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The (J, J')–Spectral Factorization of a General Discrete–Time System

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***Abstract*— For a discrete–time system whose transfer matrix function is completely general (arbitrary normal rank, strictly proper/imprope/polynomial, with poles and zeroes on the unit circle) we give existence conditions and constructive formulas for the (J, J')–spectral factor. The computation of the spectral factor is done by employing generalized state–space realizations and numerically–sound algorithms based on a preliminary unitary projection which delineates a subsystem fulfilling all regularity assumptions. En route, we give formulas for the J–lossless conjugation of an arbitrary discrete–time system and the (J, J')–lossless factorization of a marginally stable but otherwise general system.**

***Index Terms*— Discrete–time systems, (J, J')–spectral factorization, generalized systems.**

I. Introduction

A rational matrix function (rmf) with complex coefficients $\Theta(z)$ is called (J, J')–unitary if $\Theta(z)^* J \Theta(z) = J'$, at every point on the unit circle at which Θ is analytic, where J and J' are two signature matrices, i.e., $J = J^{-1} = J^*$ (* denotes conjugate transpose). By analytic continuation, $\Theta(z)^{\#} J \Theta(z) = J', \quad \forall z \in \mathbb{C}$, where $\Theta(z)^{\#} := \Theta(\frac{1}{\bar{z}})^*$. If, in addition, $\Theta(z)^* J \Theta(z) \le J'$ ($\Theta(z)^* J \Theta(z) \ge J'$) for every point of analyticity of Θ in the exterior of the closed unit disk, then Θ is called (J, J')–lossless ($(-J, -J')$–lossless). If $J = J'$, $\Theta(z)$ is called J–unitary, J–lossless, and $-J$–lossless, respectively. The normal rank of a rmf $G(z)$ is its rank for almost all $z \in \mathbb{C}$.

We consider here the following extensions of three well–known factorization problems such as to become applicable to a general $p \times m$ rmf with complex coefficients $G(z)$ (of arbitrary rank, with poles/zeros on the unit circle, possibly polynomial, strictly proper or improper):

***J*–lossless conjugation**. Find a minimal McMillan degree J–lossless rmf $\Theta(z)$ such that

$$\Pi(z) := \Theta(z) G(z) \tag{1}$$

has only marginally stable (located in the closed unit disk) poles. This is a slightly different form of Definition 3.1 in [17]. A similar statement with obvious modifications can be given for the $-J$–lossless conjugation problem.

(J, J')**–spectral factorization**. Find a rmf $\Pi(z)$ which has full row normal rank and only marginally stable zeros such that

$$G^{\#}(z) J G(z) = \Pi^{\#}(z) J' \Pi(z), \tag{2}$$

where $G(z)\Pi^{(+)}(z)$ has no poles on the unit circle. Here $\Pi^{(+)}(z)$ stands for the Moore–Penrose pseudoinverse of $\Pi(z)$ (see for example Definition 4 in [4]).

(J, J')**–lossless factorization**. Find a (J, J')–lossless rmf $\Theta(z)$ (without poles on the unit circle) and a rmf $\Pi(z)$ which has full row normal rank and only marginally stable poles and zeros such that

$$G(z) = \Theta(z) \Pi(z). \tag{3}$$

This is an extension of the definition in [17].

Due to its huge importance, the (J, J')–spectral factorization and its slightest more general (J, J')–lossless form have been investigated in tens of papers under various technical hypotheses, both in the continuous or discrete–time settings. Among the continuous–time approaches closer to the present development we mention [1], [18], [15], [11], [4], with the most general constructive solution in [25]. Until now the discrete–time (J, J')–spectral and lossless factorization problems have been solved either in different particular instances or for a particular underlying system: in [22] the unitary case (where $J = I$ and $J' = I$) has been approached for a completely general rmf, in [17] the (J, J') case is solved for a system fulfilling several regularity assumptions (full column rank, without poles at infinity or at zero and without zeroes on the unit circle or at infinity), while some of these assumptions are relaxed to include poles at zero and at infinity in [9] and zeroes on the unit circle in [10].

The purpose of this paper is to extend the ideas in [22], [25] such as to become applicable to the construction of the discrete–time factors for a completely general system $G(z)$ that allows the whole range of possible applications, i.e., is a completely general rmf. In contrast to [22], the main technical difficulty is that we need to formulate necessary and sufficient existence conditions while in contrast to [25] we should cater for the poles and zeroes at infinity which are no longer on the boundary of the stability domain. Besides the more technical machinery involved, the resulting formulas for the factors bear the same striking simplicity and the underlying algorithm resembles in details the one developed for the unitary case. Due to space limitations, we will give here the solutions to the general discrete–time J–lossless conjugation, the general (J, J')–spectral factorization and the (J, J')–lossless factorization of a stable but otherwise general system. Hence, we leave apart the more hairy construction of the completely general (J, J')–lossless factorization which is a combination of the J–lossless conjugation and the (J, J')–spectral factorization.

