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Idempotent Method for Deception Games

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***Abstract*— In recent years, idempotent methods (specifically, max-plus methods) have been developed for solution of nonlinear control problems. We extend the applicability of idempotent methods to deterministic dynamic games through usage of the min-max distributive property. However, this induces a very high curse-of-complexity. A representation of the space of max-plus hypo-convex functions as a min-max linear space is used to obtain a result which may be used to attenuate this complexity growth. We apply this approach in a game of deception, where one player is searching for certain objects, while the other player may employ deception to hinder that search. The problem is formulated as a dynamic game, where the state space is a max-plus probability simplex.**

I. Introduction

In recent years, idempotent methods have been developed for solution of nonlinear control problems. (Note that idempotent algebras are those for which $a \oplus a = a$ for all a; this class includes the well-known max-plus algebra.) Most notably, max-plus methods have been applied to deterministic optimal control problems. These consist of max-plus basis methods, exploiting the max-plus linearity of the associated semigroup [1], [6], [14], and max-plus curse-of-dimensionality-free methods which exploit the max-plus additivity and the invariance of the set of quadratic forms under the semigroup operator [13], [14]. These methods achieved truly exceptional computational speeds on some classes of problems.

In this paper, we use some similar, but more abstract, tools which bring deterministic dynamic game problems into the realm under which curse-of-dimensionality-free idempotent methods will be applicable. We will first recall how one may apply the min-max distributive property to develop curse-of-dimensionality-free methods for discrete-time, deterministic dynamic games (as indicated in [10]). Following that are the two main topics in the paper: complexity attenuation and deception games.

The difficulty with idempotent curse-of-dimensionality-free methods for game problems is an extreme curse-of-complexity. In particular, the solution complexity grows exponentially as one propagates backward in time via the idempotent distributed dynamic programming principle (IDDPP). An approach for attenuating that complexity growth extends from developments in max-plus convex analysis. Using the IDDPP, one has a representation of the value function at each time-step as a pointwise minimum of max-plus affine functionals. In this setting, the natural ordering on the range space, $\overline{I\!R} \doteq I\!R \cup \{-\infty, +\infty\}$, is downward (the reverse of our normal ordering), and therefore a very natural space is that which we refer to as the max-plus hypo-convex functions – the space of functions such that the hypograph is max-plus convex. (Note that because of the reversal of ordering, we choose not to refer to the space as a space of max-plus concave functions.) This space is a min-max vector space (more exaclty, a min-max moduloid or semi-module). This implies that our value function representation is an element of the space of max-plus hypo-convex functions. Using the results in [9], one may show that optimal complexity attenuation is achieved by pruning of the existing expansion. We note here that one may think of this step as optimal projection onto a min-max subspace of specified dimension. This will allow us to determine a surprisingly simple means by which this may be achieved.

Once we have these tools in hand, we consider the deception game. We suppose Player 1 is employing one or more sensing entities (e.g., UAVs, UUVs, humans) to search for certain assets. Player 2 may employ deception to hinder that search. Specifically, at a certain cost, Player 2 may choose to alter Player 1's observation. The cost to hide an asset may be different from the cost to employ a decoy asset. We suppose that the search domain consists of a finite set of locations. The appropriate state-space is the space of max-plus probability vectors over the set of possible asset configurations. We will indicate the dynamic program for solution of this deception game problem, the associated IDDPP, and the corresponding computations.

II. Idempotent Method for Games

We briefly describe the idempotent approach. We will keep all control spaces finite so as to simplify the analysis. We suppose the dynamics are governed by

$$\xi_{t+1} = h(\xi_t, u_t, w_t), \quad \xi_s = x \in G \subseteq I\!R^I, \qquad (1)$$

where s is the initial time. We suppose $u_t \in \mathcal{U}$ and $w_t \in \mathcal{W}$ for all t, with $W \doteq \#\mathcal{W}$ (the cardinality of $\mathcal{W}$) and $U = \#\mathcal{U}$. We assume $h(\cdot, u, w)$ maps G into G for all $u \in \mathcal{U}$, $w \in \mathcal{W}$. Time is discrete with $t \in]s, T[\doteq$

