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Optimization and Convergence of Observation Channels in Stochastic Control

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Abstract— **This paper studies the optimization of observation channels (stochastic kernels) in partially observed stochastic control problems. In particular, existence, continuity, and convexity properties are investigated. Continuity properties of the optimal cost in channels are explored under total variation, setwise convergence and weak convergence. Sufficient conditions for sequential compactness under total variation and setwise convergence are presented. It is shown that the optimization is concave in observation channels. This implies that the optimization problem is non-convex in quantization/coding policies for a class of networked control problems. Furthermore, the paper explains why a class of decentralized control problems, under the non-classical information structure, is non-convex when *signaling* is present.**

I. Introduction

In stochastic control, one is often concerned with the following problem: Given a dynamical system, an observation channel (stochastic kernel), a cost function, and an action set, when does there exist an optimal control policy, and what is an optimal control policy? The theory for such problems is advanced, and practically significant, spanning a wide variety of applications in engineering, economics, and natural sciences. In this paper, we are interested in a dual problem with the following questions to be explored: Given a dynamical system, a cost function, an action set, and a set of observation channels, how can one optimize the observation channels? What is an optimal observation channel subject to constraints on the channel? What is the right convergence notion for continuity in such observation channels for optimization purposes?

We start with the probabilistic setup of the problem. Let $\mathbb{X} \subset \mathbb{R}^n$, be a Borel set in which elements of a controlled Markov process $\{X_t,\, t \in \mathbb{Z}_+\}$ live. Here and throughout the paper $\mathbb{Z}_+$ denotes the set of nonnegative integers and $\mathbb{N}$ denotes the set of positive integers. Let $\mathbb{Y} \subset \mathbb{R}^m$ be a Borel set, and let an observation channel Q be defined as a stochastic kernel (regular conditional probability) from $\mathbb{X}$ to $\mathbb{Y}$, such that $Q(\,\cdot\,|x)$ is a probability measure on the (Borel) σ-algebra $\mathcal{B}(\mathbb{Y})$ on $\mathbb{Y}$ for every $x \in \mathbb{X}$, and $Q(A|\,\cdot\,) : \mathbb{X} \to [0,1]$ is a Borel measurable function for every $A \in \mathcal{B}(\mathbb{Y})$. Let a decision maker (DM) be located at the output an observation channel Q, with inputs X_t and outputs Y_t. Let $\mathbb{U}$ be a Borel subset of some Euclidean space. An *admissible policy* Π is a sequence of control functions $\{\gamma_t,\, t \in \mathbb{Z}_+\}$ such that γ_t is measurable with respect to the σ-algebra generated by the information variables $I_t = \{Y_{[0,t]}, U_{[0,t-1]}\}, \quad t \in \mathbb{N}, \qquad I_0 = \{Y_0\}$, where $U_t = \gamma_t(I_t), \quad t \in \mathbb{Z}_+$, are the $\mathbb{U}$-valued control actions and we used the notation

$$Y_{[0,t]} = \{Y_s,\, 0 \le s \le t\}, \quad U_{[0,t-1]} = \{U_s,\, 0 \le s \le t-1\}.$$

The joint distribution of the state, control, and observation processes is determined by the above and the following relationships:

$$\Pr\big((X_0,Y_0) \in B\big) = \int_B P(dx_0)Q(dy_0|x_0), B \in \mathcal{B}(\mathbb{X}\times\mathbb{Y}),$$

where P is the (prior) distribution of the initial state X_0, and

$$\Pr\bigg((X_t,Y_t) \in B \,\bigg|\, x_{[0,t-1]}, y_{[0,t-1]}, u_{[0,t-1]}\bigg) = \int_B P(dx_t|x_{t-1},u_{t-1})Q(dy_t|x_t), \quad B \in \mathcal{B}(\mathbb{X}\times\mathbb{Y}),$$

where $P(\cdot|x,u)$ is a stochastic kernel from $\mathbb{X}\times\mathbb{U}$ to $\mathbb{X}$.

One way of presenting the problem in a familiar setting is the following: Consider a dynamical system described by the discrete-time equations

$$\begin{aligned} X_{t+1} &= f(X_t,U_t,W_t), \\ Y_t &= g(X_t,V_t) \end{aligned}$$

for some measurable functions f, g, with $\{W_t\}$ being independent and identically distributed (i.i.d) system noise process and $\{V_t\}$ an i.i.d. disturbance process, which are independent of X_0 and each other. Here, the second equation represents the communication channel Q, as it describes the relation between the state and observation variables.