The paper is organized as follows. In Section II we review briefly a couple of definitions and notations related to matrix

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pencils, rational matrix functions and (descriptor) state–space realizations of rational matrices. In Section III we put ground for the main results by giving two spectral decompositions of the pole and system pencils associated with a descriptor realization. Section IV contains the main results. In Section V we give a numerical example for the (J, J')–spectral factorization of a polynomial rmf. We draw some conclusions in Section VI.

II. Preliminaries

A. Basic notation

We start with some notation and definitions. By $\mathbb{D}$ and $\mathbb{D}_1(0)$ we denote the open unit disk and the unit circle, respectively, and $\mathbb{D}_c = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ stands for the exterior of the closed unit disk containing the infinity. Here "overbar" denotes closure.

If a matrix A in $\mathbb{C}^{m\times n}$ is invertible, A^{-*} is its conjugate transpose inverse. A Hermitian matrix A satisfies $A = A^*$, and we denote by $A > 0$ ($A < 0$) if it is in addition positive definite (negative definite). We say A is unitary (J–unitary) if $A^*A = I$ ($A^*JA = J$). I_n will stand for the identity matrix of size $n \times n$, and we skip the dimensions whenever they are irrelevant. By $\star$ we denote irrelevant matrix entries.

B. Matrix pencils

We review a few basic notions about matrix pencils (for more details see Chapter 12 in [6]).

Let A and E be $m \times n$ matrices with elements in $\mathbb{C}$. The matrix polynomial $A-zE$ is called a *matrix pencil* or, briefly, *pencil*. The pencil is called *regular* if it is square ($m = n$) and has a non–vanishing determinant, i.e., $\det(A-zE) \not\equiv 0$. A *singular* pencil is a pencil which is not regular. The *normal rank* of the pencil – denoted $\mathrm{rank}_n(A - zE)$ – is defined as the rank of $A - zE$ for almost all $z \in \mathbb{C}$ (but a finite number of points). For an $n \times n$ regular pencil $A - zE$ the normal rank r is equal to n. If $\nu_\ell := m - r > 0$ then we say the pencil has a (nontrivial) *left singular structure*. If $\nu_r := n - r > 0$ then the pencil has a (nontrivial) *right singular structure*.

Two matrix pencils $A-zE$ and $\widetilde{A}-z\widetilde{E}$, with $A, E, \widetilde{A}, \widetilde{E} \in \mathbb{C}^{m\times n}$, are called *strictly equivalent* if there are two constant invertible matrices $Q \in \mathbb{C}^{m\times m}$, $Z \in \mathbb{C}^{n\times n}$, such that

$$Q(A - zE)Z = \widetilde{A} - z\widetilde{E}. \quad (4)$$

The equivalence relation (4) induces a canonical form (see [6]) – called the *Kronecker canonical form* – on the set of $m \times n$ pencils, $Q(A - zE)Z = A_{KR} - zE_{KR}$, where $Q \in \mathbb{C}^{m\times m}$ and $Z \in \mathbb{C}^{n\times n}$ are two invertible matrices, and

$$\begin{aligned} A_{KR} - zE_{KR} = \mathrm{diag}(&L_{\epsilon_1}, \dots, L_{\epsilon_{\nu_r}}, I_{n_\infty} - zE_\infty, \\ &A_f - zI_{n_f}, L^T_{\eta_1}, \dots L^T_{\eta_{\nu_\ell}}). \end{aligned} \quad (5)$$

Here L_k ($k \geq 0$) denotes the bidiagonal $k \times (k + 1)$ pencil

$$L_k := \begin{bmatrix} z & -1 & & \\ & \ddots & \ddots & \\ & & z & -1 \end{bmatrix},$$

A_f and E_∞ are two square matrices in the Jordan canonical form, with E_∞ nilpotent. The finite eigenstructure of $A-zE$ is determined by the eigenvalues of A_f, and the dimensions of the elementary infinite blocks of $I_{n_\infty} - zE_\infty$ determine the infinite eigenstructure of the pencil. The union of the finite and infinite eigenstructure of the pencil completely determines the regular part of the pencil and forms the spectrum of the pencil which is denoted by $\Lambda(A - zE)$. The singular part of the pencil is defined by the right and left singular Kronecker structure as follows. The $\epsilon_i \times (\epsilon_i + 1)$ blocks L_{ϵ_i}, $(i = 1, ..., \nu_r)$, are the right elementary Kronecker blocks, and $\epsilon_i \geq 0$ are called the right Kronecker indices. The $(\eta_j + 1) \times \eta_j$ blocks $L^T_{\eta_j}$, $(j = 1, ..., \nu_\ell)$, are the left elementary Kronecker blocks, and $\eta_j \geq 0$ are called the left Kronecker indices. Notice that ϵ_i and η_j can be zero.

