2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Infinite-Horizon, Multiple-Cumulant Cost Density-Shaping for Stochastic Optimal Control

Matthew Zyskowski†
matt.zyskowski@hs.utc.com

Ronald Diersing††
rwdiersing@usi.edu

Abstract— **Recently for the LQG framework, cost density-shaping control paradigms for stochastic optimizations have been proposed. These new control methods have enabled the shape of a target cost density to be transformed into a linear control law. However, the theory developed so far pertains exclusively to the finite-horizon. The purpose of this work is to develop a Multiple-Cumulant Cost Density-Shaping (MCCDS) control solution as the terminal time approaches infinity. The first-generation benchmark for seismically-excited buildings will be used to validate the infinite-horizon MCCDS control law.**

***Index Terms*— infinite horizon, stochastic optimal control, cost cumulant control, structural control, cost density-shaping**

I. Introduction

Pham's "k Cost Cumulant" (kCC) control paradigm [1] generalizes the classical Linear Quadratic Gaussian (LQG) theory through minimizing a linear combination of k initial cost cumulants. In the referenced work, the kCC control solution is shown to out-perform the best proposed control paradigms in the first-generation benchmark study for seismically-excited buildings. This success emphasizes the performance benefits of cost cumulant control. The finite-horizon kCC theory of [1] is extended to the infinite-horizon in [2], where a constant kCC control solution is derived using a Lagrange multiplier technique. This infinite-horizon kCC controller is applied to a benchmark for cable-stayed bridges with good results. A limitation of kCC control is that the weighting parameters in the performance index do not directly correspond to the cost density achieved under kCC control. This fact has largely provided the driving force behind the creation of cost density-shaping control paradigms.

Cost Density-Shaping (CDS) controls have been recently developed for the LQG framework. In [3], a general Weighted Least-Squares (WLS) paradigm has been proposed, where a Single-Degree-of-Freedom (SDOF) building problem is used to show that cost cumulants computed from a family of nominal *3*CC controls can be approximately realized with WLS-CDS controls using the aforementioned cost cumulants as targets. Mean-variance cost density-shaping controls based on Maximum Bhattacharyya Coefficient (MBC) and Minimum Kulback-Leibler Divergence (MKLD) have been given in [4], [5]. The first-generation benchmark has been used for a validation exercise in [4] to show that target cumulants resultant from a nominal *2*CC control can be realized with a MBC-CDS controller. This MBC-CDS control, computed using only the target mean and variance of the cost, preserves the control performance and robust stability properties of the *2*CC control driving the target statistics. This observation shows that desirable qualities of closed-loop system behavior are encapsulated in the cost cumulants, and thus the approximate shape of the cost density. Furthermore, a statistical target selection design methodology in [4], [5] has revealed controls that lead to alternative statistical characterizations of the random cost which have higher control performance than the same nominal *2*CC control. The correspondence of control performance to the shape of the cost density achieved under a given control law provides good reason for the continued investigation and development of MCCDS control theory [6].

Usually no fixed terminal time is specified in control applications, and it would therefore be ideal that a MCCDS controller be developed with good long-run performance. This practical situation involves the system being controlled over a time interval extending to infinity. This paper considers the aforementioned case in order to complete the theory of linear, full-observation, state-feedback MCCDS control. Simulation results will be provided to validate the derived control solution.

II. Preliminaries

Our work uses Pham's framework for deriving the infinite-horizon kCC control solution, as described in this section and the next. It pertains exclusively to the process with dynamics subject to additive white Gaussian noise,

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu(t)dt + Gdw(t), \\ x_0 &= E\{x(t_0)\},\ x_0 \in \mathbb{R}^n,\ t \in [t_0, \infty) \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times p}$ and $w(t)$ is a p-dimensional stationary Wiener process having a correlation of increments defined by

$$\begin{aligned} &E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] \\ &= W|\tau_1 - \tau_2|, W \succ \mathbf{0}^{p\times p}. \end{aligned} \tag{2}$$

It is assumed that $u \in L^2_{\mathcal{F}}(\Omega, \mathcal{C}([t_0,\infty);\mathbb{R}^m))$ so that $x \in L^2_{\mathcal{F}}(\Omega, \mathcal{C}([t_0,\infty);\mathbb{R}^n))$. That is, $E\{\int_{t_0}^{\infty} u^T(t)u(t)dt\} < \infty$ ensures $E\{\int_{t_0}^{\infty} x^T(t)x(t)dt\} < \infty$.

† Systems Engineer, Systems Engineering, Hamilton Sundstrand
†† Assistant Professor, Dept. of Engineering, Univ. of Southern Indiana

The cost J is an integral-quadratic form defined by

$$J(u) = \int_{t_0}^{\infty} \left(x(t)^T Q(t) x(t) + u(t)^T R(t) u(t) \right) dt \quad (3)$$

where $Q = Q^T \in \mathbb{S}_+^n$ and $R = R^T \in \mathbb{S}_{++}^m$. Under this membership, we have $Q \succeq \mathbf{0}^{n\times n}$ and $R \succ \mathbf{0}^{m\times m}$.