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$\{s, s+1, s+2, \dots T\}$, where for integers $a \le b$, we use $]a, b[$ to denote $\{a, a+1, \dots b\}$ throughout. Also for simplicity, we assume only a terminal cost, which will be $\phi : G \to I\!R$. We let $\mathcal{U}$ be the minimizing player's control set, and $\mathcal{W}$ be the maximizing player's control set. The payoff, starting from any $(t, x) \in]s, T[\times G$ will be

$$J_t(x, u_{]t,T-1[}, w_{]t,T-1[}) = \phi(\xi_T), \tag{2}$$

where $u_{]t,T-1[}$ denotes a sequence of controls, $\{u_t, u_{t+1}, \dots u_{T-1}\}$, with similar meaning for $w_{]t,T-1[}$. We will work with the upper value. At any time $t \in]s, T-1[$, this is

$$V_t(x) = \max_{\tilde{w}^t \in \widetilde{W}^t} \min_{u_{]t,T-1[} \in \mathcal{U}^{T-t}} J_t(x, u_{]t,T-1[}, \tilde{w}(u_{]t,T-1[})) \tag{3}$$

where $\widetilde{W}^t = \{\tilde{w}^t : \mathcal{U}^{T-t} \to \mathcal{W}^{T-t}$, nonanticipative$\}$. The associated dynamic programming equation (which we present without proof) is

$$V_t(x) = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} V_{t+1}(h(x, u, w)). \tag{4}$$

Suppose ϕ takes the form

$$\phi(x) = \min_{z_T \in \mathcal{Z}_T} g_T(x; z_T),$$

where we let $Z_T = \#\mathcal{Z}_T < \infty$. Then,

$$V_T(x) = \min_{z_T \in \mathcal{Z}_T} g_T(x; z_T). \tag{5}$$

Combining (4) and (5), one has

$$V_{T-1}(x) = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \min_{z_T \in \mathcal{Z}_T} g_T(h(x, u, w); z_T). \tag{6}$$

We now introduce the relevant idempotent algebras. The max-plus algebra (i.e., semifield) is given by

$$a \oplus b \doteq \max\{a, b\}, \qquad a \otimes b \doteq a + b,$$

operating on $I\!R^- \doteq I\!R \cup \{-\infty\}$. In the min-max algebra (i.e., semiring), the operations are defined as

$$a \wedge b \doteq \min\{a, b\}, \qquad a \vee b \doteq \max\{a, b\},$$

operating on $\overline{I\!R} \doteq I\!R \cup \{-\infty\} \cup \{+\infty\}$, where we note that $+\infty \wedge b = b$ for all $b \in \overline{I\!R}$ and $+\infty \vee b = +\infty$ for all $b \in \overline{I\!R}$ (c.f., [7]). We suppose each $g_T(\cdot; z_T)$ is max-plus affine. In other words, ϕ will be formed as the lower envelope of a finite set of max-plus affine functions. In fact, we are going to think of ϕ as a max-plus convex function. (We will have reason to reverse the ordering on the range space, and so our definition of max-plus convex functions will look directly analogous to the definition of standard-sense convex functions.) We may write these max-plus affine $g_T(\cdot; z_T)$ as

$$\begin{aligned} g_T(x; z_T) &= \alpha^{T,z_T} \odot x \oplus \beta^{T,z_T} \\ &= \left[\bigoplus_{i \in \mathcal{I}} \alpha_i^{T,z_T} \otimes x_i \right] \oplus \beta^{T,z_T} \end{aligned}$$

where $\mathcal{I} =]1, I[$. We will assume that the $h(\cdot, u, w)$ are max-plus linear. Specifically, we let

$$h(x, u, w) = A(u, w) \otimes x,$$

where here we use $\otimes$ to emphasize that this is max-plus matrix-vector multiplication. We see that

$$V_{T-1}(x) = \bigwedge_{u \in \mathcal{U}} \bigvee_{w \in \mathcal{W}} \bigwedge_{z_T \in \mathcal{Z}_T} \Big[\beta^{T,z_T} \oplus \alpha^{T,z_T} \odot A(u, w) \otimes x \Big]. \tag{7}$$

Define, for any $t \in]s+1, T[$, $\widehat{\mathcal{Z}}_t = \{\hat{z}_t : \mathcal{W} \to \mathcal{Z}_t\}$. Applying the min-max distributive property to (7) (and noting that $\oplus \equiv \vee$),

$$\begin{aligned} V_{T-1}(x) = \bigwedge_{u \in \mathcal{U}} \bigwedge_{\hat{z}_T \in \widehat{\mathcal{Z}}_T} \bigoplus_{w \in \mathcal{W}} \Big[&\beta^{T,\hat{z}_T(w)} \\ &\oplus \alpha^{T,\hat{z}_T(w)} \odot A(u, w) \otimes x \Big]. \end{aligned} \tag{8}$$

Let

$$\widetilde{\alpha}_j^{T-1,\hat{z}_T}(u) \doteq \bigoplus_{w \in \mathcal{W}} \bigoplus_{i \in \mathcal{I}} \alpha_i^{T,\hat{z}_T(w)} \otimes A_{i,j}(u, w) \ \forall j \in \mathcal{I},$$

$$\widetilde{\beta}^{T-1,\hat{z}_T} \doteq \bigoplus_{w \in \mathcal{W}} \beta^{T,\hat{z}_T(w)}.$$

