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Optimal Control of a Third Order Nonlinear System Based on an Inverse Optimality Method

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Abstract— **In this paper, a nonlinear optimal control problem for a third order system is defined and solved. The optimal control law is found using an inverse optimality approach to solve the Hamilton-Jacobi-Bellman equation, where the solution is obtained directly for the control input without needing to assume or compute a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function. The developed controller is applied to a path following control of a wheeled mobile robot.**

I. INTRODUCTION

Optimal control of nonlinear systems is one of the most challenging and difficult subjects in control theory. The control approaches can be divided into two main categories: direct optimal and inverse optimal control. In the direct nonlinear optimal control problem, a controller is developed to minimize an a priori given cost function, which ultimately results in finding a solution to a Hamilton-Jacobi-Bellman (HJB) equation. This equation is unfortunately hard to solve for a general nonlinear system. This obstacle motivated the development of inverse optimal nonlinear control design methods [1]- [10]. An interesting well known result is that when the cost is quadratic and the dynamics are affine in the input there is an explicit solution for the input as a function of the derivatives of the value function. This fact will be used in this paper together with the structure of affine dynamics in the input to develop a method to solve the Hamilton-Jacobi-Bellman equation for a class of third order systems. Another well known result is that using a control Lyapunov function (CLF), many control laws can be calculated which globally asymptotically stabilize the system and can be inverse optimal relative to a meaningful cost functional not specified beforehand by the control designer. However, the main drawback of the CLF concept, as a design tool, is that there is no systematic way to find a control Lyapunov function for general nonlinear systems [11].

Based on the concept of inverse optimality, a new solution method that can determine at the same time a controller and a sensible nonnegative cost rendering the controller optimal was developed recently in [10]. In this method, the analytical solution for the control input is obtained directly, without needing to first assume or compute any coordinate transformation, value function, or Lyapunov function. The value function and a Lyapunov function can however be computed once the optimal control input has been found. The main drawback of the new method is that it is restricted to second order systems. This paper presents an extension of the recent method to optimal control of a third order nonlinear system.

The motivation for this work is the fact that when designers are faced with a control engineering problem and want to formulate it in the optimal control framework, the choice of the most appropriate cost is a difficult task. However, quite often the following three properties are required for the design:

1) The closed loop system should be asymptotically stable to a desired equilibrium point
2) The system should have enough damping so that the trajectories do not take too long to settle around the desired equilibrium point
3) The control energy should be penalized in the cost to avoid high control inputs that can saturate actuators

The particular functions involved in the cost are not usually pre-defined, except possibly the requirement on the control energy that is usually represented by a quadratic cost on the input. This paper attempts to find a controller and a cost that together meet the requirements 1–3 and render the controller optimal relative to that cost. To that aim, the cost will be fixed to be quadratic in the input and the state plus an unknown term that shall be determined. The solution is based on the concept of inverse optimality. One special feature of this method, as compared to other methods in the literature, is the fact that the solution is obtained directly for the control input without needing to assume or compute a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function

The paper is organized as follows. First the optimal control problem will be defined and solved and then the technique will be applied to a path following problem of a Wheeled Mobile Robot (WMR) using simulations performed in MATLAB/Simulink. Some concluding remarks will close the paper.

II. PROBLEM DEFINITION AND SOLUTION

Consider the following optimal control problem

Problem 1:

$$\begin{aligned} V(x_0) = \inf \quad & \int_0^\infty \left\{q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + ru^2\right\} dt \\ s.t. \quad & \dot{x}_1(t) = f(x_2) \\ & \dot{x}_2(t) = df(x_2) + x_3 \\ & \dot{x}_3(t) = bu \\ & x(0) = x_0 \ , u \in \mathscr{U} \end{aligned} \tag{1}$$

where it is assumed that $q_1 \geq 0, q_2 \geq 0, q_3 > 0, b \neq 0,\ r > 0$, $x(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in \mathbb{R}^3$ and $u \in \mathbb{R}$. The set $\mathscr{U}$ represents the allowable inputs, which are considered to be Lebesgue integrable functions. The function f with bounded derivative $f'(x_2)$ is assumed to be measurable, bounded on any compact set with $f(0) = 0$ and with a finite or at most a countable set of zeros. The term