With the above setup, let the objective of the decision maker be the minimization of the cost

$$J_T(P,Q,\Pi) = E_P^{Q,\Pi}\left[\sum_{t=0}^{T-1} c(X_t,U_t)\right], \tag{1}$$

over the set of all admissible policies $\mathscr{P}$, where $c : \mathbb{X}\times\mathbb{U} \to \mathbb{R}$ is a Borel measurable cost function and $E_P^{Q,\Pi}$ denotes the expectation with initial state probability measure given by P under policy $\Pi \in \mathscr{P}$ and given channel Q. We adopt the convention that random variables are denoted by capital letters and lowercase letters denote their realizations. Also, given a probability measure μ the notation $Z \sim \mu$ means that Z is a random variable with distribution μ.

We are interested in the following problems:

Problem P1. Continuity on the space of channels (stochastic kernels) Suppose $\{Q_n, n \in \mathbb{N}\}$ is a

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sequence of communication channels converging in some sense to a channel Q. When does $Q_n \to Q$ imply

$$\inf_{\Pi \in \mathscr{P}} J_T(P, Q_n, \Pi) \to \inf_{\Pi \in \mathscr{P}} J_T(P, Q, \Pi)?$$

PROBLEM P2. CONCAVITY ON THE SPACE OF CHANNELS Is the optimization problem

$$J_T(P, Q) := \inf_{\Pi} E_P^{Q,\Pi}\left[\sum_{t=0}^{T-1} c(X_t, U_t)\right]$$

convex/concave on the space of channels?

PROBLEM P3: EXISTENCE OF OPTIMAL CHANNELS Let $\mathcal{Q}$ be a set of communication channels. When do there exist minimizing and maximizing channels for the problems

$$\inf_{Q \in \mathcal{Q}} J_T(P, Q), \qquad \sup_{Q \in \mathcal{Q}} J_T(P, Q).$$

If solutions to these problems exist, are they unique?

The answers may help solve problems in application areas such as the following: (1) For a partially observed stochastic control problem, sometimes we have control over the observation channels by encoding/quantization. When does there exist an optimal quantizer for such a setup? (Optimal quantization). (2) Given an uncertainty set for the observation channels, and tools available for learning the channel, can one identify a worst element/best element? (Robust control). (3) When we do not know the channel, but have statistical tools and empirical measurements to learn, typically under mild technical conditions, the empirical distributions converge to the actual distribution, in some sense. Does this imply that we could design the optimal control policies based on empirical estimates, and does the optimal cost converge to the correct limit as the number of measurements grows? (Consistency of empirical controllers). (4) Can one compare information channels with regard to the costs they induce? (Value of information channels). (5) Why are decentralized control problems, when there is an incentive for signaling, difficult? (Signaling and decentralized control).

In the following, we will address problems P1-P3 and introduce conditions under which we can provide affirmative/ conclusive answers. Proofs not given here can be found in [13].

A. Relevant literature

The problems stated are related to three main areas of research: Robust control, optimal quantizer design and design of experiments. References [2], [3], [4] have considered both optimal control and estimation and the related problem of optimal control design when the channel is unknown. Similarly, there are connections with robust detection, such as those studied by Poor [7], when the source distribution to be detected belongs to some set. Recently, [10] considered continuity and other functional properties of minimum mean square estimation problems under Gaussian channels.

We will observe that the optimal cost is concave in the space of observation channels. This is related to statistics in the context of comparison of experiments as discussed by Blackwell [1].

In most of the paper we consider the single-stage ($T = 1$) case. We will also briefly consider the technically more complex multi-stage case where further conditions on the controlled Markov chain must be imposed. The full development of this general setup is the subject of future work.

II. SOME TOPOLOGIES ON THE SPACE OF CHANNELS

One question that we wish address is the choice of an appropriate notion of convergence for a sequence of observation channels. Toward this end, we first review three notions of convergence for probability measures.