With these definitions, (8) becomes

$$V_{T-1}(x) = \bigwedge_{u \in \mathcal{U}} \bigwedge_{\hat{z}_T \in \widehat{\mathcal{Z}}_T} \Big[\widetilde{\beta}^{T-1,\hat{z}_T} \oplus \widetilde{\alpha}^{T-1,\hat{z}_T}(u) \odot x \Big].$$

Let $Z_{T-1} = U(Z_T)^W$, and let $\mathcal{Z}_{T-1} =]1, Z_{T-1}[$. Let Γ_{T-1} be a one-to-one, onto mapping from $\mathcal{U} \times \widehat{Z}_T$ to $\mathcal{Z}_{T-1}$, given by $z_{T-1} = \Gamma_{T-1}(u, \hat{z}_T)$ for each $(u, \hat{z}_T) \in \mathcal{U} \times \widehat{Z}_T$. Then,

$$V_{T-1}(x) = \bigwedge_{z_{T-1} \in Z_{T-1}} \Big[\beta^{T-1,z_{T-1}} \oplus \alpha^{T-1,z_{T-1}} \odot x \Big], \tag{9}$$

where

$$\alpha^{T-1,z_{T-1}} \doteq \widetilde{\alpha}^{T-1,\hat{z}_T}(u), \quad \beta^{T-1,z_{T-1}} \doteq \widetilde{\beta}^{T-1,\hat{z}_T},$$

and $z_{T-1} = \Gamma_{T-1}(u, , \hat{z}_T)$. Repeating this process, one easily finds the following.

Theorem 2.1: For any $t \in]s+1, T[$,

$$V_{t-1}(x) = \bigwedge_{z_{t-1} \in Z_{t-1}} \Big[\beta^{t-1,z_{t-1}} \oplus \alpha^{t-1,z_{t-1}} \odot x \Big],$$

where

$$\alpha_j^{t-1,z_{t-1}} \doteq \bigoplus_{w \in \mathcal{W}} \bigoplus_{i \in \mathcal{I}} \alpha_i^{t,\hat{z}_t(w)} \otimes A_{i,j}(u, w) \quad \forall j \in \mathcal{I},$$

$$\beta^{t-1,z_{t-1}} \doteq \bigoplus_{w \in \mathcal{W}} \beta^{t,\hat{z}_t(w)},$$

where $(u, \hat{z}_t) = \Gamma_{t-1}^{-1}(z_{t-1})$ for all $x \in I\!R^I$, $z_{t-1} \in \mathcal{Z}_{t-1}$, and Γ_{t-1} is a one-to-one, onto mapping from $\mathcal{U} \times \widehat{\mathcal{Z}}_t$ to $\mathcal{Z}_{t-1} \doteq]1, Z_{t-1}[$, with $Z_{t-1} = U(Z_t)^W$.

This is our IDDPP. The difficulty emerges through the iteration $Z_{t-1} = U(Z_t)^W$; in a naive application of this approach, the number of max-plus affine functions defining the value would grow extremely rapidly. This implies that the second piece of the algorithm must be complexity reduction in the representation at each step.

III. General Complexity Reduction Problem and Context

Certain function spaces may be spanned by infima of max-plus affine functions, that is, any element of the space may be represented as an infimum of a set of max-plus affine functions. By definition, any function in such a space as the above has an expansion, $f(x) = \inf_{\lambda\in\Lambda}\psi_\lambda(x)$, for some index set Λ, where the ψ_λ are max-plus affine. If the expansion is guaranteed to be countably infinite, we would write

$$f(x) = \inf_{i\in\mathbf{N}}\psi_i(x) = \bigwedge_{i\in\mathbf{N}}\psi_i(x) \doteq \bigwedge_{i\in\mathbf{N}}[a_i \oplus \psi_i'(x)],$$

where the ψ_i' are max-plus linear. We will refer to this as a min-max basis expansion, or simply a min-max expansion, and we think of the set of such ψ_i' as a min-max basis for the space.

Now we indicate the complexity reduction problem in a general form. Suppose we are given $f : \mathcal{X} \to I\!\!R$ with representation

$$f(x) = \bigwedge_{m\in\mathcal{M}} t_m(x) = \min_{m\in\mathcal{M}} t_m(x) = \min_{m\in]1,M[} t_m(x), \tag{10}$$

where $\mathcal{X}$ will be a partially ordered vector space. Except where noted, we will take $\mathcal{X} = I\!\!R^I$ for clarity. We are looking for $\{a_n : \mathcal{X} \to I\!\!R \,|\, n \in]1,N[\}$ with $N < M$, such that

$$g(x) \doteq \bigwedge_{n\in\mathcal{N}} a_n(x) = \min_{n\in\mathcal{N}} a_n(x) = \min_{n\in]1,N[} a_n(x) \tag{11}$$

approximates $f(x)$ from above. Note that throughout the paper, we will let $\mathcal{M} =]1,M[= \{1,2,\ldots M\}$, $\mathcal{N} =]1,N[$ and $\mathcal{I} =]1,I[$.