C. Rational matrices

We give now a short overview of some of the structural invariants of a general $p \times m$ rmf: poles, zeros and their partial multiplicities. For more details see [19].

Lemma 2.1: Let $G(z)$ be a $p\times m$ rmf having normal rank r and fix $z_0 \in \mathbb{C}$. Then there exist two square rmf U and V, analytic and invertible at z_0, such that

$$U(z)G(z)V(z) = \begin{bmatrix} D(z) & 0_{r\times(m-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(m-r)} \end{bmatrix}, \quad (6)$$

$$D(z) := \mathrm{diag}\{(z - z_0)^{k_1}, (z - z_0)^{k_2}, \dots, (z - z_0)^{k_r}\}, \quad (7)$$

where $k_1 \leq k_2 \leq \dots \leq k_r$ are integers called the indices of the local Smith–McMillan form at z_0, the matrix in the right–hand side of (6) is called the local Smith–McMillan form at z_0, and is unique.

A point $z_0 \in \mathbb{C}$ is called a *pole* (*zero*) of $G(z)$ if at least one of the indices k_i in (7) is negative (positive). In this case the set of absolute values of the negative k_i's (the set of positive k_i's) are the *partial pole (zero) multiplicities* of $G(z)$ at z_0. The *total pole (zero) multiplicity* of $G(z)$ at z_0 is the sum of the partial pole (zero) multiplicities. By definition, $z = \infty$ is a pole (zero) of $G(z)$ provided $z = 0$ is a pole (zero) of $G(\frac{1}{z})$. In this case the partial and total pole (zero) multiplicities of $G(z)$ at ∞ are the partial and total pole (zero) multiplicities at $z = 0$ of $G(\frac{1}{z})$. The McMillan degree of $G(z)$, denoted $\delta(G(z))$, is the sum of the total multiplicities of all poles (finite and infinite) of $G(z)$.

D. Realization theory for rational matrices

For any rmf $G(z)$ in $\mathbb{C}^{p\times m}$ (even improper or polynomial) one can write down a descriptor realization of the form (see for example [27], [28])

$$G(z) = D + C(zE - A)^{-1}B =: \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right], \quad (8)$$

where $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$, and $A-zE$ is a *regular* pencil. The dimension n of the square matrices A and E is called the *order of the realization* (8). With any realization (8) we associate two matrix pencils that play an important role in the sequel: the *pole pencil* $\mathcal{P}(z) =$

$A - zE$ and the *system pencil* $\mathcal{S}(z) = \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix}$. The descriptor realization (8) of $G(z)$ is called *irreducible* if it satisfies the following conditions (see [28]):

$$
\begin{array}{lll}
\text{(i)} & \operatorname{rank}\begin{bmatrix} A - zE & B \end{bmatrix} = n, & \forall z \in \mathbb{C}, \\
\text{(ii)} & \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = n, & \\
\text{(iii)} & \operatorname{rank}\begin{bmatrix} A - zE \\ C \end{bmatrix} = n, & \forall z \in \mathbb{C}, \\
\text{(iv)} & \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n. &
\end{array} \tag{9}
$$

The conditions (9) are usually known as finite and infinite controllability, and finite and infinite observability, respectively. In contrast to standard realizations, irreducibility of a descriptor realizations does not automatically imply its minimality since some simple blocks of dimension 1 at infinity (so called non–dynamic modes) which are both controllable and observable might increase indefinitely the dimension of the realization while keeping its irreducibility. Starting from an arbitrary realization (8), one can compute an irreducible realization by using solely unitary transformations.

The following result taken from [17] (see Theorem 2.4) is modified to cope with *proper* (J, J')–unitary rational matrices having a descriptor realization.