Let $\mathcal{P}(\mathbb{R}^{\mathbb{N}})$ denote the family of all probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{R}^N))$ for some $N \in \mathbb{N}$. Let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}(\mathbb{R}^N)$. Recall that $\{\mu_n\}$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ *weakly* if $\int_{\mathbb{R}^N} c(x)\mu_n(dx) \to \int_{\mathbb{R}^N} c(x)\mu(dx)$ for every continuous and bounded function $c : \mathbb{R}^N \to \mathbb{R}$. On the other hand, $\{\mu_n\}$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ *setwise* if $\mu_n(A) \to \mu(A)$, for all $A \in \mathcal{B}(\mathbb{R}^N)$ (an equivalent condition is that $\int_{\mathbb{R}^N} c(x)\mu_n(dx) \to \int_{\mathbb{R}^N} c(x)\mu(dx)$ for every measurable and bounded c). Finally, the *total variation* metric on $\mathcal{P}(\mathbb{R}^N)$ is given by $\|\mu - \nu\|_{TV} := 2\sup_{B \in \mathcal{B}(\mathbb{R}^N)} |\mu(B) - \nu(B)|$. A sequence $\{\mu_n\}$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ in total variation if $\|\mu_n - \mu\|_{TV} \to 0$.

Note that these three convergence notions are in increasing order of strength: convergence in total variation implies setwise convergence, which in turn implies weak convergence.

A. Convergence of information (observation) channels

Here $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{Y} = \mathbb{R}^m$, and $\mathcal{Q}$ denotes the set of all observation channels (stochastic kernels) with input space $\mathbb{X}$ and output space $\mathbb{Y}$. For $P \in \mathcal{P}(\mathbb{X})$ and $Q \in \mathcal{Q}$ we let PQ denote the joint distribution induced on $(\mathbb{X} \times \mathbb{Y}, \mathcal{B}(\mathbb{X} \times \mathbb{Y}))$ by channel Q with input distribution P:

$$PQ(A) = \int_A Q(dy|x)P(dx), \quad A \in \mathcal{B}(\mathbb{X} \times \mathbb{Y}).$$

Definition 2.1 (Convergence of Channels): A sequence of channels $\{Q_n\}$ converges to a channel Q at input P weakly (resp. setwise, resp. in total variation) if $PQ_n \to PQ$ weakly (resp. setwise, resp. in total variation).

If we introduce the equivalence relation $Q \equiv Q'$ if and only if $PQ = PQ'$, $Q, Q' \in \mathcal{Q}$, then the convergence notions in Definition 2.1 only induce the corresponding topologies (resp. metrics) on the resulting equivalence classes in $\mathcal{Q}$, instead of $\mathcal{Q}$. Since in most of the development the input distribution P is fixed, there should be no confusion when (somewhat incorrectly) we talk about the induced topologies (resp. metrics) on $\mathcal{Q}$.

B. Classes of assumptions

Throughout the paper the following classes of assumptions will be adopted for the cost function c and the (Borel) set $\mathbb{U} \subset \mathbb{R}^k$ in different contexts:

ASSUMPTIONS.

A1. The function $c : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is non-negative, bounded, and continuous on $\mathbb{X} \times \mathbb{U}$.

A2. The function function $c : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is non-negative, measurable, and bounded.
A3. The function $c : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is non-negative, measurable, bounded, and continuous on $\mathbb{U}$ for every $x \in \mathbb{X}$.
A4. $\mathbb{U}$ is a compact set.
A5. $\mathbb{U}$ is a convex set.

III. Problem P1: Continuity of the optimal cost in channels

In this section, we consider continuity properties. We consider the single-stage case $T = 1$, and thus investigate the continuity of the functional

$$J(P, Q) := J_1(P, Q) = \inf_{\gamma \in \mathcal{G}} \int_{\mathbb{X}\times\mathbb{Y}} c(x, \gamma(y))Q(dy|x)P(dx)$$

in the channel Q, where $\mathcal{G}$ is the collection of all Borel measurable functions mapping $\mathbb{Y}$ into $\mathbb{U}$. Note that by our previous notation, $\Pi = \gamma$ is an admissible first-stage control policy. As before, in this section $\mathcal{Q}$ denotes the set of all channels with input space $\mathbb{X}$ and output space $\mathbb{Y}$. Before proceeding further, however, we look for conditions under which an optimal control policy exists; i.e, when the infimum in $\inf_\gamma E_P^{Q,\gamma}[c(X, U)]$ is a minimum.

Theorem 3.1: Suppose assumptions A3 and A4 hold. Then, there exists an optimal control policy for any channel Q.

Remark. The assumptions that c is bounded and $\mathbb{U}$ is compact can be weakened in the preceding theorem. One may assume that $\mathbb{U} = \mathbb{R}^k$, $\lim_{\|u\|\to\infty} c(x, u) = \infty$ for all x, $c(x, u)$ is lower semi-continuous on $\mathbb{U}$ for every x, and there exists u_0 such that $\int c(x, u_0)P(dx) < \infty$.

