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Optimization and Pose Selection for a Lindy Hop Partnered Spin

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Abstract— **Swing dancers often talk about using the laws of physics in performing their physically rigorous jumps, lifts, and spins. Do expert swing dancers physically optimize their pose for a partnered spin? In a partnered spin, two dancers connect hands and spin as a unit around a single vertical axis. We describe the pose of a couple by the angles of their joints in a two-dimensional plane and compare expert and novice dancers' actual poses to the approximately ideal poses generated from a biomechanical optimization model.**

The optimization objective is to maximize rotational acceleration, by minimizing the resistance to spin, but still producing torque. The model considers only external forces and neglects internal forces. It consists of equations derived from physical principles such as Newton's laws and moment of inertia calculations. Using numerical non-linear optimization we find the pose for each couple that maximizes their rotational acceleration. Different dancers are differently sized, so every couple has a unique optimal pose. Each couple's optimal pose is compared to the pose they actually assumed for the spin.

We used motion capture to determine the angles of the joints in the couple's actual pose. The couple's actual pose is used to calculate a predicted rotational acceleration. This predicted acceleration is then compared to the optimal acceleration to determine a fraction of optimal for each couple. We hypothesized that expert swing dancers would achieve a higher fraction of their optimal acceleration than beginners. Our results did not achieve statistical significance with a simplified model and a small sample of 10 couples.

I. Why Swing Dance and Physics?

Lindy Hop is an athletic style of dancing that originated in the 1920s and is now danced recreationally and competitively. It is an American folk partner dance that originated in Harlem during the 1920's and 1930's. Today Lindy Hop is danced to very fast music and can involve aerial tricks in addition to fancy footwork. Advanced dancers often train intensely for five to ten years before reaching the top levels of competition. While other authors have examined how ballet dancers exploit physics [1] [2], and suggested dogs chase balls with near-optimal paths [4], no one has studied swing dancers.

A rhythm circle is a movement in which a couple spins as a unit around a single vertical axis. A good rhythm circle would look smooth, but also involve the dancers rotating rapidly. When discussing this movement, dancers often talk about minimizing moment of inertia or the need to create torque to spin. To our knowledge we are the first to investigate these descriptions quantitatively [6].

We developed a few candidate mathematical objectives that might represent the dancers' choices in controlling their spin. An optimal control model that captures the dynamics of the dancers' pose and motion over time would be appropriate. However, in this paper, we discuss only a static optimization problem to determine a fixed optimal pose. We considered minimizing moment of inertia, maximizing rotational acceleration, or maximizing rotational velocity as possible dancer objectives. The chosen objective, estimated rotational acceleration, takes into account the need to produce torque from the feet, along with the need to minimize moment of inertia. The optimization problem we solve varies for each couple because it is determined by their individual sizes.

We create a model to estimate the torques the dancers create to propel themselves in a circle, and their moments of inertia. Rather than directly calculate torque, we use a surrogate method to estimate the external forces acting on the dancer.

We use a numerical optimization scheme to find a pose that maximizes estimated rotational acceleration for each couple. The problem is non-convex because of the presence of trigonometric functions relating the joint angles, which determine the pose, to the objective function. Mathematica's NMaximize function implements a global non-linear optimization scheme with no performance guarantees. The optimization problem is particularly challenging because it is non-convex and has 14 decision variables, the 14 angles that define the couple's pose.

II. The Model

A. Body Segment Masses and Lengths

We measured the lengths of each represented body segment for each subject couple. Each dancer self-reported his or her weight (mass). To estimate the body segment masses, we use a system developed by Nickolova and Toshev to divide the total mass of the body into fractions of the total mass in each body segment [3].

