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A Stochastic Model Predictive Control Approach to Dynamic Option Hedging with Transaction Costs

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Abstract— **This paper proposes a stochastic model predictive control (SMPC) approach to hedging derivative contracts (such as plain vanilla and exotic options) in the presence of transaction costs. The methodology is based on the minimization of a stochastic measures of the hedging error predicted for the next trading date. Three different measures are proposed to determine the optimal composition of the replicating portfolio. The first measure is a combination of variance and expected value of the hedging error, leading to a quadratic program (QP) to solve at each trading date; the second measure is the conditional value at risk (CVaR), a common index used in finance quantifying the average loss over a subset of worst-case realizations, leading to a linear programming (LP) formulation; the third approach is of min-max type and attempts at minimizing the largest possible hedging error, also leading to a (smaller scale) linear program. The hedging performance obtained by the three different measures is tested and compared in simulation on a European call and a barrier option.**

I. Introduction

For a financial institution, hedging a derivative contract implies maintaining a self-financing portfolio of underlying assets, whose quantities need to be readjusted periodically so that at the expiration date of the contract the value of the portfolio is as close as possible to the payoff value to be paid to the customer.

The most common derivative contracts are plain vanilla options: a European *call (put) option* gives the holder the right to buy (sell) the underlying at a given *expiration date* and at a determined *strike price*. A large number of other more complex derivative contracts, called *exotic* options, are nowadays traded on the market. An example of an exotic option is the *barrier option*, a special kind of *plain vanilla* contract whose payoff is zeroed as soon as the price of the underlying asset reaches a certain barrier value.

Following the fundamental theoretical results of Black and Scholes [1], an approach to dynamically rebalancing the portfolio underlying an option that is commonly used in practice is the so called *delta hedging*, according to which the portfolio includes a quantity of stocks equal to the derivative of the option price with respect to the price of the underlying stock. Delta hedging makes the portfolio insensitive to the indeterministic evolution of the stock price, under a series of (often unrealistic) hypotheses including continuous hedging, static volatility, and the absence of transaction costs. When applied in a real market context, such assumptions may lead to intolerable hedging errors.

In [2] the hedging problem is tackled from a stochastic model predictive control (SMPC) point of-view for a plain vanilla option, for which a finite horizon constrained stochastic control problem is formulated and iteratively solved at each trading date as a semi-definite program for dynamic hedging. SMPC can be seen as a suboptimal way of solving a stochastic multi-stage dynamic programming problem: rather than solving the problem for the whole option-life horizon, a smaller problem is solved repeatedly from the current time-step t up to a certain number N of time steps in the future by suitably re-mapping the condition at the expiration date into a value at time $t+N$.

In [3] and then in [4] analytic methods based on stochastic optimization were proposed, the former to jointly determine the option price and the optimal trading strategy that reduce the total risk of writing the option, the latter formulating a scenario-based stochastic control problem where an objective function based on the expected value of a performance index is maximized and scenarios are generated according to a trinomial process. In [5] the hedging problem is formulated as a linear quadratic control problem with constraints and proposes two methods to cope with transaction costs. One involves penalizing transaction costs in the objective function, so that the problem can be solved as an unconstrained linear quadratic problem; the second method uses a receding horizon approach to solve a quadratic program over a specified horizon, exploiting the LQR solution from the first approach in the cost function.

In this paper we extend the SMPC approach to option hedging introduced in our previous works [6], [7] to handle proportional transaction costs. The approach is based on a minimum variance criterion, that we show here to be inadequate to handle transaction costs. Instead, we propose here three different approaches, the first one based on the scalarization of the multiobjective problem of minimizing both the variance and the expected value of the hedging error; the second on minimizing the Conditional Value at Risk (CVaR), a very common index used in quantitative finance for measuring the risk of great losses; the third one based on the minimization of the maximum hedging error over the set of scenarios considered in the stochastic optimization problem solved by the SMPC algorithm. The three approaches lead to, respectively, a quadratic programming (QP), a linear programming (LP), and a (smaller) LP problem to be solved at each trading date. The three SMPC formulations are tested and compared among them and to delta hedging on both

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plain vanilla and barrier exotic options.

The paper is structured as follows. In Section II we formulate the SMPC problem for option hedging based on enumeration of scenarios. In Section III we define transaction costs and describe how they affect the evolution of the portfolio. After formulating the SMPC problem, we focus on proportional transaction costs and propose the three different optimization objectives. Simulation tests on are reported in Section IV for a European call and a barrier option. Some concluding remarks are done in Section V.

II. SMPC FOR OPTION HEDGING

Consider the problem of hedging an option $\mathcal{O}$ defined over n underlying assets, whose spot prices at time τ are $s_i(\tau)$, $i = 1,\dots,n$. The simplest and most widely used model to describe the dynamics of stock prices is the log-normal model

$$ds_i(\tau) = (\mu d\tau + \sigma dz_i^x)s_i(\tau) \tag{1}$$

where $z_i^x(\tau)$ is a Wiener process, with zero mean and variance $d\tau$. More general models can be used to describe price dynamics, such as Heston's model [8]. In this paper, we focus on the log-normal model (1), whose discrete-time equivalent form is $s_i(t+1) = e^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}z_i^s(t)}s_i(t)$, where t denotes the trading instant, $t = 0, 1, \dots$. We denote by $s(t) = [s_1(t) \ \dots \ s_n(t)]' \in \mathbb{R}^n$ the overall vector of asset prices.