$$L(x_1, x_2, x_3, u) = q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + ru^2 \tag{2}$$

is called the running cost.The problem is to find if possible a control $u^\star$, and a cost L of the form (2) such that $u^\star$ will be the optimal solution of (1) with finite cost and (2) is nonnegative and has a minimum at $x_1 = x_2 = x_3 = u = 0$. If the function f is linear then from the Linear Quadratic Regulator theory [12] we know that a solution of the form $u = -k_1 x_1 - k_2 x_2 - k_3 x_3$ exists for the case $Q(x) = 0$. Motivated by this result, we will search for solutions of the form $u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3 - kg(x_2)$ where, for reasons that will become apparent in the proof of the main result, $g(x_2) = f(x_2)$. We start by presenting necessary conditions that the value function V must verify for such a solution to exist.

Theorem 1: Assume that a control solution of the form

$$u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3 - kf(x_2) \tag{3}$$

exists for problem (1) and that a class C^1 function V exists that verifies the corresponding HJB equation

$$\inf_u H(x_1, x_2, x_3, u, V_{x_1}, V_{x_2}, V_{x_3}) = 0 \tag{4}$$

where

$$\begin{aligned} H =& q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + ru^2 \\ & + V_{x_1} f(x_2) + V_{x_2}(df(x_2) + x_3) + V_{x_3}(bu) \end{aligned} \tag{5}$$

and

$$V_{x_i} = \frac{\partial V}{\partial x_i}, \text{for } i = 1, 2, 3. \tag{6}$$

and with boundary condition $V(0) = 0$. Then V must be of the form

$$V(x) = 2rb^{-1}\left(k_1 x_1 x_3 + k_2 x_2 x_3 + k_3 \frac{x_3^2}{2} + kx_3 f(x_2)\right) + h(x_1, x_2) \tag{7}$$

where $h(x_1, x_2)$ is an arbitrary integration function of class C^1 with

$$h(0,0) = 0 \tag{8}$$

Proof: Consider the HJB equation (4) associated with (1). The necessary condition on u to be a minimizer is

$$\frac{\partial V}{\partial x_3} = -2rb^{-1} u(x) \tag{9}$$

and therefore

$$V(x) = -2rb^{-1} \int u(x) dx_3 + h(x_1, x_2) \tag{10}$$

where $h(x_1, x_2)$ is an arbitrary integration function of x_1 and x_2. Searching for a solution of the form $u = u_1(x_1, x_2) + u_2(x_3)$, expression (10) becomes

$$V(x) = -2rb^{-1} x_3 u_1(x_1, x_2) - 2rb^{-1} \int u_2(x_3) dx_3 + h(x_1, x_2) \tag{11}$$

Replacing

$$u_1(x_1, x_2) = -k_1 x_1 - k_2 x_2 - kf(x_2) \tag{12}$$

$$u_2(x_3) = -k_3 x_3 \tag{13}$$

yields (7) after integration. From the boundary condition $V(0) = 0$ one obtains the constraint (8). ■

Remark 1: It is important that the value function has cross terms or otherwise, from (9), the controller will only depend on x_3, which will considerably limit the class of systems for which a solution can be found.

The main result is now stated in the next theorem.

Theorem 2: If $q_1 \geq 0,\ q_2 \geq 0,\ q_3 > 0,\ b \neq 0,\ r > 0$,

$$\begin{aligned} Q(x) = & r(k_{32}^2 - 2b^{-1} kf'(x_2)) x_3^2 + 2rk_1 k_2 x_1 x_2 + 2rk_1 k_{31} x_1 x_3 \\ & + 2rk_2 k_{31} x_2 x_3 + rk(k - 2k_{32} d + 2b^{-1} d^2 f'(x_2)) f^2 \end{aligned} \tag{14}$$