B. Defining Angles

The angles between various joints and the horizon are the only decision variables in our optimization problem. The joint angles, together with body segment lengths, define the dancer's pose. The body parts are labeled with subscript e for elbow, s for shoulder, h for hip, kg and kp for knee grind

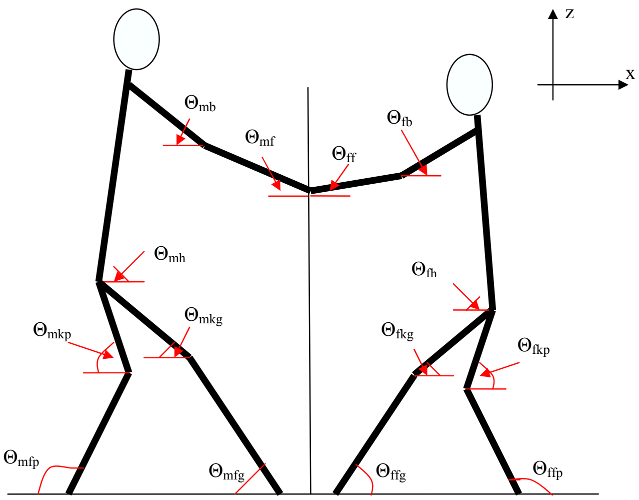


Fig. 1. Each dancer's pose is defined by seven joint angles as shown.

TABLE I

ANGLE NOTATION WITH UPPER AND LOWER BOUNDS

Angles		Minimum	Maximum
θ_{mf}	Angle of Forearms	$-\pi/4$	$\pi/2$
θ_{mb}	Angle of Biceps	0	$\pi/2$
θ_{mh}	Angle of Hip	$\pi/4$	$3\pi/4$
θ_{mkg}	Angle of Knee Grind	0	θ_{mfg}
θ_{mkp}	Angle of Knee Push	0	θ_{mfp}
θ_{mfg}	Angle of Foot Grind	0	π
θ_{mfp}	Angle of Foot Push	0	π
θ_{ff}	Angle of Forearms	$-\pi/4$	$\pi/2$
θ_{fb}	Angle of Biceps	0	$\pi/2$
θ_{fh}	Angle of Hip	$\pi/4$	$3\pi/4$
θ_{fkg}	Angle of Knee Grind	0	θ_{ffg}
θ_{fkp}	Angle of Knee Push	0	θ_{ffp}
θ_{ffg}	Angle of Foot Grind	0	π
θ_{ffp}	Angle of Foot Push	0	π

and knee push, and finally fg and fp for foot grind and foot push. The lengths and masses of each segment are labeled with similar subscripts. Table I explains the label for each angle and gives starting values for the optimization process.

Note in Table I the labeling "push" and "grind". These words distinguish between the two legs. In reviewing videos of dancers we determined that one of the feet stayed closer to the axis of rotation and is mainly used for balance. The closer "grind" foot did not contribute to the spin, but countered it by grinding on the floor. The other leg is further from the axis of rotation and is used to "push" the dancer around the circle and thus we labeled the left foot the push foot and the right foot the grind foot.

C. Calculating Distances to Axis of Rotation

We calculated the distance from each point on the body to the axis of rotation, starting at the hands which were assumed to be the locus of the axis of rotation.

The distance from the axis of rotation to each body's joints was labeled R, while the length of each body segment was labeled L, both with a subscript denoting male or female followed by a body segment, with lettering the same as used for angles. For example, the distance from the male leader's elbow to the axis of rotation is defined as $R_{me} = L_{mf} * Cos[\theta_{mf}]$. The distance to the leader's shoulder is based on the length of the upper arm and the distance to the elbow, $R_{ms} = L_{mb} * Cos[\theta_{mb}] + R_{me}$. Other distances are similarly calculated based on the distances to the body joints calculated before it. Therefore the distances between each of the feet and the axis of rotation is determined by the body segment lengths for each person and the angles of his pose.

III. CONSTRAINING THE MODEL

We developed a set of constraints to ensure physically reasonable poses in our optimization model.

The hips are prevented from thrusting inward. Even though people can place their hips inward, we know from observing dancers that it is not a pose from which one can easily start spinning. The hip constraints are represented by the following inequalities:

$$\begin{aligned} \theta_{mh} + \theta_{mkg} &\leq \pi \\ \theta_{mh} + \theta_{mkp} &\leq \pi \\ \theta_{fh} + \theta_{fkg} &\leq \pi \\ \theta_{fh} + \theta_{fkp} &\leq \pi \end{aligned}$$