A. Weak convergence

1) Absence of continuity under weak convergence: The following counterexample demonstrates that $J(P, Q)$ may not be sequentially continuous under weak convergence of channels even for continuous cost functions and compact $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{U}$. Note that the absence of continuity here is also implied by a less elementary counterexample for setwise convergence in Section III-B.2. Let $\mathbb{X} = \mathbb{Y} = \mathbb{U} = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$. Suppose the cost is given as $c(x, u) = (x - u)^2$ and assume that P is a discrete distribution with two atoms:$P = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, where δ_a is the delta measure at point a, that is, $\delta_a(A) = 1_{\{a\in A\}}$ for every Borel set A, where 1_E denotes the indicator function of event E. Let $\{Q_n\}$ be a sequence of channels given by

$$Q_n(\,\cdot\,|x) = \begin{cases} \delta_{a+\frac{1}{n}} & \text{if} \quad x \geq a + \frac{1}{n}, \\ \delta_a & \text{if} \quad x < a + \frac{1}{n}. \end{cases} \tag{2}$$

One can verify that $Q_n \to Q$ weakly at input P, $J(P, Q_n) \to 0$, and $J(P, Q) > 0$, so the optimal cost is not continuous.

2) Upper semi-continuity under weak convergence:

Theorem 3.2: Suppose assumptions A1 and A5 hold. If $\{Q_n\}$ is a sequence of channels converging weakly at input P to a channel Q, then $\limsup_{n\to\infty} J(P, Q_n) \leq J(P, Q)$, that is, $J(P, Q)$ is upper semi-continuous on $\mathcal{Q}$ under weak convergence.

Proof: Let μ be an arbitrary probability measure on $(\mathbb{X} \times \mathbb{Y}, \mathcal{B}(\mathbb{X} \times \mathbb{Y}))$ and let $\mu_\mathbb{Y}$ be its second marginal, i.e., $\mu_\mathbb{Y}(A) = \mu(\mathbb{X}\times A)$ for $A \in \mathcal{B}(\mathbb{Y})$. By a slight generalization of Lusin's theorem [9, Thm. 2.24] for any $g \in \mathcal{G}$ and $\epsilon > 0$ there is a continuous function $f : \mathbb{Y} \to \mathbb{U}$ such that $\mu_\mathbb{Y}\{y : f(y) \neq g(y)\} < \epsilon$. Then we have

$$\int \big|c(x, g(y)) - c(x, f(y))\big|\mu(dx, dy) \leq \epsilon \sup_{x,u} c(x, u).$$

Since c is bounded by assumption A1 and $\epsilon > 0$ is arbitrary, we obtain

$$\inf_{f\in\mathcal{C}} \int c(x, f(y))\mu(dx, dy) = \inf_{\gamma\in\mathcal{G}} \int c(x, \gamma(y))\mu(dx, dy)$$

where $\mathcal{C}$ denotes the set of continuous functions from $\mathbb{Y}$ into $\mathbb{U}$. Applying the above first to PQ_n and then to PQ, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \inf_{\gamma\in\mathcal{G}} \int c(x, \gamma(y))PQ_n(dx, dy) \\ &\quad= \limsup_{n\to\infty} \inf_{f\in\mathcal{C}} \int c(x, f(y))PQ_n(dx, dy) \\ &\quad\leq \inf_{f\in\mathcal{C}} \limsup_{n\to\infty} \int c(x, f(y))PQ_n(dx, dy) \\ &\quad= \inf_{f\in\mathcal{C}} \int c(x, f(y))PQ(dx, dy) \\ &\quad= \inf_{\gamma\in\mathcal{G}} \int c(x, \gamma(y))PQ(dx, dy) \end{aligned}$$

where the next to last equality holds since PQ_n converges weakly to PQ. ■

B. Continuity properties under setwise convergence

1) Upper semi-continuity under setwise convergence:

Theorem 3.3: Under assumption A2 the optimal cost

$$J(P, Q) := \inf_\gamma E_P^{Q,\gamma}[c(X, U)]$$

is sequentially upper semi-continuous on the set of communication channels $\mathcal{Q}$ under setwise convergence.