With $x(t)$ being random the cost J is also random so statistics of (3) can be considered, such as its cumulants.

III. Cost Cumulants

The form of the cost cumulants on the infinite-horizon is now given.

***Definition** 3.1:* (Stabilizing Control)
A control $u(t) = k(t, x(t))$ is called stabilizing if the system (1) is bounded-input/bounded-state stable. In addition, if $w = 0$ then the origin $x = 0$ is asymptotically stable.

***Definition** 3.2:* (Stabilizing Gain)
A feedback gain $K \in \mathbb{R}^{n\times m}$ is stabilizing if the following state-feedback control law is stabilizing, $u(t) = Kx(t),\ t \in [t_0, \infty)$.

***Theorem** 3.3:* (Infinite-Horizon Cost Cumulants)
Let $r \in \mathbb{Z}^+$, and the matrices A, B, G, Q and R be as defined previously. Suppose (A, B) is stabilizable, so that $\exists K$ such that the closed-loop matrix $(A + BK)$ has only eigenvalues with negative real parts and the controlled system (1) in the absence of disturbances is exponentially stabilized. Under these conditions, the r cost cumulants are defined by

$$\kappa_{\infty,l} = \text{Tr}(\mathcal{H}_l G W G^T),\ 1 \le l \le r$$

where the matrices $\{\mathcal{H}_l\}_{l=1}^r$ are such that $\mathcal{H}_l \succeq \mathbf{0}^{n\times n}$ and satisfy the equations,

$$\begin{aligned} \mathcal{F}_1(\boldsymbol{\mathcal{H}}, K) &\triangleq (A+BK)^T \mathcal{H}_1 + \mathcal{H}_1(A+BK) \\ &\quad + Q_\infty + K^T R_\infty K = \mathbf{0}^{n\times n} \\ \mathcal{F}_l(\boldsymbol{\mathcal{H}}, K) &\triangleq (A+BK)^T \mathcal{H}_l + \mathcal{H}_l(A+BK) \\ &\quad + 2\sum_{j=1}^{l-1} \begin{pmatrix} l \\ j \end{pmatrix} \mathcal{H}_j G W G^T \mathcal{H}_{l-j} = \mathbf{0}^{n\times n} \end{aligned} \quad (4)$$

where $Q_\infty = \lim_{t\to\infty} Q(t)$ and $R_\infty = \lim_{t\to\infty} R(t)$.

Proof: See Theorem 4.2.2 on pg. 106 in [7]. ■

IV. Notation

We introduce some notation to make restatements of the above equations more concise in the development. This notation is heavily inspired by what Pham originated in [7]. Begin by defining the state variable $\boldsymbol{\mathcal{H}} \in \mathbb{R}^{rn\times n}$ as below

$$\boldsymbol{\mathcal{H}} \triangleq (\mathcal{H}_1, \ldots, H_r).$$

Using these state variables, define the function

$$\mathcal{F}(\boldsymbol{\mathcal{H}}, K) \triangleq (\mathcal{F}_1(\boldsymbol{\mathcal{H}}, K), \ldots, \mathcal{F}_r(\boldsymbol{\mathcal{H}}, K)).$$

where $\mathcal{F}_i(\cdot)$ in the above definition is defined as in (4).

Also, let the vector of cumulants $\boldsymbol{\kappa}_\infty \in \mathbb{R}^r$ be defined as

$$\boldsymbol{\kappa}_\infty = (\kappa_{\infty,1}, \ldots, \kappa_{\infty,r}).$$

Where appropriate, the dependence of $\boldsymbol{\kappa}_\infty$ on $\boldsymbol{\mathcal{H}}$ will be indicated by $\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}})$.

V. Target Cost Statistics

Given matrices for a system characterization $(A,\ B,\ G)$, an integral-quadratic cost characterization (Q, R), and the second-order statistics of the noise (W), consider the cost cumulants as a result of the alternative (and unknown) *stabilizing* linear state-feedback control $\tilde{u}(t) = \tilde{K}\tilde{x}(t)$, where $\tilde{K} \in \mathbb{R}^{m\times n}$. Given the preceding results, r cost cumulants are given by

$$\tilde{\kappa}_{\infty,l} = \text{Tr}(\tilde{\mathcal{H}}_l G W G^T),\ 1 \le l \le r$$

where the positive semi-definite matrices $\{\tilde{\mathcal{H}}_l\}_{l=1}^r$ satisfy the algebraic equations $\mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$. We will refer to the quantities $\{\tilde{\kappa}_{\infty,l}\}_{l=1}^r$ as the *target cost cumulants.*

VI. Problem Formulation

The optimization problem can be formulated by defining a control space over which the infinite-horizon MCCDS performance index can be minimized. The appropriate definitions precede the problem statement.

***Definition** 6.1:* (Well-Posed Control Law)
A feedback gain K is well-posed if $\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n}$ admits unique solutions $\mathcal{H}_l,\ 1 \le l \le r$.