and if the gains

$$k_1 = \pm\sqrt{\frac{q_1}{r}} \tag{15}$$

$$k_2 = \pm\sqrt{\frac{q_2}{r}} \tag{16}$$

$$k_3 = \pm\sqrt{\frac{q_3}{r}} + b^{-1}\sqrt{\frac{q_2}{q_3}} \tag{17}$$

$$k = b^{-1}\left(\sqrt{\frac{q_1}{q_3}} + d\sqrt{\frac{q_2}{q_3}}\right) \tag{18}$$

verify

$$(k_{32}^2 - 2b^{-1} kf'(x_2)) x_3^2 + k(k - 2k_{32} d + 2b^{-1} d^2 f'(x_2)) f^2 \geq 0 \tag{19}$$

with

$$k_{31} = \pm\sqrt{\frac{q_3}{r}}, \quad k_{32} = b^{-1}\sqrt{\frac{q_2}{q_3}} \tag{20}$$

then the control input (3) is a solution of the HJB equation (4) associated with (1) with value function

$$V(x) = r\left(\sqrt{k_1(k-dk_{32})}x_1 + \sqrt{k_2k_{32}}x_2 + \sqrt{b^{-1}k_{31}}x_3\right)^2 + \\ + rb^{-1}k_{32}x_3^2 + 2rb^{-1}kx_3f(x_2) + \\ - rb^{-1}kdf^2(x_2) + (2rkk_{32})\int f(x_2)dx_2 + \alpha \tag{21}$$

where α is an integration constant verifying

$$\alpha = -(2rkk_{32})\left[\int f(x_2)dx_2\right]_{x_2=0} \tag{22}$$

The function V is also a local Lyapunov function provided it is locally positive definite in a region around the origin. Moreover, if V is globally positive definite and radially unbounded then it is a Lyapunov function. Finally, the trajectories will converge to one of the minimizers of $L(x_1,x_2,x_3,u(x_1,x_2,x_3))$, i.e, to a point (x_1,x_2,x_3) such that $L(x_1,x_2,x_3,u(x_1,x_2,x_3)) = 0$. If $L(x_1,x_2,x_3,u(x_1,x_2,x_3))$ is convex, then the trajectories will converge to the origin for all initial conditions.

Proof: Using the results of theorem 1 and replacing

$$k_3 = k_{31} + k_{32} \tag{23}$$

the HJB equation (4) yields after rearranging

$$0 = (q_1 - rk_1^2)x_1^2 + (q_2 - rk_2^2)x_2^2 + (q_3 - rk_{31}^2 - rk_{32}^2 + 2rb^{-1}k_2 \\ - 2rk_{31}k_{32} + 2rb^{-1}kf')x_3^2 + Q(x) - 2rk_1k_2x_1x_2 \\ - 2rk_1k_{31}x_1x_3 - 2rk_1k_{32}x_1x_3 - 2rk_2k_{31}x_2x_3 - 2rk_2k_{32}x_2x_3 \\ - rk^2f^2 - 2rkk_{31}x_3f - 2rkk_{32}x_3f + 2rb^{-1}k_1x_3f \\ - 2rkk_1x_1f - 2rkk_2x_2f + \frac{\partial h}{\partial x_1}f + \frac{\partial h}{\partial x_2}x_3 + 2rb^{-1}dk_2x_3f \\ + 2rb^{-1}dkf'fx_3 + \frac{\partial h}{\partial x_2}df \tag{24}$$

where the arguments of the functions were omitted for simplicity. Making

$$\frac{\partial h}{\partial x_2} = 2rk_1k_{32}x_1 + 2rk_2k_{32}x_2 + 2rkk_{32}f(x_2) - 2rb^{-1}dkf'f \tag{25}$$

and using (25) together with $(15)-(18)$ and (20) in (24) and noting that (18) and (20) can also be expressed as

$$kk_{31} - b^{-1}k_1 - b^{-1}dk_2 = 0 \\ b^{-1}k_2 - k_{31}k_{32} = 0 \tag{26}$$

then the expression (24) can be written as

$$0 = (-rk_{32}^2 + 2rb^{-1}kf')x_3^2 + Q(x) - 2rk_1k_2x_1x_2 - 2rk_1k_{31}x_1x_3 \\ - 2rk_2k_{31}x_2x_3 - rk^2f^2 - 2rkk_1x_1f - 2rkk_2x_2f + \frac{\partial h}{\partial x_1}f \\ + 2rk_2k_{32}dx_2f + 2rk_1k_{32}dx_1f + 2rkk_{32}df^2 \\ - 2rb^{-1}kd^2f'f^2 \tag{27}$$