Also, human knees cannot bend backwards. Thus, we require the angle at each foot($\theta_{ffg}, \theta_{mfg}$) be greater than the angle of the knee ($\theta_{fkg}, \theta_{mkg}$). These constraints are:

$$\begin{aligned} \theta_{mkp} &\leq \theta_{mfp} \\ \theta_{mkg} &\leq \theta_{mfg} \\ \theta_{fkp} &\leq \theta_{ffp} \\ \theta_{fkg} &\leq \theta_{ffg} \end{aligned}$$

The elbow also has a limited range of motion. We limit the movement of the shoulder to the 2-dimensional yz-plane by treating the shoulder as a hinge joint rather than a ball and socket. We also rule out any pose with the elbow bent backward. The elbow constraints are:

$$\begin{aligned} \theta_{mb} &\geq \theta_{mf} \\ \theta_{fb} &\geq \theta_{ff} \end{aligned}$$

Finally, we constrain the hip angle. Early in our optimization attempts we would occasionally get a hip angle that was negative or near zero. An angle at or below zero creates a pose where the dancer is entirely bent forward with her torso nearly level with the floor. While this pose creates a small moment of inertia, it is biomechanically unreasonable. This hip constraint is:

$$\theta_{mh} \geq \frac{\pi}{4}$$
$$\theta_{fh} \geq \frac{\pi}{4}$$

The above constraints ensure that the poses chosen via optimization are biologically reasonable. We tried not to overly constrain our solution so as to pre-judge which pose was best.

We also lower bound the distances from the feet and each of the joints to the axis of rotation. An optimization scheme might place a dancer's feet on the opposite side of the axis of rotation crossed over his partner's feet. This pose is a difficult position from which to begin rotating and is not particularly safe, since dancers might trip.

To eliminate this pose and any other poses where the dancers might be so close that their lower body ends up invading their partner's body space, we require that the distances to each dancer's joints defined in the model are positive. For example the distance to the leader's grind foot is R_{mfg}. The m notes that it is the leader, while the fg indicates that it is the distance to the grind foot. The R notes that the variable is distance. The constraints on distances are listed:

$$\begin{aligned} R_{mfg} &\geq 0 & R_{ffg} &\geq 0 \\ R_{mfp} &\geq 0 & R_{ffp} &\geq 0 \\ R_{mkg} &\geq 0 & R_{fkg} &\geq 0 \\ R_{kp} &\geq 0 & R_{fkp} &\geq 0 \\ R_{mh} &\geq 0 & R_{fh} &\geq 0 \\ R_{ms} &\geq 0 & R_{fs} &\geq 0 \\ R_{mE} &\geq 0 & R_{fE} &\geq 0 \end{aligned}$$

The dancers are connected at the hands, but the height of each dancer's hands depends on his or her pose. Therefore, we require that the height of the dancers' hands be equal, so they can hold hands:

$$H_{fHand} = H_{mHand}$$

The height of the hip is determined by the pose of the grind leg. However, the height of the hip might also be separately determined by the pose of the push leg. These two values must be equal:

$$H_{fh} = H_{fHip}$$

The H_{fh} variable represents the location of the hip as defined by the grind foot and H_{fHip} represents the height of the hip as defined from the push foot.

We assume the dancers pull on their partner's hands. We do not believe that the dancers pose in such a way that they are pushing on one another. Thus, in order to maintain their balance and not push on their partner the dancers grind feet must be in front of their respective center of mass:

$$\begin{aligned} CoM_{fx} &\geq R_{ffg} \\ -CoM_{mx} &\geq R_{ffg} \end{aligned}$$

These constraints do not allow the dancers to lean into one another.

IV. Calculating and Maximizing Rotational Acceleration

In modeling the spinning motion of dancers, we use their size parameters to determine the best pose for a couple by maximizing their rotational acceleration. This model yields somewhat realistic poses for the Lindy Hop rhythm circle. The ideal pose is deemed to be a pose that maximizes the rotational acceleration of the dancers. Let $\Theta = [\theta_{ff}, \theta_{fb}, \theta_{fh}, \theta_{fkg}, \theta_{fkp}, \theta_{ffg}, \theta_{ffp}, \theta_{mf}, \theta_{mb}, \theta_{mh}, \theta_{mkg}, \theta_{mkp}, \theta_{mfg}, \theta_{mfp}]$. Theta, Θ, is a vector of angles that define both dancers' poses. The rotational acceleration, $\alpha[\Theta]$, is calculated as:

$$\alpha[\Theta] = \frac{\tau[\Theta]}{I[\Theta]}$$

where tau,$\tau[\Theta]$, is the scalar torque produced by the dancer in the direction perpendicular to the floor as a function of Θ. The moment of inertia, $I[\Theta]$, is the dancers' resistance to initiating a spin.