A. Min-max spaces

As indicated earlier, it is well-known that it is useful to apply max-plus basis expansions to solve certain HJB PDEs and their corresponding control problems. In particular, the solutions are represented as max-plus sums of affine or quadratic functions. In fact, the spaces of standard-sense convex and semiconvex functions have max-plus bases (more properly, max-plus spanning sets) consisting of linear and quadratic functions, respectively,

We will be applying the analogous concept, where the standard algebra will be replaced by the max-plus, and the max-plus will be replaced by the min-max. On $I\!\!R^I$, we will define the partial order $x \preceq y$ if $x_i \le y_i$ for all $i \in \mathcal{I}$. Let $\mathcal{O}^I$ denote the closed first octant, i.e., $\mathcal{O}^I = [0,\infty)^I \doteq \{x \in I\!\!R^I \,\big|\, x \ge 0\}$. For $\delta \in \mathcal{O}^I$, let $\|\delta\|^\oplus \doteq \max_{i\in\mathcal{I}}\delta_i = \bigoplus_{i\in\mathcal{I}}\delta_i$. Let $\mathbf{1}$ denote a generic-length vector all of whose elements are 1's. Let

$$\mathcal{S}^{\mathbf{1}}(I\!\!R^I) \doteq \left\{ f : I\!\!R^I \to \overline{I\!\!R} \,\middle|\, 0 \le f(x+\delta) - f(x) \right. \tag{12}$$
$$\left. \le \|\delta\|^\oplus, \forall x \in I\!\!R^I, \delta \in \mathcal{O}^I \right\},$$

For $a \in \overline{I\!\!R}$ and $f, g \in \mathcal{S}^{\mathbf{1}}(I\!\!R^I)$, we define the inherited operations $[f \wedge g](x) = \min\{f(x), g(x)\}$ and $[a \vee f](x) = \max\{a, f(x)\}$.

It is not difficult to show that $\mathcal{S}^{\mathbf{1}}(I\!\!R^I)$ is also exactly the space of sub-topical functions [16] from $I\!\!R^I$ to $\overline{I\!\!R}$. We will refer to a space as a min-max vector space if it satisfies the standard conditions (c.f. [14]).

Theorem 3.1: $\mathcal{S}^{\mathbf{1}}$ is a min-max vector space.

One of the most useful aspects of looking at the spaces of convex and semiconvex spaces as max-plus vector spaces was that these spaces had countable max-plus bases. For example the space of convex functions has the set of (standard-algebra) linear functionals with rational coefficients as a countable max-plus basis. We are interested in analogous results here.

We take $\psi(x,z) : I\!\!R^I \times I\!\!R^I \to I\!\!R$ to be

$$\psi(x,z) \doteq z \odot x \doteq \bigoplus_{i\in\mathcal{I}} z_i \otimes x_i = \max_{i\in\mathcal{I}}\{z_i + x_i\}. \tag{13}$$

We will be taking $\phi_k(x) = \psi(x, z^k)$ where the z^k will form a countable dense subset of $I\!\!R^I$. The result will follow if we have

$$f(x) = \inf_{z\in I\!\!R^I}\big\{\max[c(z), \psi(x,z)]\big\},$$

where c has sufficient continuity properties. Note that this would imply that f was the lower envelope of a set of functions. Further, note that

$$c(z) \vee \psi(x,z) = c(z) \oplus \psi(x,z)$$

where the $\psi(\cdot, z)$ are max-plus linear. In other words, f would be an infimum of max-plus affine functions.

B. Min-max basis representation and max-plus convexity

Given $\overline{x} \in I\!\!R^I$, let $z^{\overline{x}} \in I\!\!R^I$ and $c(z^{\overline{x}})$ be given by

$$z_i^{\overline{x}} = f(\overline{x}) - \overline{x}_i \quad \forall\, i \in \mathcal{I}, \text{ and } c(z^{\overline{x}}) \doteq f(\overline{x}).$$

Note that this may not define c on all of $I\!\!R^I$. However, the composite mapping $\overline{x} \mapsto c(z^{\overline{x}})$ is defined on all of $I\!\!R^I$. See [9] for the proofs of the following set of results.