Finally, adding and subtracting $2rk_1k_{32}x_2f(x_2)$ to the right hand side of the above expression, making

$$\frac{\partial h}{\partial x_1} = 2rk_1(k - k_{32}d)x_1 + 2rk_1k_{32}x_2 \tag{28}$$

and choosing Q as (14), one finds that all terms in (27) vanish and therefore the HJB equation is verified. This is a sufficient condition for the control input (3) to be a solution that minimizes the cost of problem (1) because the second derivative of the Hamiltonian (5) with respect to u is equal to $2r > 0$. The running cost is a sensible cost because from (2), $(15)-(18)$ and (20) it is given by

$$L = r(k_1x_1 + k_2x_2 + k_{31}x_3)^2 + r(k_{32}^2 - 2b^{-1}kf'(x_2))x_3^2 \\ + rk(k - 2k_{32}d + 2b^{-1}d^2f'(x_2))f^2 + ru^2 \tag{29}$$

and it is non-negative with a minimun at $x_1 = x_2 = x_3 = u = 0$ under the assumption (19). Replacing $(12)-(13)$, (23) and the integral of resulting expressions from (25) and (28) in (11) yields the value function (21) after integration, considering (26). Furthermore, the boundary condition $V(x(\infty)) = 0$ yields (22). Observe that

$$V(x(0)) = \int_0^\infty L(x_1,x_2,x_3,u^\star)\,dt = -[V(x(\infty)) - V(x(0))] \\ = -\int_0^\infty \dot{V}(x)\,dt \tag{30}$$

so when the optimal control law is used, L and $-\dot{V}$ coincide. Hence,

$$\dot{V} = -L(x_1,x_2,x_3,u^\star) \leq 0 \tag{31}$$

which makes V a local Lyapunov function for the system if it is positive definite in a region around the origin. If V is globally positive definite and radially unbounded then it is a Lyapunov function. Finally, since the optimal cost (21) is finite for all initial conditions, then the trajectories will converge to one of the minimizers of $L(x_1,x_2,x_3,u(x_1,x_2,x_3))$ because $L \geq 0$ and $\lim_{t\to\infty} L = 0$ for integrability. If L is convex, then the trajectories must converge to the origin because the origin is the only minimizer of L. This finishes the proof. ■

Remark 2: It is interesting that the square of the nonlinearity comes naturally as a term in the cost, although this would be difficult to predict based on a general tendency to always construct costs that have only quadratic terms on the state.

III. EXAMPLE: PATH FOLLOWING OF A WMR

In this section, the developed controller is applied to a path following problem of a WMR.

A. Dynamic Model

Fig. 1 shows a schematic of the WMR, which is assumed to be rigid and to be driven by a torque T to control the heading angle ψ which is measured from the positive x-axis in the inertial frame. The forward velocity $V_0 = 1m/s$ is assumed to be already made constant by the proper design