Proof: Let $\{Q_n\} \to Q$ setwise at P. Then

$$\begin{aligned} &\limsup_{n\to\infty} \inf_{\gamma\in\mathcal{G}} \int c(x, \gamma(y))PQ_n(dx, dy) \\ &\quad\leq \inf_{\gamma\in\mathcal{G}} \limsup_{n\to\infty} \int c(x, \gamma(y))PQ_n(dx, dy) \\ &\quad= \inf_{\gamma\in\mathcal{G}} \int c(x, \gamma(y))PQ(dx, dy), \end{aligned}$$

where the equality holds since c is bounded. ■

2) Absence of continuity under setwise convergence: $J(P,Q)$ may not be sequentially continuous under setwise convergence of channels even for continuous cost functions and compact $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{U}$, as we observe in the following.

Let $\mathbb{X} = \mathbb{Y} = \mathbb{U} = [0,1]$. Assume that X has distribution $P = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Let $Q(\,\cdot\,|x) = U([0,1])$ for all x, so that if $(X,Y) \sim PQ$, then Y is independent of X and has the uniform distribution on $[0,1]$. Let $c(x,u) = (x-u)^2$.

By independence, $E[X|Y] = E[X] = 1/2$, so

$$\begin{aligned} J(P,Q) &= \min_{\gamma\in\mathcal{G}} E[(X-\gamma(Y))^2] = E[(X-E[X|Y])^2] \\ &= \frac{1}{2}\left(1-\frac{1}{2}\right)^2 + \frac{1}{2}\left(0-\frac{1}{2}\right)^2 = \frac{1}{4}. \end{aligned}$$

For $n \in \mathbb{N}$ and $k = 1,\dots,n$ consider the intervals

$$L_{nk} = \left[\frac{2k-2}{2n}, \frac{2k-1}{2n}\right), \quad R_{nk} = \left[\frac{2k-1}{2n}, \frac{2k}{2n}\right) \tag{3}$$

and define the "square wave" function $h_n(t) = \sum_{k=1}^n \big(1_{\{t\in L_{nk}\}} - 1_{\{t\in R_{nk}\}}\big)$. Since $\int_0^1 h_n(t)\,dt = 0$ and $|h_n(t)| \le 1$, the function

$$f_n(t) = \big(1 + h_n(t)\big)1_{\{t\in[0,1]\}}$$

is a probability density function. Furthermore, the proof of the Riemann-Lebesgue lemma can be used almost verbatim to show that the sequence of probability measures induced by the sequence $\{f_n\}$ converges setwise to $U([0,1])$.

Now, for every n, define a channel as

$$Q_n(\,\cdot\,|x) = \begin{cases} U([0,1]), & x = 0 \\ \sim f_n, & x = 1. \end{cases}$$

Then $Q_n(\cdot|x) \to Q$ setwise for $x = 0$ and $x = 1$, and thus $PQ_n \to PU([0,1])$ setwise. However, letting $(X,Y_n) \sim PQ_n$, the optimal policy for PQ_n is

$$\gamma_n(y) = E[X|Y_n = y] = \begin{cases} 0, & y \in \bigcup_{k=1}^n R_{nk} \\ \frac{2}{3}, & y \in \bigcup_{k=1}^n L_{nk} \end{cases}$$

and therefore for every $n \in \mathbb{N}$

$$\begin{aligned} &J(P,Q_n) = \min_{\gamma\in\mathcal{G}} E[(X-\gamma(Y_n))^2] \\ &= \frac{1}{2}\left(\int_0^1 (\gamma_n(y))^2 dy + \int_0^1 (1-\gamma_n(y))^2 f_n(y) dy\right) = \frac{1}{6}. \end{aligned}$$

Thus, the optimal cost value is not continuous under setwise convergence. $\diamond$

C. Continuity under total variation

We have the following theorem, the proof of which is omitted [13].

Theorem 3.4: Under assumption A2 the optimal cost $J(P,Q)$ is continuous on the set of communication channels $\mathcal{Q}$ under the topology of total variation.

IV. Problem P2: Concavity on the space of channels

The following is a result which has important consequences in decentralized stochastic control problems as will be elaborated on later.

Theorem 4.1: The expression $J(P,Q) = \inf_\gamma E_P^{Q,\gamma}[c(X,U)]$ is a concave function of Q.