Theorem 3.2: Let $\{x_k\}_{k\in\mathbf{N}}$ be a countable dense subset of $I\!\!R^I$. Let $\phi_k(x) \doteq \psi(x, \hat{z}^k)$, where $\hat{z}^k \doteq z^{x_k}$, for all $x \in I\!\!R^I$ and all $k \in \mathbf{N}$. For any $f \in \mathcal{S}^{\mathbf{1}}$,

$$f(x) = \bigwedge_{k\in\mathbf{N}} c_k \vee \phi_k(x) \quad \forall\, x \in I\!\!R^I, \tag{14}$$

where $c_k \doteq c(\hat{z}^k)$ for all $k \in \mathbf{N}$.

In [12], a problem similar to that described above was formulated, but in that case the max-plus algebra was replaced by the standard field, and the min-max algebra was replaced by the max-plus algebra. In solving that problem, we used a certain optimization criterion which was convex and increasing. Below, we will use a similar technique. Consequently, we will be dealing here with the analog of the convex functions – the max-plus hypo-convex functions; the optimization criterion will be max-plus hypo-convex. As the min-max algebra suggests a natural order on the range space, $\overline{I\!R}$, which is the opposite of the standard order, this will lead us to a definition of max-plus hypo-convex functions in which the set *below* the function is max-plus convex.

We begin with the definition of max-plus convex sets. A set, $\mathcal{C} \subseteq I\!R^I$ is *max-plus convex* if

$$\lambda_1 \otimes x^1 \oplus \lambda_2 \otimes x^2 \in \mathcal{C}$$

for all $x^1, x^2 \in \mathcal{C}$ and all $\lambda_1, \lambda_2 \in [-\infty, 0]$ such that $\lambda_1 \oplus \lambda_2 = 0$. See [3], [16]. We now turn to max-plus hypo-convex functions. We would like the set of such functions to form a min-max vector space. Consequently, we define the ordering on the range space, $\overline{I\!R}$, by $y_1 \preceq^R y_2$ if $y_1 \geq y_2$, and $y_1 \prec^R y_2$ if $y_1 > y_2$; relations $\succeq^R$ and $\succ^R$ are defined analogously. We henceforth refer to this as the *range order.* Suppose $f: I\!R^I \to \overline{I\!R}$, and define the max-plus epigraph as

$$\text{epi}^{\oplus} f \doteq \{(x,y) \in I\!R^I \times \overline{I\!R} \,|\, y \succeq^R f(x)\}. \tag{15}$$

Alternatively, f may be referred to as the hypograph [16], but due to the natural reversal of order in the range space here, the term max-plus epigraph is more appropriate in this context. Lastly, we say f is *max-plus hypo-convex* if $\text{epi}^{\oplus} f$ is max-plus convex. With some work, one obtains:

Theorem 3.3: Let $Z \subseteq I\!R^I$, $c: Z \to \overline{I\!R}$, and

$$f(x) = \int_Z^{\wedge} c(z) \oplus \psi(x,z)\, dz = \inf_{z \in Z} \{c(z) \oplus x \odot z\}, \tag{16}$$

for all $x \in I\!R^I$.. Then f is max-plus hypo-convex.

Corollary 3.4: Suppose

$$f(x) = \bigwedge_{k \in \mathcal{K}} c_k \oplus z^k \odot x \quad \forall\, x \in I\!R^I, \tag{17}$$

where $\mathcal{K} \subseteq \mathbf{N}$, and $c_k \in \overline{I\!R}$ and $z^k \in I\!R^I$ for all $k \in \mathcal{K}$. Then, f is max-plus hypo-convex.

Theorem 3.5: $f \in \mathcal{S}^{\mathbf{1}}$ if and only if f is max-plus hypo-convex.

C. Complexity reduction

Recall that our originating problem was complexity reduction in a min-max expansion; see (10),(11). The a_n and t_m will now be selected from a specified class of functions, the max-plus linear functions. We take $\mathcal{X} \doteq I\!R^I$ throughout.

We will use a measure of approximation quality which is monotonic and max-plus hypo-convex. Specifically, we wish to minimize

$$J(A) \doteq \int_G^{\oplus} \left\{ \left[\bigwedge_{n \in \mathcal{N}} \alpha^n \odot x \right] - \left[\bigwedge_{m \in \mathcal{M}} \tau^m \odot x \right] \right\} dx, \tag{18}$$

conditioned on

$$\alpha^n \cdot x \geq \bigwedge_{m \in \mathcal{M}} \tau^m \odot x \quad \forall\, x \in I\!R^I, \forall n \in \mathcal{N}, \tag{19}$$

where we let A denote the set of coefficients $\{\alpha^n\}_{n \in \mathcal{N}}$.

The following result is obtained by embedding problem (18)/(19) in a larger class of problems. A proof appears in [9], and we note that the proof in analogous (but in a min-max sense) to a proof in [12].