***Definition** 6.2:* (Admissible Control Gain)
A feedback gain K is called admissible if $\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n}$ admits unique solutions $\mathcal{H}_l,\ 1 \le l \le r$ that are positive semi-definite. Denote this set of gains as $\mathcal{K}_\infty$.

***Definition** 6.3:* (Infinite-Horizon Performance Index)
Consider a function $g : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ denoted as $g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)$ which satisfies the following properties:

- The function g is analytic in both of its vector arguments
- The function g is convex in $\boldsymbol{\kappa}_\infty$ and the domain $\textbf{dom}\ g_{\tilde{\boldsymbol{\kappa}}_\infty}$ (e.g. g's restriction to $\tilde{\boldsymbol{\kappa}}_\infty$) is a convex set
- The function $g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)$ is non-negative in $\boldsymbol{\kappa}_\infty$ on some neighborhood of $\tilde{\boldsymbol{\kappa}}_\infty$
- The function $g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)$ is strictly non-decreasing in $\boldsymbol{\kappa}_\infty$

The MCCDS infinite-horizon performance index is non-negative, and convex in the cost cumulants. As such, it is well-suited for the objective function in the following optimization problem.

***Definition** 6.4:* (MCCDS Infinite-Horizon Optimization)
The infinite-horizon MCCDS optimization can be stated as,

$$\min_{K\in\mathcal{K}_\infty} g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}}))$$

subject to:

$$\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n},\ \mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}.$$

VII. Infinite-Horizon MCCDS Solution

We follow the same Lagrange multiplier approach that Pham used in [7] to derive our control solution. To do so, first a regularity condition must be introduced that guarantees a notion of $2r$-fold smoothness is present in the manifold formed by the constraint sets $\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n},\ \mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$.

Definition 7.1: (Regularity Condition, Matrix Case) Let the optimization point $(\boldsymbol{\mathcal{H}}, K)$ satisfy the equations of motion $\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n}$ and the point $(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})$ satisfy the equations $\mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$. Then it is said that $(\boldsymbol{\mathcal{H}}, K)$ and $(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})$ are regular if the equations

$$\text{grad}\left\{ \sum_{k=1}^{r} \text{Tr}\left(\mathcal{F}_k(\boldsymbol{\mathcal{H}}, K)\Lambda_k^T\right) + \sum_{k=1}^{r} \text{Tr}\left(\mathcal{F}_k(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})\tilde{\Lambda}_k^T\right)\right\} = \mathbf{0}^{2rn\times n}$$

admit the unique solutions $\boldsymbol{\Lambda} = \mathbf{0}^{rn\times n}$ and $\tilde{\boldsymbol{\Lambda}} = \mathbf{0}^{rn\times n}$.

Theorem 7.2: (Necessary Conditions for Optimality) Assume $(\boldsymbol{\mathcal{H}}^*, K^*) \in (\mathbb{S}^n)^r \times \mathcal{K}$ and $(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})$ are regular points for the respective constraint hyper-surfaces

$$\mathcal{F}(\boldsymbol{\mathcal{H}}, K) = \mathbf{0}^{rn\times n},\ \mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$$

for which the functional $g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}}))$ is minimized. Then there exists matrix multipliers $\boldsymbol{\Lambda}^* = (\Lambda_1^*, \ldots, \Lambda_r^*)$ and $\tilde{\boldsymbol{\Lambda}}^* = (\tilde{\Lambda}_1^*, \ldots, \tilde{\Lambda}_r^*)$ such that the gradient of the Lagrangian

$$\begin{aligned}\mathcal{L}(\boldsymbol{\mathcal{H}}, K, \boldsymbol{\Lambda}, \tilde{\boldsymbol{\Lambda}}) &= g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}})) \\ &+ \sum_{i=1}^{r} \text{Tr}(\mathcal{F}_i(\boldsymbol{\mathcal{H}}, K)\Lambda_i^T) + \sum_{i=1}^{r} \text{Tr}(\mathcal{F}_i(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})\tilde{\Lambda}_i^T)\end{aligned}$$

vanishes for the optimal 4-tuple $(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*, \tilde{\boldsymbol{\Lambda}}^*)$, that is $\nabla\mathcal{L}_{\boldsymbol{\mathcal{H}},K,\boldsymbol{\Lambda},\tilde{\boldsymbol{\Lambda}}}(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*, \tilde{\boldsymbol{\Lambda}}^*) = \mathbf{0}$.

The technical lemma below follows directly from Pham's controller derivation, and holds for control solutions having the structure of the kCC controller. In the following, we use $S \triangleq GWG^T$.

Lemma 7.3: (Cancellation Properties) Let $\mathcal{H}_l^\dagger,\ 1 \le l \le r$ be the solutions of the following system of equations $\mathcal{F}_l(\boldsymbol{\mathcal{H}}^\dagger, K^\dagger) = \mathbf{0}^{n\times n}, 1 \le l \le r$ under $K^\dagger$ where the gain $K^\dagger$ for $c_l > 0, 1 \le l \le r$ is defined by

$$K^\dagger = -R^{-1}B^T\left(\mathcal{H}_1^\dagger + \sum_{l=2}^{r}\frac{c_l}{c_1}\mathcal{H}_l^\dagger\right).$$

Let (A, B) be stabilizable and (Q, A) be detectable, and the solution Γ exists to the following Lyapunov equation

$$(A + BK^\dagger)^T\Gamma + \Gamma(A + BK^\dagger) = -c_1 S.$$

Then the matrices $\mathcal{H}_l^\dagger$ satisfy

$$\Gamma\mathcal{H}_l^\dagger S + S\mathcal{H}_l^\dagger\Gamma = \mathbf{0}^{n\times n}, 1 \le l \le r-1. \tag{5}$$

Proof: See [8]. ■

We are now ready to state the main result of the paper.

