tilde{\gamma}\|\tilde{e}\|.$$

Then, we have $\tilde{A} = TAT^{-1}$, $\tilde{C} = CT^{-1}$, and the new Lipschitz constant $\tilde{\gamma} = 0.333$. Fig. 2 shows the quadratic stability margin (α_k) with respect to ϵ_k for the given $\tilde{\gamma}$. It is found that $\alpha_k = 0.3630$ is greater than $\tilde{\gamma}$ when $\epsilon_k = 4.8329$ for $k = 17$ in Fig. 2. it is summarized that $\alpha > \gamma$ for all ϵ satisfying $\epsilon \le 4.8329$. Therefore, by Theorem 4, it is guaranteed that there exist a positive definite matrix P satisfying LMI (19) for all $\epsilon \in (0, 4.8329]$.

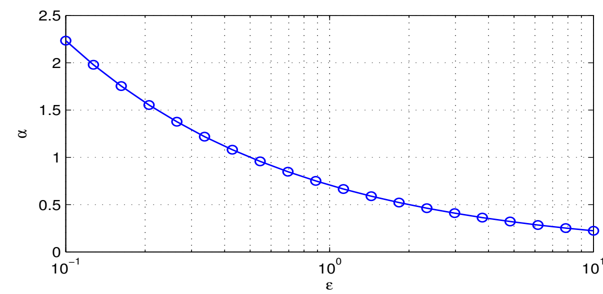

Fig. 2. Quadratic stability margin with respect to ϵ_k for the given $\tilde{A}$ and $\tilde{C}$

c) Step 3. Design of observer gain matrix L: Suppose $d = 0$ in (36). If COP (26) is solved numerically via cxv for an $\epsilon \in (0, 4.8329]$, we can calculate the observer gain matrix in (33). When $\epsilon = 0.1$ and 1 respectively, the corresponding gain matrices L_1 and L_2 are calculated as

$$L_1 = \begin{bmatrix} 1.6477 & 0.1965 \\ 0.1965 & 11.5338 \\ 1.5185 & 3.1468 \\ 2.9224 & 10.8291 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0.8155 & 0.4422 \\ 0.4422 & 6.1759 \\ 0.8080 & 1.2933 \\ 0.6979 & 2.5587 \end{bmatrix}$$

When a sinusoidal input with frequency of 1 Hz is used to drive the system dynamics, the state estimates for the given gain matrices L_1 and L_2 are compared in Fig. 3. The state estimate with L_1 approaches the true value faster than one with L_2 although the estimation errors with both gain matrices converge zero asymptotically. It is noted that the gain matrix L_2 is quite similar with the value of an observer gain matrix in [4] as expected from results of Theorem 2.

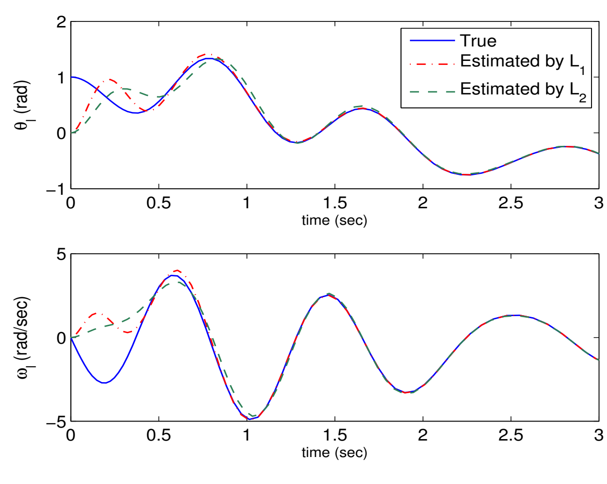

Fig. 3. Estimation of the link angle and angular velocity by different gain matrices L_1 and L_2 respectively

Alternatively, if COP (27) is solved iteratively for the given $\delta \in [0, 1]$, *i.e.*, for various relative weights given to two objective functions, the corresponding values of two objective functions for a fixed δ are shown in Fig. 4. It is shown that $1/\epsilon$ is proportional to $\lambda_{min}(P)$ as δ varies from 0.1 to 0.99 and a set of solutions can produce the feasible observer gain matrices. Therefore, a specific observer gain matrix can be obtained for the given δ.