Lemma 2.2: Let $G(z)$ be a proper rational matrix given by the minimal realization

$$G(z) := \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right]. \tag{10}$$

$G(z)$ is (J, J')–unitary ((J, J')–lossless) if and only if there is a (semipositive) hermitian matrix X such that

$$A_x^* X A_x - E_x^* X E_x + C_x^* J C_x = 0, \tag{11a}$$

$$D_x^* J C_x + B_x^* X A_x = 0, \tag{11b}$$

$$D_x^* J D_x + B_x^* X_s B_x = J'. \tag{11c}$$

III. Spectral decompositions

In this section we introduce two particular spectral decompositions of the pole and system pencils with respect to the partition $\overline{\mathbb{C}} = \overline{\mathbb{D}} \cup \mathbb{D}_c$. The decompositions can be achieved by unitary transformations and will play a capital role in expressing our main results in the next section.

Without restricting generality, we may assume that $G(z)$ is given by an irreducible realization of the form

$$G(z) = \left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_b - zE_b & A_{bg} - zE_{bg} & B_b \\ 0 & A_g - zE_g & B_g \\ \hline C_b & C_g & D \end{array}\right], \tag{12}$$

where $A_b - zE_b$ contains the n_b poles of $G(z)$ in $\mathbb{D}_c$, $\operatorname{rank}\begin{bmatrix} E_b & E_{bg} \end{bmatrix} = n_b$ and $A_g - zE_g$ contains all poles of $G(z)$ in $\overline{\mathbb{D}}$ and the nondynamic modes. In particular, notice that A_b and $A_b - E_b$ are invertible. Starting from an arbitrary realization (8) it is always possible to get (12) by employing unitary equivalence transformations only [20].

We give next the unitary decomposition of the system pencil which has been previously used to get the solutions in the unitary case (see [22]).

Lemma 3.1: Let $G(z)$ be a $p \times m$ real rational matrix given by a controllable realization (12), i.e., fulfilling (i) and (ii) in (9). Then there exist two constant unitary matrices U and Z such that

$$
\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc} A - zE & B \\ \hdashline C & D \end{array}\right] Z
= \left[\begin{array}{cccc} A_{rg} - zE_{rg} & \star & \star & \star \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zE_{\ell n} \\ 0 & 0 & 0 & B_n \\ \hdashline 0 & C_{b\ell} & D_{b\ell} & D_n \end{array}\right], \tag{13}
$$

where:

(I) The pencil $A_{rg} - zE_{rg}$ has full row rank in $\mathbb{D}_c$ and E_{rg} has full row rank.

(II) $E_{b\ell}$ and B_n are invertible, the pencil

$$\begin{bmatrix} A_{b\ell} - zE_{b\ell} & B_{b\ell} \\ C_{b\ell} & D_{b\ell} \end{bmatrix} \tag{14}$$

has full column rank in $\overline{\mathbb{D}}$, the pencil $A_{b\ell} - zE_{b\ell}$ is regular, and the pair $\begin{bmatrix} A_{b\ell} - zE_{b\ell} & B_{b\ell} \end{bmatrix}$ is controllable.

See [22] for a numerically stable algorithm to compute the matrices U and Z.

The above lemma constructs a projection of the original system (8)

$$G_p(z) = \left[\begin{array}{c|c} A_{b\ell} - zE_{b\ell} & B_{b\ell} \\ \hline C_{b\ell} & D_{b\ell} \end{array}\right], \tag{15}$$

which fulfills all standard assumptions in the literature [16] (is proper, has full column normal rank, and has no zeros on the unit circle). In the next section we will see that it is enough to solve the factorization problems for $G_p(z)$ to get the solution to the corresponding factorization for $G(z)$.

IV. Main results

Once the spectral decompositions obtained, we have the coefficients of two equations, a Stein and a Riccati, whose solutions (when they exists) can be used directly to write down state–space formulas for the factors $\Pi(z)$ and $\Theta(z)$ solving the factorization problems under investigation. Due to space limitation the proofs are only sketched.

A. *J–lossless conjugation*

Lemma 4.1: Let $G(z)$ be a general rmf given by an irreducible realization (12). The J–lossless ($-J$–lossless) conjugation (1) has a solution if and only if the Stein equation

$$E_b^* Y E_b - A_b^* Y A_b + C_b^* J C_b = 0 \tag{16}$$

has an invertible solution $Y_s < 0$ ($Y_s > 0$). The class of solutions to the J–lossless ($-J$–lossless) conjugation problem is given by

$$\Pi(z) := \left[\begin{array}{cc|c} \widetilde{A}_b - z\widetilde{E}_b & \widetilde{A}_{bg} - z\widetilde{E}_{bg} & \widetilde{B}_b \\ 0 & A_g - zE_g & B_g \\ \hline WC_b & WC_g & W\widetilde{D} \end{array}\right], \tag{17}$$

$$\Theta(z) := \left[\begin{array}{cc|c} A_b - zE_b & K(1-z) & 0 \\ 0 & I & I \\ \hline WC_b & 0 & W \end{array}\right], \tag{18}$$

where W is any constant J–unitary matrix,

$$K := Y_s^{-1}(E_b - A_b)^{-*}C_b^* J, \tag{19}$$

and $\widetilde{A}_b = A_b + KC_b$, $\widetilde{E}_b = E_b + KC_b$, $\widetilde{A}_{bg} = A_{bg} + KC_g$, $\widetilde{E}_{bg} = E_{bg} + KC_g$, $\widetilde{B}_b = B_b - Y^{-1}A_b^{-*}C_b^* D$, $\widetilde{D} = [I - C_b(A_b - E_b)^{-1}Y^{-1}A_b^{-*}C_b^*]D$.