Theorem 7.4: (Infinite-Horizon MCCDS Control) Assume that (A, B) is stablizable and (Q, A) is detectable. Fix $r \in \mathbb{N}$ and suppose there exist r cost cumulants of (3). Under these conditions, the optimal solution to the Infinite-Horizon MCCDS optimization concerning the process (1) and the cost (3) is given by $u^*(t) = K^*x(t)$ where the extremalizing gain K^* is

$$K^* = -R^{-1}B^T\sum_{i=2}^{r}\left(\mathcal{H}_1^* + \left(\frac{\dfrac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,i}}}{\dfrac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,1}}}\right)\mathcal{H}_i^*\right)$$

which is well-defined when the following equations admit a solution $\mathcal{F}(\boldsymbol{\mathcal{H}}^*, K^*) = \mathbf{0}^{rn\times n},\ \mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$.

Proof: The regularity condition can be verified for feasible optimization points, following the same approach as in [7]. Now the necessary condition of optimality can be applied to the Lagrange functional $\mathcal{L}(\boldsymbol{\mathcal{H}}, K, \boldsymbol{\Lambda}, \tilde{\boldsymbol{\Lambda}})$ defined by

$$\begin{aligned}\mathcal{L}(\boldsymbol{\mathcal{H}}, K, \boldsymbol{\Lambda}, \tilde{\boldsymbol{\Lambda}}) &= g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}})) \\ &+ \sum_{k=1}^{r} \text{Tr}\left(\mathcal{F}_k(\boldsymbol{\mathcal{H}}, K)\Lambda_k^T\right) + \sum_{k=1}^{r} \text{Tr}\left(\mathcal{F}_k(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K})\tilde{\Lambda}_k^T\right)\end{aligned}$$

which is $\nabla\mathcal{L}_{\boldsymbol{\mathcal{H}},K,\boldsymbol{\Lambda},\tilde{\boldsymbol{\Lambda}}}(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*, \tilde{\boldsymbol{\Lambda}}^*) = \mathbf{0}$.

A key assumption for the fixed control $\tilde{K}$ is that it satisfies the constraints $\mathcal{F}(\tilde{\boldsymbol{\mathcal{H}}}, \tilde{K}) = \mathbf{0}^{rn\times n}$, so that $\mathcal{L}(\cdot)$ is minimized for any choice of $\tilde{\boldsymbol{\Lambda}}$ with the optimal triple $(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)$. We can thus choose $\tilde{\boldsymbol{\Lambda}} = \mathbf{0}^{rn\times n}$, and write $\mathcal{L}^\dagger(\boldsymbol{\mathcal{H}}, K, \boldsymbol{\Lambda}) = \mathcal{L}(\boldsymbol{\mathcal{H}}, K, \boldsymbol{\Lambda}, \mathbf{0}^{rn\times n})$ and consider that $\nabla\mathcal{L}^\dagger_{\boldsymbol{\mathcal{H}},K,\boldsymbol{\Lambda}}(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*) = \mathbf{0}$ will yield the extremal point of the original functional $\mathcal{L}(\cdot)$. This condition requires that

$$\frac{\partial\mathcal{L}^\dagger(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)}{\partial\Lambda_l} = \mathbf{0}^{n\times n},\ 1 \le l \le r \tag{6}$$

$$\frac{\partial\mathcal{L}^\dagger(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)}{\partial\mathcal{H}_m} = \mathbf{0}^{n\times n},\ 1 \le m \le r \tag{7}$$

$$\frac{\partial\mathcal{L}^\dagger(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)}{\partial K} = \mathbf{0}^{n\times n} \tag{8}$$

The equations (6) are just the constraints $\mathcal{F}(\boldsymbol{\mathcal{H}}^*, K^*) = \mathbf{0}^{rn\times n}$. The equations (7) give the relations $1 \le m \le r-1$,

$$\begin{aligned}&\frac{\partial}{\partial\mathcal{H}_m}\left(g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}^*), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}})) + \sum_{k=1}^{r}\text{Tr}\left(\mathcal{F}_k(\boldsymbol{\mathcal{H}}^*, K^*)\Lambda_k^{*T}\right)\right) \\ &= (A + BK^*)\Lambda_m^* + \Lambda_m^*(A + BK^*)^T + \frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,m}}S \\ &+ 2\sum_{k=m+1}^{r-1}\begin{pmatrix} k \\ m\end{pmatrix}\left(\Lambda_k^*\mathcal{H}_{k-m}^*S + S\mathcal{H}_{k-m}^*\Lambda_k^*\right) \\ &= \mathbf{0}^{n\times n}\end{aligned} \tag{9}$$

and for $m = r$ the relation,

$$\begin{aligned}&\frac{\partial}{\partial\mathcal{H}_r}\left(\sum_{k=1}^{r}\text{Tr}\left(\mathcal{F}_k(\boldsymbol{\mathcal{H}}^*, K^*)\Lambda_k^{*T}\right)\right) \\ &= (A + BK^*)\Lambda_r^* + \Lambda_r^*(A + BK^*)^T + \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,r}}S \\ &= \mathbf{0}^{n\times n}.\end{aligned} \tag{10}$$