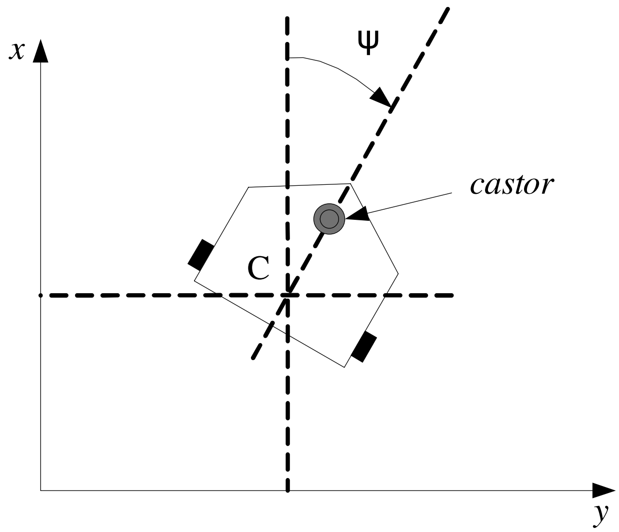

Fig. 1. Schematic of the Wheeled Mobile Robot (WMR)

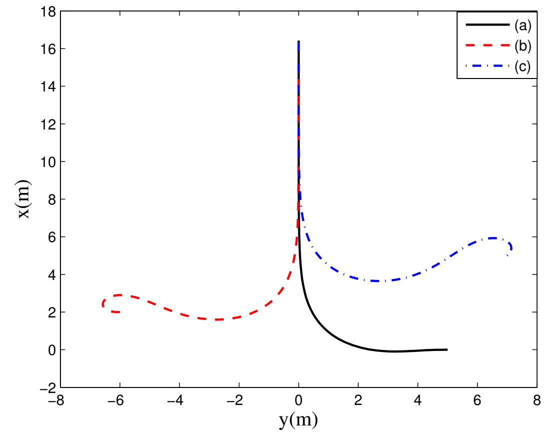

Fig. 2. Trajectories of WMR following path $y = 0$

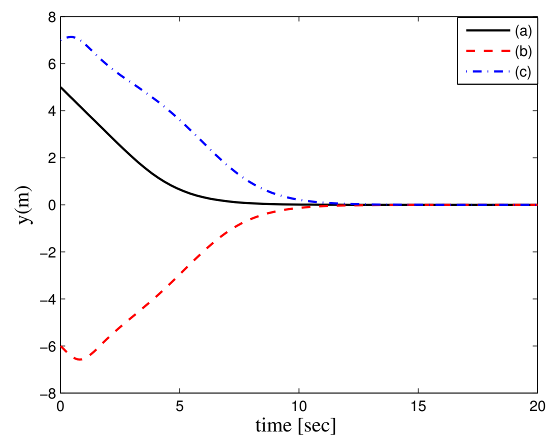

Fig. 3. Position y

of a cruise controller. The dynamic model of the WMR is composed of two parts: kinematics and dynamics. The kinematics equations are

$$\begin{aligned} \dot{y} &= V_0 \sin\psi \\ \dot{\psi} &= R \end{aligned} \tag{32}$$

and the dynamics equation is

$$\dot{R} = \frac{1}{I} T \tag{33}$$

where T is the input torque generated by the DC motors. The moment of inertia of the WMR with respect to the center of mass is $I = 1kg.m^2$. It is desired that the WMR follows the path y = 0.

B. Controller Design and Simulation Results

The differential equations (32) and (33) are cast in the form of *Problem 1* with $f(x_2) = \sin(x_2)$, $b = 1$, where the states are defined by $x_1 = y$, $x_2 = \psi$ and $x_3 = R$. If $q_1 = q_3 = r = 1$ and $q_2 = 4$, then the optimal controller is

$$u = -x_1 - 2x_2 - 3x_3 - sin(x_2) \tag{34}$$

the running cost is

$$L = (x_1 + 2x_2 + x_3)^2 + (4 - 2cos(x_2))x_3^2 + \sin^2(x_2) + u^2 \tag{35}$$

and the value function is

$$V(x) = (x_1 + 2x_2 + x_3)^2 + 2x_3^2 + 2x_3 \sin(x_2) - 4\cos(x_2) + 4 \tag{36}$$