The first step in determining $\alpha[\Theta]$ is to determine the moment of inertia. To find the moment of inertia $I[\Theta]$ of the dancers, we find the moment of inertia of each part of the body and then sum the individual moments of inertia. We treat the body segments as non-right cylinders consisting of stacked thin disks. We calculate the moment of inertia of each body segment in a given pose by twice applying the parallel axis theorem.

For example, the above calculations result in this moment of inertia for the follower's torso:

$$\begin{aligned} &InertiaTorso_f = \\ &\quad \frac{1}{2} m_{ft} * r_{ft}^2 + m_{ft} * (R_{fs} + \frac{1}{2} * R_{fh})^2 + \\ &\quad \frac{1}{12} * L_{ft}^2 * m_{ft} * Cos[\theta_{fh}]^2 * Sin[\theta_{fh}] \end{aligned}$$

where $\frac{1}{2} m_{ft} * r_{ft}^2$ is the inertia for a single thin disk around its center of mass. The $\frac{1}{12} * L_{ft}^2 * m_{ft} * Cos[\theta fh]^2 * Sin * [\theta_{fh}]$ is the result of the integral that sums all of the disks over the length of the cylinder. Finally the $m_{ft} * (R_{fs} + \frac{1}{2} * R_{fh})^2$ term shifts the entire moment of inertia from rotating around its own center to rotating around the axis some distance away. The angle variables are defined in Table I.

A. Calculating Torque

Calculating the correct tau, $\tau[\Theta]$, is the most challenging part of building the model. Torque, τ, the rotational analog of force causes an object to spin and produces rotational acceleration. Though torque is a vector, we consider only the torque component relating to the dancers' partnered spin around each other on the floor and thus treated it as a scalar. Torque can not be arbitrarily large because the force is generated by the dancer pushing against the floor, and there is an upper limit determined by friction, and above this limit the dancer's foot will slip. Figure 2 illustrates the external forces acting on the dancers. There are four external forces at work as each dancer spins: the force of gravity acting through the center of mass, the force acting at her hands from her partner pulling on her, and the force from the floor acting on each of her two feet. We neglect entirely the dynamic aspect of alternating weight between the two feet as the dancer takes steps.

Figure 2 shows the non-zero components of the external forces. Gravity acts through the calculated center of mass of each dancer. $fxFhands$ is the force in the x-direction on the follower from her partner pulling on her hands while

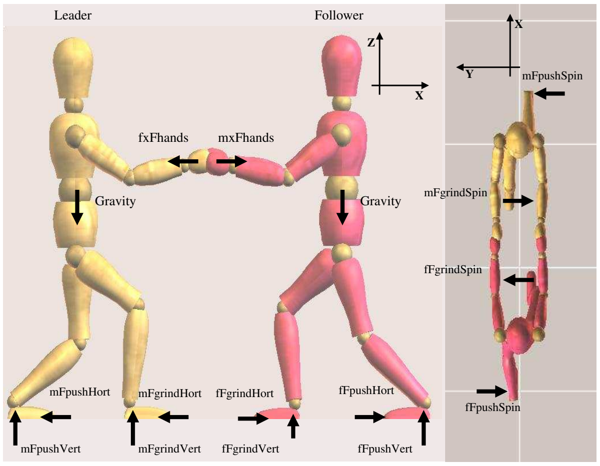


Fig. 2. All of the forces acting on the system to cause it to rotate.