The derivatives are used above,

$$\begin{aligned}\frac{\partial g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}}))}{\partial \mathcal{H}_l} &= \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}} \cdot \frac{\partial \kappa_{\infty,l}}{\partial \mathcal{H}_l} \\ &= \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}} \cdot \frac{\partial \mathrm{Tr}(\mathcal{H}_l S)}{\partial \mathcal{H}_l} \\ &= \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}} S.\end{aligned}$$

Now propose the multipliers $\Lambda_l,\ 2 \le l \le r$,

$$\Lambda_l^* = \left(\frac{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}}}{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}}} \right) \Lambda_1^*,\ 2 \le l \le r. \tag{11}$$

From (10), clearly Λ_1^* must satisfy

$$(A + BK^*)\Lambda_1^* + \Lambda_1^*(A + BK^*)^T = -\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}} S. \tag{12}$$

Above the standard assumption on the finite-horizon has been invoked, more precisely

$$\left(\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}} \right) \neq 0.$$

Note that for $1 \le m \le r-1$, the expression below

$$(A + BK^*)\Lambda_m^* + \Lambda_m^*(A + BK^*)^T + \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,m}} S$$

by (11) becomes

$$\begin{aligned}&(A + BK^*)\Lambda_1^* + \Lambda_1^*(A + BK^*)^T + \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}} S \\ &= -\left(\frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}} S \right) + \frac{\partial g(\boldsymbol{\kappa}_\infty, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}} S = \mathbf{0}^{n\times n}.\end{aligned}$$

Using the above relations in (9) reduces the equations to

$$\begin{aligned}&\frac{\partial}{\partial \mathcal{H}_m} \left(g(\boldsymbol{\kappa}_\infty(\boldsymbol{\mathcal{H}}^*), \tilde{\boldsymbol{\kappa}}_\infty(\tilde{\boldsymbol{\mathcal{H}}})) + \sum_{k=1}^{r} \mathrm{Tr}\left(\mathcal{F}_k(\boldsymbol{\mathcal{H}}^*, K^*)\Lambda_k^{*T}\right) \right) \\ &= 2 \sum_{k=m+1}^{r-1} \begin{pmatrix} k \\ m \end{pmatrix} \underbrace{\left(\Lambda_k^* \mathcal{H}_{k-m}^* S + S\mathcal{H}_{k-m}^* \Lambda_k^*\right)}_{\left(\frac{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,k}}}{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}}} \right)\left(\Lambda_1^* \mathcal{H}_{k-m}^* S + S\mathcal{H}_{k-m}^* \Lambda_1^*\right)} \\ &= \mathbf{0}^{n\times n}\end{aligned} \tag{13}$$

Introduce the notation

$$c_{k,m} \triangleq \underbrace{2 \begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,k}}}{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}}} \right)}_{\neq 0}.$$

The non-zero ratio of derivatives above reflects the assumption that $g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)$ is convex and strictly non-decreasing in $\boldsymbol{\kappa}_\infty$. Re-write the equations (13) as

$$\sum_{k=m+1}^{r-1} c_{k,m}\left(\Lambda_1^* \mathcal{H}_{k-m}^* S + S\mathcal{H}_{k-m}^* \Lambda_1^*\right) = \mathbf{0}^{n\times n} \tag{14}$$

Using the multipliers (11), consider now the optimality condition (8),

$$\frac{\partial \mathcal{L}(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)}{\partial K} = 2B^T \sum_{l=1}^{r} \mathcal{H}_l^* \Lambda_l^* + 2RK^*\Lambda_1^*$$

becomes

$$\left(2B^T \sum_{l=2}^{r} \left(\mathcal{H}_1^* + \left(\frac{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}}}{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}}} \right) \mathcal{H}_l^* \right) + 2RK^* \right) \Lambda_1^* = \mathbf{0}^{n\times n}$$

For any Λ_1^*, this equation is satisfied when the gain is

$$K^* = -R^{-1}B^T \sum_{l=2}^{r} \left(\mathcal{H}_1^* + \left(\frac{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}}}{\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,1}}} \right) \mathcal{H}_l^* \right). \tag{15}$$

Since $\frac{\partial g(\boldsymbol{\kappa}_\infty^*, \tilde{\boldsymbol{\kappa}}_\infty)}{\partial \kappa_{\infty,l}} > 0,\ 1 \le l \le r$ are just positive constants, from the form of our control gain above, and it's assumed that (12) has a unique solution, from the previous lemma it must be that the following relations are true.