Proof: The proof follows with minor modifications from the proof of Theorem 6.2 in [20]. ∎

Since $\Lambda(A_b - zE_b) \subset \mathbb{D}_c$, A_b is invertible. Moreover, (16) has a unique solution and can be converted into a standard Stein equation which can be solved by any numerically–sound algorithm. If (16) has an invertible solution Y_s, the pencil $A_b + KC_b - z(E_b + KC_b)$ is regular with the spectrum located in the unit disk, where K is given by (19). Lemma 4.1 is an extension to the case of an arbitrary rmf of the J–conjugation introduced in [17] for the case of a proper rmf (compare to (3.4), (3.5) and (3.7) in [17]). Even in the standard proper case, our method is more efficient since we solve a Stein equation – rather than a Riccati one –, and the dimension of the Stein equation is smaller than that of the Riccati equation in [17] whenever $G(z)$ has marginally stable poles.

B. (J,J')–spectral and lossless factorizations

Theorem 4.2: Let $G(z)$ be a general rmf given by a controllable realization (8), and let U and Z be two constant unitary matrices such that (13) holds.

(I) The (J, J')–spectral factorization problem (2) has a solution if and only if the following conditions are fulfilled:

1) The Riccati equation

$$\begin{gathered} A_{b\ell}^* X A_{b\ell} - E_{b\ell}^* X E_{b\ell} - (A_{b\ell}^* X B_{b\ell} + C_{b\ell}^* J D_{b\ell}) \\ \times (D_{b\ell}^* J D_{b\ell} + B_{b\ell}^* X B_{b\ell})^{-1}(B_{b\ell}^* X A_{b\ell} + D_{b\ell}^* J C_{b\ell}) \\ + C_{b\ell}^* J C_{b\ell} = 0 \end{gathered} \tag{20}$$

has an invertible stabilizing solution X_s, i.e., $\Lambda(A_{b\ell} + B_{b\ell}F - zE_{b\ell}) \subset \overline{\mathbb{D}}$, $F := -(D_{b\ell}^* J D_{b\ell} + B_{b\ell} X_s B_{b\ell})^{-1}(B_{b\ell}^* X_s A_{b\ell} + D_{b\ell}^* J C_{b\ell})$.

2)
$$D_{b\ell}^* J D_{b\ell} + B_{b\ell}^* X_s B_{b\ell} = V^* J' V \tag{21}$$

for an appropriate constant invertible matrix V;

(II) Assume in addition $G(z)$ is marginally stable. The (J, J')–lossless factorization problem (3) has a solution if and only if conditions 1) and 2) at (I) are fulfilled for $X_s \geq 0$.

(III) In both factorizations $\Pi(z)$ is given by

$$\Pi(z) := \left[\begin{array}{c|c} A - zE & B \\ \hline \widetilde{C} & \widetilde{D} \end{array}\right], \left[\widetilde{C}\ \widetilde{D}\right] := V\left[0\ {-F}\ I\ 0\right]Z^*, \tag{22}$$

while for the (J, J')–lossless factorization

$$\Theta(z) := \left[\begin{array}{c|c} A_{b\ell} + B_{b\ell}F - zE_{b\ell} & B_{b\ell}V^{-1} \\ \hline C_{b\ell} + D_{b\ell}F & D_{b\ell}V^{-1} \end{array}\right]. \tag{23}$$

Proof: We show first the sufficiency of conditions 1) and 2) simultaneously with proving that (22) together with (23) form a solution to the (J, J')–lossless factorization (3) of a marginally stable $G(z)$. Incidentally, it will follow that (22) is a solution to the (J, J')–spectral factorization (2) of a general $G(z)$.