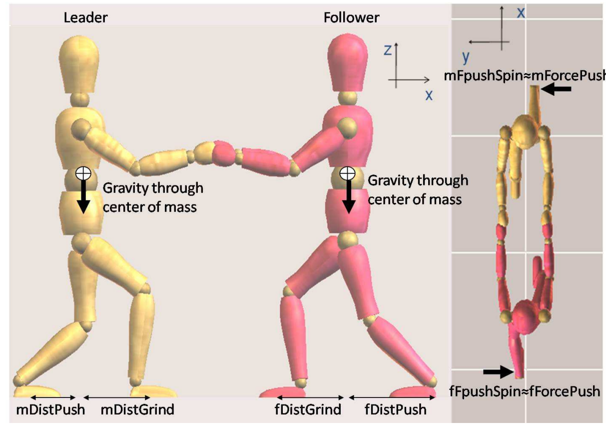


Fig. 3. This figure illustrates the calculations for the surrogate model.

$mxFhands$ is the force on the leader from the follower pulling on his hands. $fFgrindHort$ is the force in the x-direction on the follower's grind foot that is a result of friction and represents her tendency to slide toward or away from her partner. $fFpushHort$ is the force in the x-direction on the follower's push foot. $fFgrindVert$ and $fFpushVert$ are the normal forces acting on each of the follower's feet.

Finally, $fFpushSpin$ is the force on the follower's push foot in the y-direction that induces motion and initiates the spin. This and $mFpushSpin$ are crucial forces because they are the forces that induce the spin. Our goal is to estimate these two forces from motion capture and size parameters. These forces are countered by $fFgrindSpin$ and $mFgrindSpin$.

While there are only 8 scalar quantities to estimate, estimating these from the dancers' pose is a challenging problem. First, we calculate the location of the center of mass for a dancer in a given pose [7], using our segmented body model.

We attempted to solve the physical equations of motion (this system of equations is not shown) to estimate the forces in Figure 2. Unfortunately we were unable to solve these equations, even after trying several numerical solvers and various formulations of the system. Instead, we use a surrogate method for estimating the external forces on the dancers.

V. Surrogate Force Model

Ultimately to model the dancers we use a simpler model that neglects the forces at the hands and on the grind foot, and uses a surrogate, $NormalPush$ for $FpushSpin$. We assume the reaction forces from the floor are equal to the weight of the person over their push foot, which we estimate using the location of their center of mass. We calculate the distance from the axis of rotation to each of the feet. For the follower, these distances are:

$$fDistPush = \sqrt{(fxCoM - fRfp)^2 + frt^2} \quad (1)$$
$$fDistGrind = \sqrt{(fxCoM - fRfg)^2 + frt^2} \quad (2)$$

The variables above represent the location of the center of mass minus the distance to the axis of rotation, plus the radius of the body segment, which takes into account the physical size of the body segment. From these distances, we estimate what fraction of each dancer's weight is supported by his push foot:

$$fWeightPush = \frac{fDistGrind}{fDistGrind + fDistPush} \quad (3)$$

If the dancer is standing mostly over her push foot, then the distance between her center of mass and grind foot is large, so a larger fraction of her weight is on her push foot. Conversely, if $fDistGrind$ is small, then most of the dancer's weight is over her grind foot, so a small value for $fDistGrind$ corresponds to a smaller value for $fWeightPush$. Taking the product of the fraction of weight over the push foot and the dancer's weight, we determine the normal force acting on the dancer's foot:

$$fNormalPush = fWeightPush * fMass * g \quad (4)$$

Using these normal forces and an estimated μs (coefficient of static friction), we claim that the normal force on the push foot is proportional to $fFpushSpin$:

$$fForcePush = fRfp * fNormalPush * \mu s \quad (5)$$

Since both the leader and follower contribute to the force that causes the couple to spin, we sum these forces and divide by $InertiaTotal$ to estimate the rotational acceleration:

$$\alpha = \frac{fForcePush + mForcePush}{InertiaTotal} \quad (6)$$

This surrogate model performs relatively well, yielding plausible optimal poses. Section VII includes images of the actual and optimal poses, and values for α.

VI. Numerical Optimization

In addition to estimating the forces involved in the actual recorded movements of the dancers, we solve an optimization problem for each couple to estimate how large a rotational acceleration they could have achieved. We use the NMaximize routine in Mathematica as our numerical optimization algorithm. Our system does not admit an analytic solution. We use individual size length and mass parameters to determine each couple's optimal and achieved accelerations.