$$\Lambda_1^* \mathcal{H}_l^* S + S\mathcal{H}_l^* \Lambda_1^* = \mathbf{0}^{n\times n}, 1 \le l \le r-1.$$

With the above relation, clearly (14) is rendered true, which is the ultimate simplification of (7) given the consequences of the selection (11). The gain (15) is thus extremalizing in that (6), (7), and (8) are all satisfied with the triple $(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)$. When $\Lambda_1^* \succeq \mathbf{0}^{n\times n}$ this gain is minimizing, as seen by

$$\frac{\partial^2 \mathcal{L}(\boldsymbol{\mathcal{H}}^*, K^*, \boldsymbol{\Lambda}^*)}{\partial K^2} = 2R \otimes \Lambda_1^* \succeq \mathbf{0}.$$

■

VIII. Simulation Results

The first-generation benchmark problem will be used to validate the infinite-horizon MCCDS control algorithm. Due to length limitations, the details of benchmark study cannot be provided here but can be found in [9]. Also, readers are directed to Chapter 5 of [8] for additional information on this numerical experiment, as this description is fairly concise.

The system matrices (A, B, G) and output matrices (C, D) of the reduced-order model for control design in the benchmark are used for this computation. Furthermore, weighting matrices (Q, R) are employed that appear in the original benchmark study. The target *4*CC control is calculated using iterative techniques. The infinite-horizon averaged cost cumulants are realized in solving the family of algebraic Riccati equations to arrive at the $\tilde{\boldsymbol{\mathcal{H}}}$ matrices. In particular, we solve

$$\begin{aligned}
&(A+B\tilde{K})^T\tilde{\mathcal{H}}_1+\tilde{\mathcal{H}}_1(A+B\tilde{K})+\tilde{K}^TR\tilde{K}+\tilde{Q}=\mathbf{0}^{n\times n}\\
&(A+B\tilde{K})^T\tilde{\mathcal{H}}_2+\tilde{\mathcal{H}}_2(A+B\tilde{K})+4\tilde{\mathcal{H}}_1GWG^T\tilde{\mathcal{H}}_1=\mathbf{0}^{n\times n}\\
&(A+B\tilde{K})^T\tilde{\mathcal{H}}_3+\tilde{\mathcal{H}}_3(A+B\tilde{K})+6\tilde{\mathcal{H}}_1GWG^T\tilde{\mathcal{H}}_2\\
&+6\tilde{\mathcal{H}}_2GWG^T\tilde{\mathcal{H}}_1=\mathbf{0}^{n\times n}\\
&(A+B\tilde{K})^T\tilde{\mathcal{H}}_4+\tilde{\mathcal{H}}_4(A+B\tilde{K})+8\tilde{\mathcal{H}}_1GWG^T\tilde{\mathcal{H}}_3\\
&+12\tilde{\mathcal{H}}_2GWG^T\tilde{\mathcal{H}}_2+8\tilde{\mathcal{H}}_3GWG^T\tilde{\mathcal{H}}_1=\mathbf{0}^{n\times n}
\end{aligned}\tag{16}$$

using Pham's iterate solutions technique with the control gain

$$\tilde{K}=-B^TR^{-1}\left(\tilde{\mathcal{H}}_1+\frac{\mu_2}{\mu_1}\tilde{\mathcal{H}}_2+\frac{\mu_3}{\mu_1}\tilde{\mathcal{H}}_3+\frac{\mu_4}{\mu_1}\tilde{\mathcal{H}}_4\right)$$

with

$$(\mu_1,\mu_2,\mu_3,\mu_4)=(1,1.0\times10^{-5},9.0\times10^{-12},2.0\times10^{-20})$$

This yields the vector of targets $\tilde{\boldsymbol{\kappa}}_\infty\in\mathbb{R}^4$ having components

$$\tilde{\kappa}_{\infty,i}=\mathrm{Tr}(\tilde{\mathcal{H}}_iGWG^T),\ 1\le i\le 4. \tag{17}$$