We check first $G(z) = \Theta(z)\Pi(z)$. Let

$$\left[\begin{array}{cc} \widehat{C} & \widehat{D} \end{array}\right] := V^{-1}\left[\begin{array}{cc} \widetilde{C} & \widetilde{D} \end{array}\right] \overset{(22)}{=} \left[\begin{array}{cccc} 0 & -F & I & 0 \end{array}\right] Z^*. \tag{24}$$

Using (22) and (23) we have

$$\begin{aligned} \Theta(z)\Pi(z) &= \left[\begin{array}{cc|c} A_{b\ell} + B_{b\ell}F - zE_{b\ell} & B_{b\ell}\widehat{C} & B_{b\ell}\widehat{D} \\ 0 & A - zE & B \\ \hline C_{b\ell} + D_{b\ell}F & D_{b\ell}\widehat{C} & D_{b\ell}\widehat{D} \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_{b\ell} + B_{b\ell}F - zE_{b\ell} & 0 & B_{b\ell}\widehat{D} - X_1 B \\ 0 & A - zE & B \\ \hline C_{b\ell} + D_{b\ell}F & C_e & D_{b\ell}\widehat{D} \end{array}\right] \end{aligned} \tag{25}$$

$$= \left[\begin{array}{c|c} A - zE & B \\ \hline C_e & D_{b\ell}\widehat{D} \end{array}\right] \tag{26}$$

where $C_e := D_{b\ell}\widehat{C} + (C_{b\ell} + D_{b\ell}F)X_2$, X_1 and X_2 are two matrices such that

$$\left[X_1\ {-B_{b\ell}}\right]\left[\begin{array}{cc} A - zE & B \\ \widehat{C} & \widehat{D} \end{array}\right] = (A_{b\ell} + B_{b\ell}F - zE_{b\ell})\left[X_2\ 0\right]. \tag{27}$$

The existence of matrices X_1 and X_2 results from the following identity which can be checked directly

$$\begin{gathered} \left[\begin{array}{cccc} 0 & I & X_{13} & -B_{b\ell} \end{array}\right] \times \\ \left[\begin{array}{cccc} A_{rg} - zE_{rg} & B_1 - zF_1 & B_2 - zF_2 & B_3 - zF_3 \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zF_{\ell n} \\ 0 & 0 & 0 & B_n \\ \hdashline 0 & -F & I & 0 \end{array}\right] \\ = (A_{b\ell} + B_{b\ell}F - zE_{b\ell}) \left[\begin{array}{cccc} 0 & I & 0 & E_{b\ell}^{-1}F_{\ell n} \end{array}\right] \end{gathered} \tag{28}$$

where $X_{13} := [-B_{\ell n} + (A_{b\ell} + B_{b\ell}F)E_{b\ell}^{-1}F_{\ell n}]B_n^{-1}$. Indeed, multiplying (28) to the right with Z^* and using (13), (22) and (24) we get (27) for $X_1 := \left[\begin{array}{ccc} 0 & I & X_{13} \end{array}\right] U$ and

$$\left[\begin{array}{cc} X_2 & 0 \end{array}\right] = \left[\begin{array}{cccc} 0 & I & 0 & E_{b\ell}^{-1}F_{\ell n} \end{array}\right] Z^*. \tag{29}$$

We check that $G(z) = \Theta(z)\Pi(z)$ by showing that the two system pencils associated with the realizations (8) and (26) are related by a transformation to the left that leaves the transfer function invariant, i.e.,

$$\left[\begin{array}{cc} I & 0 \\ \widehat{U} & I \end{array}\right]\left[\begin{array}{cc} A - zE & B \\ C & D \end{array}\right] = \left[\begin{array}{cc} A - zE & B \\ D_{b\ell}\widehat{C} + (C_{b\ell} + D_{b\ell}F)X_2 & D_{b\ell}\widehat{D} \end{array}\right]$$

(see for example [26]). Indeed, multiplying this to the left with $\left[\begin{array}{cc} U & 0 \\ 0 & I \end{array}\right]$ and to the right with Z, using (13), (24),

(29) and the partition $\widehat{U}U^* := \left[U_{11}\, U_{12}\, U_{13}\right]$, we get

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ U_{11} & U_{12} & U_{13} & I \end{bmatrix} \times$$
$$\begin{bmatrix} A_{rg} - zE_{rg} & B_1 - zF_1 & B_2 - zF_2 & B_3 - zF_3 \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zF_{\ell n} \\ 0 & 0 & 0 & B_n \\ 0 & C_{b\ell} & D_{b\ell} & D_n \end{bmatrix}$$
$$= \left[\begin{array}{cccc} A_{rg} - zE_{rg} & B_1 - zF_1 & B_2 - zF_2 & B_3 - zF_3 \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zF_{\ell n} \\ 0 & 0 & 0 & B_n \\ \hdashline 0 & C_{b\ell} & D_{b\ell} & (C_{b\ell} + D_{b\ell}F)E_{b\ell}^{-1}F_{\ell n} \end{array}\right]. \tag{30}$$

But (30) is clearly satisfied for $U_{11} = 0$, $U_{12} = 0$ and $U_{13} := [-B_{\ell n} + (C_{b\ell} + D_{b\ell}F)E_{b\ell}^{-1}F_{\ell n}]B_n^{-1}$ which proves the existence of $\widehat{U}$ and ends the proof of $G(z) = \Theta(z)\Pi(z)$.