We attempt to maximize α, the rotational acceleration estimate for the couple, subject to biological feasibility constraints on the pose. For a list of constraints, see Section III. The decision variables are the fourteen pose angles in Table I.

NMaximize is very sensitive to the starting pose used for the optimization. The starting pose is specified as a range of values for each decision variable, from which the algorithm starts its search. The search for the global optimum may get stuck in a local optimum near the starting solution. We start each search near the pose the couple actually held. While the pose generated by the algorithm is very different from the initial pose, we observed that the value of the objective at "optimal" varies with even slight changes in the initial pose.

We seek the global maximum for the estimated rotational acceleration, subject to the above-detailed constraints. Finding the global optimum is much more difficult than finding a local optimum, which is just the nearest peak or valley in the solution. Methods for global optimization generally combine multiple random start points with local optimization techniques to find the global optimum, but have no performance guarantees. Mathematica did not reliably reach a global optimum. The variation in the "optimal" solution, as described in the previous paragraph, is evidence of failure to consistently find the true global optimum.

VII. Data Analysis and Results

Our biomechanical model predicted the best pose and highest achievable rotational acceleration for each pair of dancers. The measured actual pose for each couple was analyzed with the same model to compute an estimated acceleration in that pose. We calculated the ratio of each couple's estimated acceleration in the observed pose to that of the best pose found. A larger ratio means that the pair achieved a higher fraction of their potential acceleration. Table II lists the achieved and optimal rotational accelerations for each couple along with the fraction of optimal. In comparing each couple's performance to their individual optimum, we hypothesized that the expert dance couples would achieve a higher fraction of their optimum than less skilled dancers.

Couple H is excluded because we were not able to garner reasonable data from the couple. Couple H's recorded pose was infeasible by the limitations on the rotation of the hip joint. This could have been caused by inaccurate marker placement.

With our motion capture system, we could have recorded an observed rotational acceleration. Instead, we estimated acceleration based on our surrogate model, because we calculated the fraction of optimal performance based on the optimal acceleration from that same model. Using the estimated acceleration in the actual poses calculated from the same model provided a metric for comparing the couples' performances.

TABLE II

Achieved and optimal rotational acceleration for each couple, and fraction of optimal acceleration achieved.

Couple	Class	Achieved	Optimal	Fraction of Optimal
A	Expert	4.50093	45.9221	0.0980
B	Beginner	3.22323	44.6527	0.0722
C	Expert	6.44338	48.0171	0.1368
D	Beginner	3.49177	48.6595	0.0718
E	Expert	3.49527	49.8348	0.0701
F	Expert	4.972	47.3274	0.1056
G	Beginner	3.96729	49.8875	0.0795
I	Beginner	3.95054	45.0883	0.0876
J	Beginner	4.28396	42.431	0.1010

To test our hypothesis, we used a one-tailed Mann-Whitney statistical test to compare these numbers across couples. We categorized the couples as beginners or experts. We ranked each couples' fraction of optimal estimated acceleration from largest to smallest. Using a table of test statistics from Rice [5], we did not find a difference between the two categories at an $\alpha = .05$ level of statistical significance.

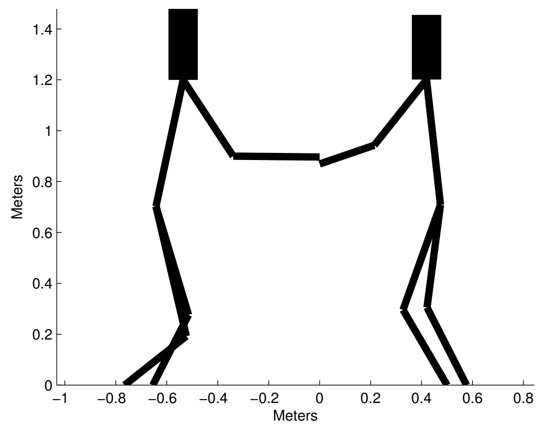
We cannot conclude from our small dataset that dancers choose poses predicted by our optimization model. Anecdotally, all of the couples' predicted poses are similar and seem intuitively logical. To spin fast, the couples should be close together and with feet close to the center of the circle. The push foot should be placed some distance away from the axis of rotation to produce the torque that sustains the spin.