Consider the function

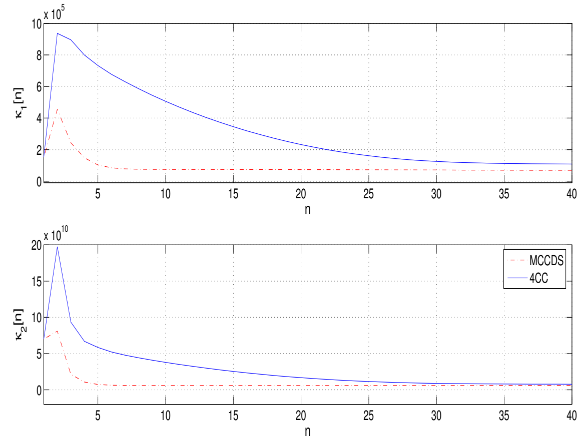
$$\begin{aligned}
g(\boldsymbol{\kappa}_\infty,\tilde{\boldsymbol{\kappa}}_\infty)=&\frac{\kappa_{\infty,3}^2+\tilde{\kappa}_{\infty,3}^2-2\kappa_{\infty,3}\tilde{\kappa}_{\infty,3}}{24\tilde{\kappa}_{\infty,2}^4}\\
&+\frac{\kappa_{\infty,4}^2+\tilde{\kappa}_{\infty,4}^2-2\kappa_{\infty,4}\tilde{\kappa}_{\infty,4}}{92\tilde{\kappa}_{\infty,2}^5}\\
&+\frac{21\tilde{\kappa}_{\infty,3}^4+7\kappa_{\infty,3}^4-28\kappa_{\infty,3}\tilde{\kappa}_{\infty,3}^3}{92\tilde{\kappa}_{\infty,2}^7}\\
&+\frac{(\kappa_{\infty,3}\tilde{\kappa}_{\infty,3}-\tilde{\kappa}_{\infty,3}^2)\cdot\tilde{\kappa}_{\infty,4}}{8\tilde{\kappa}_{\infty,2}^6}\\
&+\frac{(\tilde{\kappa}_{\infty,3}^2-\kappa_{\infty,3}^2)\cdot\kappa_{\infty,4}}{16\tilde{\kappa}_{\infty,2}^6}\\
&+\frac{1}{2}\left(\frac{\kappa_{\infty,2}}{\tilde{\kappa}_{\infty,2}}-1-\log\left(\frac{\kappa_{\infty,2}}{\tilde{\kappa}_{\infty,2}}\right)\right)\\
&+\frac{1}{2}\frac{(\kappa_{\infty,1}-\tilde{\kappa}_{\infty,1})^2}{\tilde{\kappa}_{\infty,2}}.
\end{aligned}\tag{18}$$

that expresses the distance between 4-cumulant Edgeworth approximations to the cost density and the target cost density in terms of $\boldsymbol{\kappa}_{\boldsymbol{\infty}}$ and $\tilde{\boldsymbol{\kappa}}_{\boldsymbol{\infty}}$. This function is a cumulant-representation of a hybrid variant for the KLD derived using techniques described in [10], and will be positive under the assumption of convexity. This assumption seems reasonable since $g(\cdot)$ is comprised of functions related to KLD, which is convex. Consider the optimization

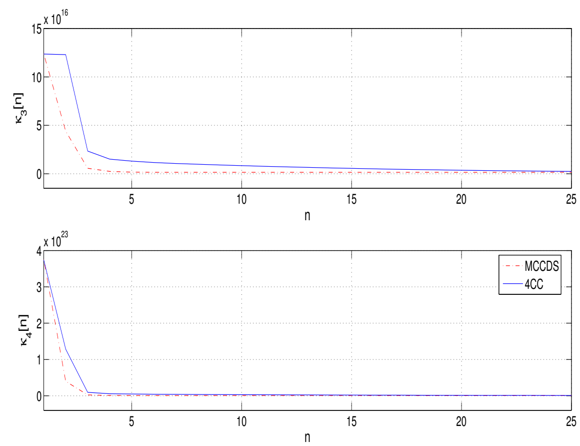
$$\begin{aligned}
&\min_K\left\{g(\boldsymbol{\kappa}_\infty,\tilde{\boldsymbol{\kappa}}_\infty)\right\}\\
dx(t)=&(A+BK(t))x(t)+Gdw(t),\ t\in[t_0,\infty).
\end{aligned}$$

From the proof, the solution is

$$K=-R^{-1}B^T\left(\mathcal{H}_1^*+\gamma_{\infty,2}\mathcal{H}_2^*+\gamma_{\infty,3}\mathcal{H}_3^*+\gamma_{\infty,4}\mathcal{H}_4^*\right).$$



(a) Cost Mean & Variance with Targets



(b) Cost Skew & Kurtosis with Targets

Fig. 1: Infinite-Horizon Cost Density-Shaping

with

$$\gamma_{\infty,i}=\left(\frac{\dfrac{\partial g(\boldsymbol{\kappa}_\infty^*,\tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,i}}}{\dfrac{\partial g(\boldsymbol{\kappa}_\infty^*,\tilde{\boldsymbol{\kappa}}_\infty)}{\partial\kappa_{\infty,1}}}\right),\ 2\le i\le 4$$

where $\mathcal{H}_i^*,\ 1\le i\le 4$ are determined by the algebraic equations

$$\begin{aligned}
&(A+BK^*)^T\mathcal{H}_1^*+\mathcal{H}_1^*(A+BK^*)+K^{*T}RK^*+Q=\mathbf{0}^{n\times n}\\
&(A+BK^*)^T\mathcal{H}_2^*+\mathcal{H}_2^*(A+BK^*)+4\mathcal{H}_1^*GWG^T\mathcal{H}_1^*=\mathbf{0}^{n\times n}\\
&(A+BK^*)^T\mathcal{H}_3^*+\mathcal{H}_3^*(A+BK^*)+6\mathcal{H}_1^*GWG^T\mathcal{H}_2^*\\
&+6\mathcal{H}_2^*GWG^T\mathcal{H}_1^*=\mathbf{0}^{n\times n}\\
&(A+BK^*)^T\mathcal{H}_4^*+\mathcal{H}_4^*(A+BK^*)+8\mathcal{H}_1^*GWG^T\mathcal{H}_3^*\\
&+12\mathcal{H}_2^*GWG^T\mathcal{H}_2^*+8\mathcal{H}_3^*GWG^T\mathcal{H}_1^*=\mathbf{0}^{n\times n}.
\end{aligned}\tag{19}$$