Further, we show that $\Pi(z)$ has full row normal rank and the zeros in $\overline{\mathbb{D}}$, while $\Theta(z)$ is (J, J')–unitary (or (J, J')–lossless if in addition $X_s \geq 0$). Indeed, we have

$$\begin{bmatrix} U & 0 \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} A - zE & B \\ \widetilde{C} & \widetilde{D} \end{bmatrix} Z \overset{(13)(22)}{=}$$
$$\left[\begin{array}{cccc} A_{rg} - zE_{rg} & B_1 - zF_1 & B_2 - zF_2 & B_3 - zF_3 \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zF_{\ell n} \\ 0 & 0 & 0 & B_n \\ \hdashline 0 & -F & I & 0 \end{array}\right],$$

in which $A_{rg} - zE_{rg}$ has full row rank in $\mathbb{D}_c$ and $\begin{bmatrix} A_{b\ell} - zE_{b\ell} & B_{b\ell} \\ -F & I \end{bmatrix}$ is square, has full normal rank, and its zeros are located in $\overline{\mathbb{D}}$. $\Theta(z)$ is (J, J')–unitary (or (J, J')–lossless) as follows directly by using its expression (23) and (20) in Lemma 2.2. Hence, provided $G(z)$ is marginally stable, (3) is indeed a (J, J')–lossless factorization. Finally, if $G(z)$ is arbitrary, some tedious but otherwise elementary algebraic manipulations show that $G(z)\Pi^{(+)}(z)$ has no poles on the unit disk, concluding in this way that (2) is a (J, J')–spectral factorization. This ends the whole sufficiency part.

We sketch now the necessity of conditions 1) and 2). Notice first in complete analogy with [25] that if $G(z)$ has a (J, J')–lossless factorization then there is an invertible factor $R(z)$ which cancels in the product $R(z)G(z)$ all left minimal indices and the unstable zeros of $G(z)$, and $R(z)$ is in particular $(-J)$–lossless. Such an invertible factor has the form (deduced from [23] with minor modifications)

$$R^{-1}(z) = \left[\begin{array}{c|c} A_{b\ell} - zE_{b\ell} + B_{b\ell}F_x & B_x D_x^{-1} \\ \hline C_{b\ell} + D_{b\ell}F_x & D_x^{-1} \end{array}\right], \tag{31}$$

where F_x is such that $\Lambda(A_{b\ell} - zE_{b\ell} + B_{b\ell}F_x) \subset \mathbb{D} \cup \mathbb{D}_1(0)$, D_x is any invertible matrix and B_x is such that $B_{b\ell} - B_x D_{b\ell} = 0$. If we add on (31) the conditions (11) of being additionally J–unitary we get straightforwardly (20) and (21). Similar arguments apply for the (J, J')–spectral case. This ends the whole proof. ■

In particular, the above result may be applied to a polynomial matrix $G(z)$, and provides a numerically sound state–space construction of the (J, J')–spectral factor (see [18], [1]). Theorem 4.2 is also an extension of the (J, J')–lossless factorization to the case of an improper marginally stable rmf having arbitrary normal rank, with poles and zeros on the unit circle (compare with Theorem 4.2 in [17]).

The existence of the stabilizing solution to the Riccati equation (20) can be checked and the equation solved by using any existing numerical algorithm that copes with indefinite sign matrix coefficients (see for example [14] and the references therein).

V. Numerical Example

We exemplify the proposed approach on a simple but relevant polynomial system. For illustrative simplicity we use nonunitary transformations as well. Let $J = \begin{bmatrix} 1 & 0 \\ 0 & J' \end{bmatrix}$, $J' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$G(z) = \begin{bmatrix} 3z^4 - 6z^3 - 3z + 6 & z^4 - 2z^3 - z + 2 & 2z^4 - 4z^3 - 2z + 4 \\ 3z^5 + 3z^4 - 3z^3 - 3z^2 + 12z + 11 & z^5 + z^4 - z^3 - z^2 + 4z + 5 & 2z^5 + 2z^4 - 2z^3 - 2z^2 + 8z + 10 \\ 6z^5 + 6z^3 + 3z^2 + 3z + 28 & 2z^5 + 2z^3 + z^2 + z + 12 & 4z^5 + 4z^3 + 2z^2 + 2z + 24 \end{bmatrix}$$