All the dancers' actual poses differ in a systematic way from the estimated optimal poses. In the observed poses, the dancers' feet are both at a greater distance from the axis of rotation than in the predicted poses. The dancers' feet are also actually closer together and more underneath the dancers, so there appears to be less force at the hands as dancers lean away from each other than in the predicted poses. See Figure 4 for an example of an optimal and actual pose for an expert couple.

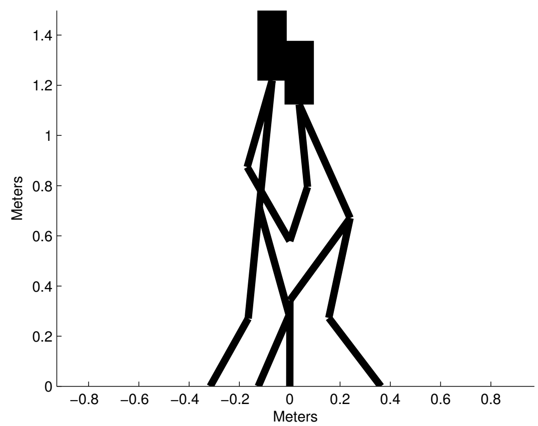
A. Model Shortcomings

We neglected many significant aspects of pose selection for the partnered spin: the ease or difficulty with which people are able to hold various poses (internal forces), the need to see one's partner, the social norms requiring a certain amount of personal space for each dancer, the two asymmetric arm connection points, the freedom of many joints like the shoulder and hip to move in more than a hinge fashion, the alternation of weight from one foot to another as dancers take steps during the spin, and the possibility that the dancers might prioritize aesthetic considerations over physical efficiency in selecting a pose. Perhaps these simplifications explain why none of the couples in our study adopt a pose that is close to the pose predicted by our model.

1) Hand simplification: Treating the various points of connection between the leader and follower as a single link might be problematic. Recording the closed arm connection, between the leader's right arm and the follower's back, would have made our "actual" poses seem much closer together than they appeared in our calculations. The arm around the



(a) The actual pose the dancers held



(b) The optimal pose as calculated by the model

Fig. 4. The actual and optimal poses for couple A.

back connection is a stronger connection in this spin than the connection at the hands. By describing the dancers' actual performance in terms of the distance between their open hand-to-hand connection, we may have chosen a very noisy observation of the true distance between their torsos. Figure 5 illustrates the pose dancers spin in and shows how much closer the dancers are on their closed shoulder side. Our model recorded only the large distance between the dancers' open hands.

Our model is superficially 3D but essentially planar. Future work should incorporate a truly three-dimensional model. The dancers do not actually face one another as in our model. The dancers' shoulders on the closed side, with the arm around the back connection, are much closer together than their shoulders on the open hand-to-hand connection side. A planar model neglects this twist.

2) Changing feet: The distinction between the push foot and the grind foot is probably overdrawn in our model. In fact, as the dancers turn, they both take steps, shifting their weight entirely from one foot to another. Both feet could exert forces that encourage the couple to rotate at different times during the spin, so the dancers' choice to have both feet at some distance from the axis of rotation makes sense.

Fig. 5. The pose dancers assume to spin. One arm is around their partners' shoulder (closed arm) and one grasping their partners' hand (open hand).

VIII. Conclusion

We hoped to understand the pose a swing dancer selects to complete a rhythm circle. We built a simplified biomechanical model to predict the optimal pose for a dance couple based on the leader's and follower's sizes. We estimated the external forces on the system, the moment of inertia of the couple, and the rotational acceleration of the couple's actual dance. Using numerical optimization with sensible pose constraints, we predicted the "best" pose and compared it to that couple's recorded pose.

Qualitatively the optimum poses that we found agree with what expert dancers would teach students about this partnered spin. Dance teachers usually advise that this spin works better the closer one can get to one's partner and that the right (grind) foot should be at or close to the axis of rotation while the left (push) foot should be farther away. We did not find a difference between the fraction of optimality achieved by beginners and expert dancers. The rotational acceleration achieved by the dancers was roughly a factor of ten less than the predicted optimal acceleration.

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