The components of the vector $\boldsymbol{\kappa}_\infty^*\in\mathbb{R}^4$ can then be computed by

$$\kappa_{\infty,i}^*=\mathrm{Tr}(\mathcal{H}_i^*GWG^T),\ 1\le i\le 4. \tag{20}$$

The results of the numerical experiment are captured in Figure 1 (a) and (b), which show the iterate value n on the x-axis and corresponding iterate value $\kappa_i[n]$ on the y-axis. The iterate cumulants $\kappa_i[n]$ and iterate targets iterate cumulants $\tilde{\kappa}_i[n]$ are both time-invariant, being computed according to a procedure analogous to the iterate solutions technique for solving algebraic Riccati equations outlined in [7]. It is clear that the cost cumulants align well with the targets, approximately achieving the target statistical characterization for the MCCDS control design. This evidences that the infinite-horizon MCCDS control uses the target information to successfully achieve the target cost cumulants.

IX. Summary and Conclusion

The recent development of cost density-shaping control paradigms has stirred-up some interest in the controls community, and lends reason to the continued development of the theory. These controllers enable the designer to translate the shape of a target density for the random cost functional into a linear control law, and have produced clear gains in control performance [3-6].

This paper focuses on the infinite-horizon extension of the MCCDS theory. A cancellation property and the same adaptation of Lagrange multiplier theory for the infinite-horizon MCCDS problem, as originally made by Pham, have been used together to establish optimality of a linear state-feedback control with an optimal constant control gain. The solution takes the same form as the finite-horizon MCCDS controller, and confirms that higher-order cost cumulant-generating variables in linear optimal controls are weighted by constants that depend on the rates of change for the MCCDS optimization's performance index with respect to its cost cumulant arguments.

The theoretical MCCDS controller derivation and the cumulant-tracking validation exercise presented in this work leave further areas of investigation for infinite-horizon MCCDS control that have yet be studied. A separate investigation of stability properties and performance for infinite-horizon MCCDS controls constitute excellent next steps in the continued development of MCCDS control theory.

References

[1] K. D. Pham, M. K. Sain, and S.R. Liberty, "Cost Cumulant Control: State-Feedback, Finite-Horizon Paradigm with Application to Seismic Protection", *Journal of Optimization Theory and Applications*, Vol. 115, No. 3, pp. 685-710, 2002.

[2] K. D. Pham, M. K. Sain, and S.R. Liberty, "Infinite Horizon Robustly Stable Seismic Protection of Cable Stayed Bridges Using Cost Cumulants", *Proceedings of the American Control Conference*, pp. 692-696, 2004.

[3] M. J. Zyskowski, M.K. Sain, and R.W. Diersing, "Weighted Least-Squares, Cost Density-Shaping Stochastic Optimal Control: A Step Towards Total Probabilistic Control Design", accepted for the *49th IEEE Conference on Decision and Control*, December 2010.

[4] M. J. Zyskowski, M.K. Sain, and R.W. Diersing, "Maximum Bhattacharyya Coefficient, Cost Density-Shaping: A New Cumulant-Based Control Paradigm with Applications to Seismic Protection". *5th World Conference on Structural Control and Monitoring*, 5WCSCM-10402, July 2010.

[5] M. J. Zyskowski, M.K. Sain, and R.W. Diersing, "Minimum Kullback-Leibler Divergence, Cost Density-Shaping Stochastic Optimal Control with Applications to Vibration Supression". *3rd ASME Dynamic Systems and Controls Conference*, September 2010.

[6] M. J. Zyskowski, M.K. Sain, and R.W. Diersing, "State-Feedback, Finite-Horizon, Cost Density-Shaping Control for the Linear Quadratic Gaussian Framework", to appear in the *Journal of Optimization Theory and Applications*.

[7] K. D. Pham,"Statistical Control Paradigms for Structural Vibration Suppression", *Ph.D. Dissertation*, Department of Electrical Engineering, University of Notre Dame, 2004.

[8] M. J. Zyskowski,"Cost Density-Shaping for Stochastic Optimal Control", *Ph.D. Dissertation*, Department of Electrical Engineering, University of Notre Dame, 2010.

[9] B. F. Spencer, S. Dyke, and H. Deoskar, "Benchmark Problems in Structural Control: Part I - Active Mass Driver System", *Earthquake Engineering and Structural Dynamics*. Vol. 27, 1127-1139, 1998.

[10] J. J. Lin, N. Saito, and R. A., Levine, "On approximation of the Kullback-Leibler information by Edgeworth expansion" Technical Report. Dept. of Statistics, University of California-Davis, 2001.