Theorem 3.6: There exists $A^* = \{\alpha^{*,n}\}_{n \in \mathcal{N}}$ minimizing (i.e., range-order maximizing) J subject to constraints (19). Further, there exist $\{m_n\}_{n \in \mathcal{N}} \subset \mathcal{M}$ such that $A^* = \{\tau^{m_n}\}_{n \in \mathcal{N}}$.

Remark 3.7: Note that the above result covers only the max-plus linear case. We may extend this to the affine case on $G' \subseteq \overline{I\!R}^{I'}$ with $I' \doteq I - 1$ by letting $G = G' \times \{0\}$. Then with $\alpha^n = ([\alpha']^n, \beta) \in \overline{I\!R}^I$, for any $x' \in G'$ there exists unique $x \in G$ given by $x = (x', 0)$ such that

$$[\alpha']^n \odot x' \oplus \beta = \alpha^n \odot (x', 0) = \alpha^n \odot x.$$

With this equivalence, one extends our result to affine functionals.

D. Application to the game problem

We will now see how this can be used for our game problem. Recall that using Theorem 2.1, if the value function for the game at any time, t, took the form

$$V_t(x) = \bigwedge_{z_t \in Z_t} \left[\beta^{t,z_t} \oplus \alpha^{t,z_t} \odot x \right], \tag{20}$$

then at time $t-1$, one had

$$V_{t-1}(x) = \bigwedge_{z_{t-1} \in Z_{t-1}} \left[\beta^{t-1,z_{t-1}} \oplus \alpha^{t-1,z_{t-1}} \odot x \right],$$

where the computation of the constants was given there. Although this avoided the curse-of-dimensionality, there was a very high "curse-of-complexity", where in particular, $Z_{t-1} = \#\mathcal{Z}_{t-1} = U(Z_t)^W$. Consequently, after each iteration of the algorithm, we approximate in order to reduce complexity. That is, given any V_t of the form (20), we seek a smaller set of max-plus affine functionals that yields the best approximation. Theorem 3.6 and Remark 3.7 tell us that this is optimally achieved by pruning of the $\mathcal{Z}_t$ set (as opposed to using a different set of max-plus affine functionals). *That is, in optimally reducing from $\mathcal{Z}_t$ to some smaller set of say, N, functionals, we do not need to search over all possible sets of size N of affine functionals, but only over subsets of $\mathcal{Z}_t$.* Further, (18) gives us a criterion by which we may measure the quality of any pruning option.

To generalize this to the affine case, we use Remark 3.7. For simplicity of notation, we replace $\mathcal{Z}_t =]1, Z_t[$ with $\mathcal{M} =]1, M[$, and pairs $(\alpha^{t,z_t}, \beta^{t,z_t})$ with (α^m, β^m). Also, we let $\tau^m \doteq (\alpha^m, \beta^m)$ for $m \in \mathcal{M}$. Let $\hat{G} \doteq G \times \{0\} \subseteq I\!R^I \times \{0\}$, and given $x \in I\!R^I$, let $\hat{x} = \hat{x}(x) \doteq (x, 0) \in \hat{G}$. Then,

$$\beta^{t,z_t} \oplus \alpha^{t,z_t} \odot x = \beta^m \oplus \alpha^m \odot x = \tau^m \odot \hat{x}.$$

Noting that, by Theorem 3.6, the optimal solution of (18)/(19) is a subset of $\mathcal{M}$, which we will denote by $\mathcal{M}'$, optimization criterion (18) may be replaced by

$$\hat{J}(\mathcal{M}') \doteq \int_{\hat{G}}^{\oplus} \left\{ \left[\bigwedge_{m' \in \mathcal{M}'} \tau^{m'} \odot \hat{x} \right] - \left[\bigwedge_{m \in \mathcal{M}} \tau^m \odot \hat{x} \right] \right\} dx. \qquad (21)$$

Using this, we find the solution surprisingly easy to compute. One can understand, heuristically, why this might be so, by noting that the maximum vertical amount that a single max-plus affine function contributes to the pointwise minimum of a set of max-plus affine functions always occurs at what we refer to as the "crux" of the max-plus affine function, where the crux is a point in $I\!R^{I+1}$ where all the hyperplanes comprising the graph of the function intersect. Due to paper-length considerations, we do not include the details.

IV. A Deception Game

We now consider a deception game. Player 1 will search for what we will refer to as the assets of Player 2 over a series of time-steps, $\mathcal{T}^- \doteq]0, T-1[\doteq \{0, 1, \ldots T-1\}$. Then, at time T, Player 1 takes an action, $a \in \mathcal{A} =]1, A[$. The true Player 2 asset configuration is $x \in \mathcal{X} =]1, X[$. (In the case of a single asset hidden among L possible locations, one would take $\mathcal{X} =]1, L[$.) Given true asset configuration x, Player 2 would receive a loss, $c(x, a)$. Here, we use the convention that Player 2 wishes to maximize (make less negative) the loss, $c(x, a)$. We assume a zero-sum game. Let $C(a)$ be the vector of length X with components $c(x, a)$. It is natural to use the max-plus probability structure (c.f., [2], [4], [5], [15] and the references therein) for deterministic games.