Moreover, the derivative of the value function

$$\begin{aligned} \dot{V} = &-(4 - 2cos(x_2))x_3^2 - (x_1 + 2x_2 + x_3)^2 - \sin^2(x_2) \\ &- (x_1 + 2x_2 + 3x_3 + sin(x_2))^2 \end{aligned} \tag{37}$$

is negative definite for $x_2 \in (-\pi, \pi)$. Therefore, the value function is a local Lyapunov function in the largest invariant set contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, |x_2| < \pi\} \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 | V > 0\}$ where > 0 stands for positive definite. Note that one cannot guarantee convergence to the origin from any initial condition because L is not convex. Simulations were performed using the optimal controller (34) for the following different initial conditions:

case (a): $x_0 = 0, y_0 = 5,\ \psi_0 = -\frac{\pi}{2}$ and $R_0 = 0$
case (b): $x_0 = 2, y_0 = -6,\ \psi_0 = -\frac{\pi}{2}$ and $R_0 = 0$
case (c): $x_0 = 5, y_0 = 7,\ \psi_0 = \frac{\pi}{6}$ and $R_0 = 0$

Convergence to the desired path is clearly seen in Figure 2. Moreover, Figures $3 - 5$ show the time variations of the position, heading angle and angular velocity. The input signal is depicted in Figure 6.

IV. CONCLUSIONS

The solution to a third order nonlinear optimal control problem has been presented in this paper extending an inverse optimality method originally developed for second order systems. The important feature of this approach is that the analytical solution for the control input is obtained directly without needing to assume or compute a coordinate transformation, value function or Lyapunov function. The value function and a Lyapunov function can however be computed after the control input has been found. The con-

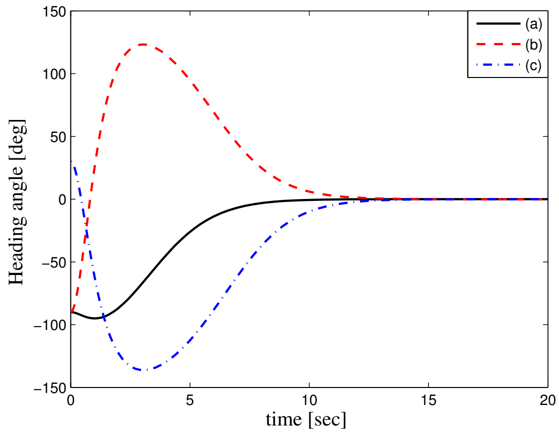

Fig. 4. Heading angle ψ

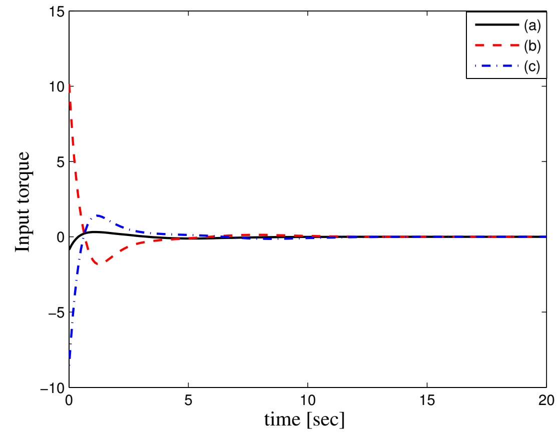

Fig. 6. Input Signal

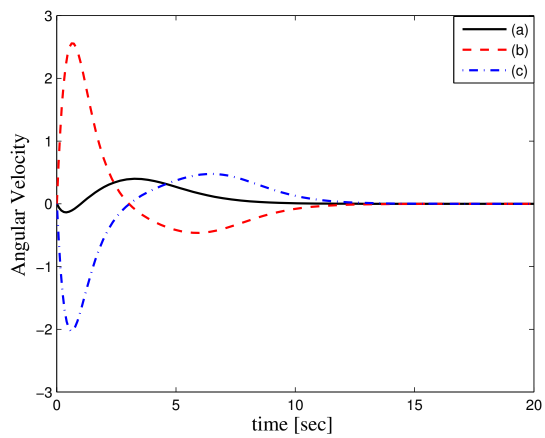

Fig. 5. Angular Velocity

troller was applied to a path following problem of a Wheeled Mobile Robot (WMR). The simulation results verified the effectiveness of the method.

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