Proof of Theorem 4.1. For $\alpha \in [0,1]$ and $Q', Q'' \in \mathcal{Q}$, let $Q = \alpha Q' + (1-\alpha)Q'' \in \mathcal{Q}$, i.e.,

$$Q(A|x) = \alpha Q'(A|x) + (1-\alpha)Q''(A|x)$$

for all $A \in \mathcal{B}(\mathbb{Y})$ and $x \in \mathbb{X}$. Noting that $PQ = \alpha PQ' + (1-\alpha)PQ''$, we have

$$\begin{aligned} &J(P,Q) = J(P, \alpha Q' + (1-\alpha)Q'') \\ &= \inf_\gamma E_P^{Q,\gamma}[c(X,U)] = \inf_{\gamma\in\mathcal{G}} \int c(x,\gamma(y))PQ^*(dx,dy) \\ &= \inf_{\gamma\in\mathcal{G}} \Big(\alpha \int c(x,\gamma(y))PQ'(dx,dy) \\ &\qquad + (1-\alpha)\int c(x,\gamma(y))PQ''(dx,dy)\Big) \\ &\ge \inf_{\gamma\in\mathcal{G}} \Big(\alpha \int c(x,\gamma(y))PQ'(dx,dy)\Big) \\ &\qquad + \inf_{\gamma\in\mathcal{G}} \Big((1-\alpha)\int c(x,\gamma(y))PQ''(dx,dy)\Big) \\ &= \alpha J(P,Q') + (1-\alpha)J(P,Q'') \end{aligned} \tag{4}$$

proving that $J(P,Q)$ is concave in Q. $\diamond$

A. Concavity and connection with comparison of experiments

The following is a folk theorem in statistical decision theory whose proof is similar to that of Theorem 4.1.

Proposition 4.1: The function

$$V(P) := \inf_{u\in\mathbb{U}} \int P(dx)c(x,u),$$

is concave in P, under assumption A2.

We will use the above observation to revisit a classical result in statistical decision theory and comparison of experiments [1]. In a single decision maker setup, we refer to the probability space induced on $\mathbb{X}\times\mathbb{Y}$ as an information structure.

Definition 4.1: An information structure induced by some channel Q_2 is weakly stochastically degraded with respect to another one, Q_1, if there exists a channel Q' on $\mathbb{Y}\times\mathbb{Y}$ such that

$$Q_2(B|x) = \int_{\mathbb{Y}} Q'(B|y)Q_1(dy|x), \quad B \in \mathcal{B}(\mathbb{Y}), \;\; x \in \mathbb{X}.$$

In view of Proposition 4.1, we obtain the following.

Theorem 4.2 (Blackwell [1]): If Q_1 is weakly stochastically degraded with respect to Q_1, then the information structure induced by channel Q_1 is more informative with respect to the one induced by channel Q_2 in the sense that

$$\inf_\gamma E_P^{Q_2,\gamma}[c(X,U)] \ge \inf_\gamma E_P^{Q_1,\gamma}[c(X,U)],$$

for all cost functions c satisfying assumption A2.

Sketch of Proof. Let $(X, Y_1) \sim PQ_1$ and let Y_2 be such that $\Pr(Y_2 \in B|X = x, Y_1 = y_1) = Q_2(B|x)$ for all $B \in \mathcal{B}(\mathbb{Y})$, $y_1 \in \mathbb{Y}$, and $x \in \mathbb{X}$. Then X, Y_1, and Y_2 form a Markov chain in this order and therefore $P(dy_2|y_1, x) = P(dy_2|y_1)$ and $P(x|dy_2, y_1) = P(x|y_1)$. (Note that we used the somewhat sloppy notation where, for example, $P(dy_2|y_1, x)$ means $P_{Y_2|Y_1,X}(dy_2|y_1, x)$).

Thus we have

$$\begin{aligned} J(P, Q_2) &= \int V\big(P(\,\cdot\,|y_2)\big)P(dy_2) \\ &= \int V\bigg(\int P(\,\cdot\,|y^1)P(dy_1|y_2)\bigg)P(dy_2) \\ &\geq \int \bigg(\int P(dy_1|y_2)V\big(P(\,\cdot\,|y_1)\big)\bigg)P(dy_2) \quad (5) \\ &= \int V\big(P(\,\cdot\,|y_1)\big)\bigg(\int P(dy_1|y_2)P(dy_2)\bigg) \\ &= \int V\big(P(\,\cdot\,|y_1)\big)P(dy_1) = J(P, Q_1) \end{aligned}$$

where we used Proposition 4.1 and Jensen's inequality in (5). ◇

V. Problem P3: Existence of optimal channels

Here we study characterizations of compactness (or sequential compactness) which will be useful in obtaining existence results. The discussion on weak convergence showed us that weak convergence does not induce a strong enough topology, i.e., under which useful continuity properties can be obtained. In the following, we will obtain conditions for sequential compactness for the other two convergence notions, that is, for setwise convergence and total variation.