Suppose that Player 1's knowledge of the true asset configuration is described by max-plus probability distribution, $q \in S^{\oplus X}$, where

$$S^{\oplus X} \doteq \Big\{ q = \in [-\infty, 0]^X \,\Big|\, \bigoplus_{x \in \mathcal{X}} q_x = 0 \Big\},$$

where $[-\infty, 0]$ denotes $(-\infty, 0] \cup \{-\infty\}$ and the X superscript indicates outer product X times. (Recall that $[-\infty, 0]$ is analogous to $[0, 1]$ in the standard algebra.) We may interpret each component, q_x, as the (relative) cost to Player 2 to cause Player 1 to believe that the asset configuration is x. This will be become more clear below. The expected payoff for action $a \in \mathcal{A}$ given max-plus distribution q at terminal time T, is as follows. Letting max-plus random variable ξ be distributed according to q, and $\mathbf{E}_q^{\oplus}$ denote max-plus expectation according to this q, the expected payoff is

$$\hat{J}(q, a) = \mathbf{E}_q^{\oplus}[c(a, \xi)] = \bigoplus_{x \in \mathcal{X}} c(a, x) \otimes q_x = C(a) \odot q. \qquad (22)$$

Given that Player 1 wants to minimize (make more negative) the loss to Player 2, the value of information q at time T is

$$\phi(q) \doteq \min_{a \in \mathcal{A}} J(q, a) = \bigwedge_{a \in \mathcal{A}} [C(a) \odot q]. \qquad (23)$$

We see that if information is represented by a max-plus probability distribution over a finite set (and one has a finite set of controls), then *the value of information takes the form of a min-max sum of max-plus linear functionals over a max-plus probability simplex.*

We will view ϕ as the terminal payoff in the deception game. Now we describe the actual deception game. At each time, $t \in \mathcal{T}^-$, Player 1 may task sensing entities. The possible Player 1 sensing controls at each time step are denoted by $u \in \mathcal{U} =]1, U[$. Each sensing step results in an observation (or set of observations) denoted by $y \in \mathcal{Y}$. Again, in the max-plus probability structure, one may associate max-plus probabilities with costs. Let the max-plus probability of observing y given sensing control u and true asset state x be denoted by $p^{\oplus}(y|x; u) \in [-\infty, 0]$. These may be associated with Player 2's deception actions. We suppose that at each time step, Player 2 may use a combination of decoys, stealth and "no action". Here, each use of a decoy or stealth will have associated costs. (Note that the use of stealth may be associated with a cessation of activity which would otherwise be benefiting Player 2.) That is, we may interpret $p^{\oplus}(y|x; u)$ as the (non-positive) cost to Player 2 to cause Player 1 to observe y given true state x and sensing control u.

Suppose $q(t)$ is the max-plus probability distribution after observation at time t. Suppose that at time $t+1$, Player 1 employs control $u(t+1) = \hat{u} \in \mathcal{U}$ with resulting observation $y \in \mathcal{Y}$ (which we recall may be at least partially controlled by Player 2). The resulting cost for any true state $x \in \mathcal{X}$ would be

$$\hat{q}_x(t+1) = p^{\oplus}(y|x;\hat{u}) + q_x(t) = p^{\oplus}(y|x;\hat{u}) \otimes q_x(t).$$

In solving the optimization problem, we are concerned only with the relative costs, and so we may normalize so that the max-plus sum over $x \in \mathcal{X}$ is zero. Let $q(t+1)$ denote the normalized cost, where we want $\bigoplus_{x\in\mathcal{X}} q_x(t+1) = 0$. The normalized cost is

$$\begin{aligned} q_x(t+1) &= p^{\oplus}(y|x;\hat{u}) \otimes q_x(t) \\ &\quad - \Big\{ \bigoplus_{\zeta\in\mathcal{X}} \big[p^{\oplus}(y|\zeta;\hat{u}) \otimes q_\zeta(t) \big] \Big\} \qquad (24) \\ &= p^{\oplus}(y|x;\hat{u}) \otimes q_x(t) \oslash \Big\{ \bigoplus_{\zeta\in\mathcal{X}} \big[p^{\oplus}(y|\zeta;\hat{u}) \otimes q_\zeta(t) \big] \Big\}, \end{aligned}$$

where $\oslash$ indicates max-plus division (standard-sense subtraction). One sees that this is directly analogous to Bayes rule in standard-algebra probability. We may interpret each component of the resulting max-plus probability at time r, $q_x(r)$, as the maximal (least negative) relative cost to Player 2 for modification of the observation process to yield observed sequence $\{y(0), y(1), \ldots y(r)\}$ given true state x.