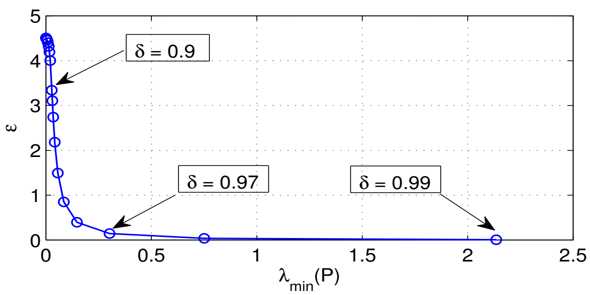

Fig. 4. Minimum eigenvalue of P and $\epsilon = 1/\sigma$ with respect to δ

d) Step 4. Consideration of exogenous unknown input: Suppose $d = \sin(10\omega_m)$ and $B_d = [0\ 2\ 0\ 0]^T$ in (35) and (36). When $\epsilon = 0.1$ which is used for the calculation of L_1 in Step 3, the gain matrix L_3 is computed by minimizing the induced $\mathcal{L}_2$ gain, κ in LMI (30) or solving COP (34) for $\delta = 0$. In addition, the gain matrix L_4 is computed by solving COP (34) for $\delta = 0.2$ respectively as follows:

$$L_3 = \begin{bmatrix} 44727 & -27.15 \\ -27.15 & 63.058 \\ 0.0496 & 17.314 \\ 11.092 & 39.441 \end{bmatrix},\ L_4 = \begin{bmatrix} 4.5906 & -1.312 \\ -1.312 & 16.197 \\ 3.1660 & 3.8467 \\ 9.1013 & 9.6936 \end{bmatrix}$$

As discussed in Algorithm 1, the magnitude of elements of L_3 are large when the $\mathcal{L}_2$ gain is only minimized. However, when the additive disturbance d is considered, the state estimate with L_3 goes to the true value closer and faster than one with L_1 (compare a dash line with a dash-dot line in Fig. 5). When L_4 is used, it is shown that the performance of estimation is close to one with L_3 (compare a dash-dot line with a solid line in Fig. 5) and the magnitude of elements of L_4 is relative small.

VI. CONCLUSIONS

This paper presented a nonlinear observer design method for Lipschitz nonlinear systems via convex optimization. By checking whether the quadratic stability margin of the error dynamics is greater than the Lipschitz constant, the existence of the observer gain matrix to stabilize the estimation error dynamics was guaranteed. Furthermore, the observer gain matrix was designed optimally to minimize an induced $\mathcal{L}_2$ gain between the disturbance to the estimation error as well as the effect of measurement noise.

VII. ACKNOWLEDGMENTS

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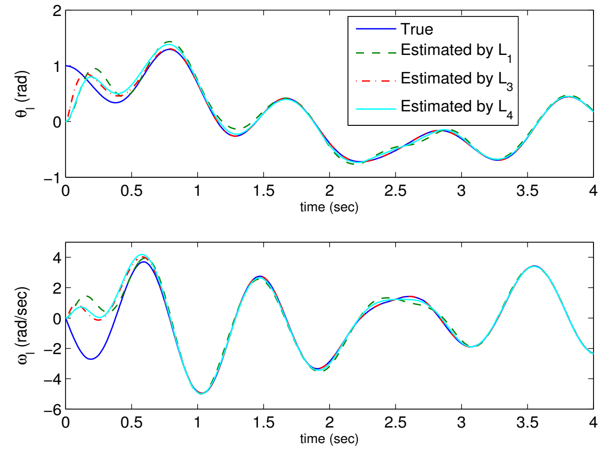

Fig. 5. Estimation of the link angle and angular velocity with consideration of exogenous unknown input

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