We first discuss setwise convergence. A set of probability measures $\mathcal{M}$ on some measurable space is said to be *setwise sequentially precompact* if every sequence in $\mathcal{M}$ has a subsequence converging setwise to a probability measure (not necessarily in $\mathcal{M}$). For two finite measures ν and μ defined on the same measurable space we write $\nu \leq \mu$ if $\nu(A) \leq \mu(A)$ for all measurable A.

As before, $PQ \in \mathcal{P}(\mathbb{X} \times \mathbb{Y})$ denotes the joint probability measure induced by input P and channel Q, where $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{Y} = \mathbb{R}^m$. The following is a simple consequence of a majorization criterion for setwise sequential precompactness of a family of probability measures (see, e.g., [5], p. 305-306 or [6], p. 7).

Lemma 5.1: Let ν be a finite measure on $\mathcal{B}(\mathbb{X} \times \mathbb{Y})$ and let P be a probability measure on $\mathcal{B}(\mathbb{X})$. Suppose $\mathcal{Q}$ is a set of channels such that

$$PQ \leq \nu, \quad \text{for all } Q \in \mathcal{Q}.$$

Then $\mathcal{Q}$ is setwise sequentially precompact at input P in the sense that any sequence in $\mathcal{Q}$ has a subsequence $\{Q_n\}$ such that $Q_n \to Q$ setwise at input P for some channel Q.

Lemma 5.2: Let $\mathcal{Q}$ be a set of channels such that $\{PQ : Q \in \mathcal{Q}\}$ is a precompact set of probability measures under total variation. Then $\mathcal{Q}$ is precompact under total variation at input P.

We can combine the preceding results with the following theorem to obtain sufficient conditions for the existence of best and worst channels when the given family of channels $\mathcal{Q}$ is closed under the appropriate convergence notion.

Theorem 5.1: Recall problem P2.

(i) There exist a worst channel in $\mathcal{Q}$, that is, a solution for the maximization problem

$$\sup_{Q \in \mathcal{Q}} J(P, Q) = \sup_{Q \in \mathcal{Q}} \inf_{\gamma} E_P^{Q,\gamma} E[c(X, U)]$$

when the set $\mathcal{Q}$ is weakly sequentially compact and assumptions A1, A4, and A5 hold.

(ii) There exist a worst channel in $\mathcal{Q}$ when the set $\mathcal{Q}$ is setwise sequentially compact and assumption A2 holds.

(iii) There exist best and worst channels in $\mathcal{Q}$, that is, solutions for the minimization problem $\inf_{Q \in \mathcal{Q}} J(P, Q)$ and the maximization problem $\sup_{Q \in \mathcal{Q}} J(P, Q)$ when the set $\mathcal{Q}$ is compact under total variation and assumption A2 holds.

Proof: Under the stated conditions, we have sequential upper semi-continuity or continuity (Theorems 3.2, 3.3, and 3.4) under the corresponding topologies. By sequential compactness, the existence of the cost maximizing (worst) channel follows when $J(P, Q)$ is upper-semicontinuous, while the existence of the cost minimizing (best) channel follows when $J(P, Q)$ is continuous in Q. ■

VI. Revisiting non-classical team decision problems: The optimal signaling problem is non-convex

In a dynamic decentralized control system, multiple controllers are present. We say that the information structure in such a decentralized control system is non-classical if one decision maker DM^k can affect the information of DM^j through his actions, but DM^j cannot have access to the information available to DM^k. When the information structure is non-classical, it is known that there may be an incentive for *signaling*: Signaling is the action of communication of the decision makers via their control actions. That is, under signaling, the decision makers apply their actions to affect the information available at the other decision makers actively [12]. It is an implication of Theorem 4.1 that the stochastic control problems are difficult when signaling is present. In this case, the problem becomes partly a communication problem; and as we observed; this problem is *non-convex*. The property on concavity in Theorem 4.1 is the reason why even linear-quadratic-Gaussian (LQG) type control problems lose their convexity properties, when signaling or communication/quantization is involved. For example, the well-known Witsenhausen's counterexample is non-convex due to such an effect. To make this important issue more explicit, let us consider the following example, taken from [12]. Consider a two-controller system evolving in $\mathbb{R}^n$ with the following description:

$$\begin{aligned} X_{t+1} &= AX_t + B^1 U_t^1 + B^2 U_t^2 + W_t, \\ Y_t^1 &= C^1 X_t + V_t^1, \qquad Y_t^2 = C^2 X_t + V_t^2, \end{aligned}$$

with W, V^1, V^2 zero-mean, i.i.d. disturbances, and A, B^1, B^2, C^1, C^2 matrices of appropriate dimensions. For $\rho_1, \rho_2 > 0$, let the goal be the minimization of

$$J = E\left[\left(\sum_{t=0}^{T-1} \|X_t\|^2 + \rho_1 \|U_t^1\|^2 + \rho_2 \|U_t^2\|^2\right) + \|X_2\|^2\right]$$

over the control policies of the form:

$$u_t^i = \mu_t^i(y_{[0,t]}^i), \quad i = 1,2; \quad t = 0,1.$$

It was observed by Radner [8] that a static LQG team problem (for $T = 1$) with a non-nested information structure admits an optimal solution which is linear. The proof for this result follows from the observation that the team cost is convex in the joint strategies of the DMs, and it suffices to find the unique fixed point (under a verification of differentiability). This, in turn, is satisfied by a linear set of solutions for each DM. For a two-stage problem ($T = 2$), however, the cost is in general no-longer *convex* in the policy of the controllers acting in the first stage $t = 0$, by Corollary 4.1 or Theorem 4.1. This is because the actions might affect the estimation quality of the other controller in the second stage, if one DM can signal information to the other DM in one stage. We note that this condition is equivalent to $C^1 A^l B^2 \neq 0$ or $C^2 A^l B^1 \neq 0$ ([11], Lemma 3.1), with $l+1$ denoting the delay in signaling with $l = 0$ in the problem considered. In particular, if the controller is allowed to apply a randomized policy, this induces a conditional probability measure (channel) from the external variables and the initial state of the system to the observation variables at the other decision maker. The optimization problem, as such, is not jointly convex in such policies, and finding a fixed point does not lead to the conclusion that such policies are optimal.

VII. Multi-stage case

We consider the general case $T \in \mathbb{N}$. It should be observed that the effects of a control policy applied any given time-stage presents itself in two ways, in both the cost occurred at the given time-stage and the effect on the process distribution at future time-stages, which is known as the dual effect of control. The next theorem shows the continuity of the optimal cost in the observation channel under some regularity conditions. Note that the existence of best and worst channels follows under an appropriate compactness condition as in Theorem 5.1 (iii). We need the following definition.

Definition 7.1: A sequence of channels $\{Q_n\}$ converges to a channel Q *uniformly* in total variation if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{X}} \left\| Q_n(\,\cdot\,|x) - Q(\,\cdot\,|x) \right\|_{TV} = 0.$$

Theorem 7.1: Consider the cost function (1) with arbitrary $T \in \mathbb{N}$. Suppose assumption A2 holds. Then, the optimization problem P1 is continuous in the observation channel in the sense that if $\{Q_n\}$ is a sequence of channels converging to Q uniformly in total variation, then

$$\lim_{n\to\infty} J_T(P, Q_n) = J_T(P, Q).$$

We obtained the continuity of the optimal cost on the space of channels equipped with a more stringent notion for convergence in total variation. This result and its proof indicate that further technical complications emerge in multi-stage problems. Likewise, upper semi-continuity under weak convergence and setwise convergence require more stringent uniformity assumptions, which we leave for future research.

On the other hand, the concavity property applies directly to the multi-stage case. That is, the optimization problem P1 is concave in the space of channels. The proof of this fact follows that of Theorem 4.1.

VIII. Concluding remarks and extensions

This paper studied the structural and topological properties of some optimization problems in stochastic control in the space of observation channels. The main problem we considered is how to approach appropriate notions of convergence and distance while studying communication channels in the context of stochastic control problems. It was observed that the optimization problem is concave in such channels. One implication of this observation is that in a decentralized control problem, if signaling is present, the original convex problem (which may be convex under a nested, partially nested or a stochastically nested information structure) loses its convexity.

The restriction to Euclidean state spaces is not essential and many (but not all) of the results can be extended to the case where $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{U}$ are Polish spaces. In particular, all the results (except for Theorem 3.2) in Sections III and IV carry through without change.

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