For simplicity, we assume that the sensing entities can move from any sensing control to any other in one time-step. Consequently, the state process for the game is simply $q(t)$. One may also easily include a second controller for Player 2 which allows the assets to change configuration, with an associated cost (analogous to a Markov chain transition matrix), but we do not include this here. The payoff will be the terminal value of information, ϕ, above plus the deception costs.

One may use the method of Section II to solve this problem. Although the dynamics of q do not quite fit the general form given there, we nonetheless obtain a similar IDDPP due to the max-plus expectation operation. Due to space limitations, we do not include the details. Further, one may use the result in Theorem 3.6 to generate complexity attenuation algorithms. That is, pruning at each step is optimal for complexity attenuation, and the calculations required for the pruning reduce to a quite small set of max-plus linear (in this case) function evaluations.

The algorithms have been coded and tested on some simple examples. A plot of a solution along a three-dimensional sub-manifold of the max-plus probability simplex appears in Figure 1. (Recall that the max-plus probability simplex is different in shape than a standard-algebra probability simplex; for obvious reasons, we truncate the figure, extending only to -10 in each component.) The value at each point is denoted by color.

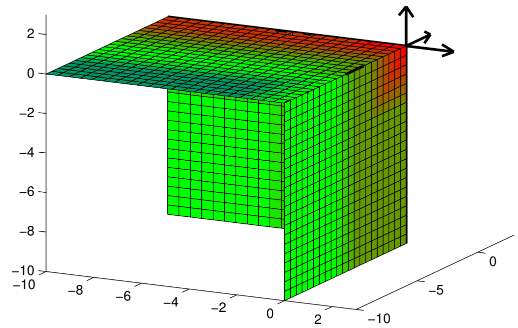

Fig. 1. Value on $S^{\oplus 3}$ sub-manifold.

References

[1] M. Akian, S. Gaubert and A. Lakhoua, *The max-plus finite element method for solving deterministic optimal control problems: basic properties and convergence analysis,* SIAM J. Control and Optim., 47 (2008), 817–848.

[2] M. Akian, *Densities of idempotent measures and large deviations,* Trans. Amer. Math. Soc., 109 (1999), 79–111.

[3] G. Cohen, S. Gaubert, J.-P. Quadrat and I. Singer, *Max-Plus Convex Sets and Functions,* preprint.

[4] P. Del Moral and M. Doisy, *Maslov idempotent probability calculus,* Theory Probab. Appl., 43 (1999), 562–576.

[5] W.H. Fleming, H. Kaise and S.-J. Sheu, *Max-Plus Stochastic Control and Risk-Sensitivity,* Applied Math. and Optim., 62 (2010), 81–144.

[6] W.H. Fleming and W.M. McEneaney, *A max-plus based algorithm for an HJB equation of nonlinear filtering,* SIAM J. Control and Optim., 38 (2000), 683–710.

[7] B. Heidergott, G.J. Olsder and J. van der Woude, *Max-Plus at Work: Modeling and Analysis of Synchronized Systems,* Princeton Univ. Press, 2006.

[8] V.N. Kolokoltsov and V.P. Maslov, *Idempotent Analysis and Its Applications,* Kluwer, 1997.

[9] W.M. McEneaney and S. Gaubert, *Min-Max Spaces and Complexity Reduction in Min-Max Expansions,* Applied Math. and Optim., (submitted).

[10] W.M. McEneaney, *Idempotent Method for Dynamic Games and Complexity Reduction in Min-Max Expansions,* Proc. IEEE CDC 2009.

[11] W.M. McEneaney, *Distributed Dynamic Programming for Discrete-Time Stochastic Control, and Idempotent Algorithms*, Automatica (to appear).

[12] W.M. McEneaney, *Complexity Reduction, Cornices and Pruning,* Proc. of the International Conference on Tropical and Idempotent Mathematics, G.L. Litvinov and S.N. Sergeev (Eds.), Contemporary Math. 495, (2009), 293–303.

[13] W.M. McEneaney, A. Deshpande, S. Gaubert, *Curse-of-Complexity Attenuation in the Curse-of-Dimensionality-Free Method for HJB PDEs,* Proc. ACC 2008, Seattle (2008).

[14] W.M. McEneaney, *Max-Plus Methods for Nonlinear Control and Estimation*, Birkhauser, Boston, 2006.

[15] A. Puhalskii, *Large Deviations and Idempotent Probability,* Chapman and Hall/CRC Press, 2001.

[16] A.M. Rubinov and I. Singer, *Topical and Sub-Topical Functions, Downward Sets and Abstract Convexity,* Optimization, 50 (2001), 307–351.