$G(z)$ has a realization (12) given by

$$\left[\begin{array}{c|c} A - zE & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & -z & 0 & 0 & 0 \\ -z & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -z & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -\frac{1}{3} & -\frac{2}{3} \\ \hline -3 & 0 & -6 & 3 & 0 & 1 & 5 & \frac{5}{3} & \frac{10}{3} \\ 12 & -3 & -3 & 3 & 3 & 0 & 11 & 5 & 10 \\ 3 & 3 & 6 & 0 & 6 & 0 & 28 & 12 & 24 \end{array}\right].$$

The structural elements of $G(z)$ are: one pole at ∞ with multiplicity 5, one zero at 2 with multiplicity 1, one zeros at 1 with multiplicity 1, one zero at ∞ with multiplicity 1, one left minimal index equal to 2, one right minimal index equal to 0 and normal rank $r = 2$. With

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & -2 & -3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & -1 & -5 & -4 & 12 & -3 & 0 \\ 0 & 4 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 27 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ -18 & -69 & 0 & 0 & 3 & 3 & -27 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get the decomposition (13) in the form

$$\left[\begin{array}{c:c:c:c} A_{rg} - zE_{rg} & B_1 - zF_1 & B_2 - zF_2 & B_3 - zF_3 \\ \hdashline 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zF_{\ell n} \\ \hdashline 0 & 0 & 0 & B_n \\ \hdashline 0 & C_{b\ell} & D_{b\ell} & D_n \end{array}\right] =$$

$$\left[\begin{array}{cc:cccc:cc:c} 0 & 4-4z & 1 & 0 & -2 & -3+z & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1-2z & -3+2z & -1-z & 12 & -3 & 0 \\ 0 & 0 & -1-z & 1 & 2+2z & 3+2z & 0 & 0 & 0 \\ 0 & 0 & -1 & 2-z & 0 & 3-z & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2-z & 2-2z & 0 & 0 & -z \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & -18 & 0 \end{array}\right].$$

The Riccati equation (20) has a stabilizing positive solution

$$X_s = \left[\begin{array}{cccc} 167.138438763306 & -38.78460969082739 & -161.138438763306 & 5.999999999999 \\ -38.784609690827 & 226.4922678357841 & -81.746133917891 & 17.999999999999 \\ -161.138438763306 & -81.74613391789126 & 301.492267835785 & -11.999999999999 \\ 5.999999999999 & 17.999999999999 & -11.999999999999 & 47.999999999999 \end{array}\right].$$

With $V = \begin{bmatrix} -157.628430595195 & 36.603010518552 \\ 27.906754558497 & 8.861965366188 \end{bmatrix}$ which fulfils (21) we get $\Pi(z)$ as in (32). A direct check shows that $\Pi(z)$ fulfils (2), has full row rank, the same poles as $G(z)$, zeros at $0, 1, \frac{1}{2}, -0.267949259275$ and -0.267949125586, each with multiplicity 1, and one right minimal index equal to 0.

VI. Conclusions

We have solved several essential factorization problems formulated for a completely general discrete–time system: J–lossless conjugation, (J, J')–spectral factorization and (J, J')–lossless factorization (for a marginally stable system). The results coping with the (J, J')–lossless factorization of a completely general discrete–time system (not necessarily marginally stable) can be readily obtained and will be reported elsewhere.

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$$\Pi(z) = \left[\begin{array}{ccc} -12.201003506184z^5 + 12.352585696233z^4 + 4.605982123644z^3 - 1.170350330080z^2 - 1.094683201171z + 4.6734952170768 & -4.067001168728z^5 + 4.117528565400z^4 + 1.535327374548z^3 - 0.390116776693z^2 - 0.364894400390z + 1.9732535360991 & -8.134002337456z^5 + 8.235057130800z^4 + 3.070654749096z^3 - 0.780233553387z^2 - 0.729788800781z + 3.946507072198 \\ -2.953988455396z^5 - 0.482228483501z^4 - 9.338900969778z^3 - 4.906870493510z^2 + 3.603184841463z - 26.397756676354 & -0.984662818465z^5 - 0.160742827833z^4 - 3.112966989926z^3 - 1.635623497836z^2 + 1.201061613821z - 11.145719485572 & -1.969325636930z^5 - 0.321485655667z^4 - 6.225933979852z^3 - 3.271246995673z^2 + 2.402123227642z - 22.291438971144 \end{array}\right] \quad (32)$$