In general, the option price $p(t)$ of $\mathcal{O}$ at a generic instant t is considered as the expected value of the payoff $\mathcal{P}(m(T))$ at expiration date in the risk-neutral measure, given the market state $m(t)$ at time t ($m(t) = s(t)$ for plain vanilla options). Denoting by T the maturity of an option $\mathcal{O}$ in terms of number of sampling steps of duration Δ_T, the price of the hedged option at a generic intermediate date $t\Delta_T$ is $p(t) = (1 + r)^{t-N}\tilde{E}[\mathcal{P}(m(T))|m(t)]$. For European call options the payoff is

$$\mathcal{P}(m(T)) = p(T) = \max\{s(T) - K, 0\} \tag{2}$$

while for "barrier" options

$$\begin{aligned} p(T) &= \begin{cases} \max(s(T)-K,0) & \text{if } s(t) < s_u, \ \forall t \le T \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \max(s(T)-K,0) & \text{if } s_\ell(t) = 0 \\ 0 & \text{if } s_\ell(t) = 1 \end{cases} \end{aligned} \tag{3}$$

where s_u define the upper barrier level, and $s_\ell(t) \in \{0, 1\}$ is a logic state with dynamics $s_\ell(t+1) = s_\ell(t)$ OR $[s(t) \ge s_u]$, $s_\ell(0) = 0$ (in this case $m(t) = \{x(t), x_\ell(t)\}$).

Assume that there are no transaction costs, and that the standard *self-financing* condition holds, i.e., that the wealth $w(t)$ of the portfolio replicating option $\mathcal{O}$ is always totally reinvested. Then, the dynamics of the wealth $w(t)$ of the portfolio is

$$w(t+1) = (1+r)w(t) + \sum_{i=1}^{n} b_i(t)u_i(t) \tag{4}$$

where $u_i(t)$ is the quantity of asset i held at time t and $b_i(t) \triangleq s_i(t+1) - (1+r)s_i(t)$ is the excess return, i.e. how much the asset gains (or loses) with respect to the risk-free rate. The initial condition $w(0)$ is set equal to the price paid by the customer to purchase option $\mathcal{O}$, $w(0) = (1 + r)^{-N}\tilde{E}[p(T)|m(0)]$.

Dynamic hedging aims at making the final wealth $w(T)$ as close as possible to $p(T)$ for all possible market realizations. The hedging problem can be restated as a stochastic control problem, where the wealth $w(t) \in \mathbb{R}$ represents the state and output of the regulated process, the traded asset quantities $u(t) \in \mathbb{R}^n$ are the inputs, the option price $p(t)$ the target reference for $w(t)$. By defining the tracking error $e(t) \triangleq w(t) - p(t)$, the objective can be restated as the one of minimizing $e(t)$ for all possible asset price realizations. As shown in [6], [7], in the absence of transaction costs and under the lack of arbitrage, a way to achieve this is to minimize the variance of the hedging error

$$J(e(T)) = E\left[(e(T) - E[e(T)])^2\right] \tag{5}$$

by solving the one-step ahead minimum-variance problem

$$\min_{\{u(t)\}} \ \mathrm{Var}_{m_{t+1}}\left[w(t+1,m_{t+1}) - p(t+1,m_{t+1})\right] \tag{6a}$$
$$\text{s.t.} \ w(t+1,m_{t+1}) = (1+r)w(t) + \sum_{i=0}^{n} b_i(t,m_{t+1})u_i(t) \tag{6b}$$

at each trading date $t\Delta_T$ with respect to the portfolio composition $u(t)$. Note that expectations and variances are conditioned to the particular market realization m_t at time t; we omit here the conditional notation for simplicity. Furthermore, since now on we will use the notation $w(t+1)$ as a shortcut for the future wealth $w(t+1, m_{t+1})$. The formulation in (6) is equivalent to a stochastic model predictive control approach with prediction horizon $N = 1$, under the terminal condition of perfect hedging between prediction time $t+N$ and expiration date T. Problem (6) can be solved by enumerating a certain number M of scenarios, each one corresponding to a different realization of a certain sequence of prices, and optimize the resulting sample variance. Each scenario j has probability π_j of occurring, $j = 1,\dots,M$, $\pi_j > 0$, $\pi_j \le 1$, $\sum_{k=1}^{M}\pi_j = 1$. Scenarios can be generated through Monte Carlo simulation [6], where $\pi^j = \frac{1}{M}$, or by discretizing a given probability density function that describes the disturbance process $z_i(t)$ [7]. Note that by restricting the prediction horizon to $N = 1$, the number M of considered scenarios can be quite large without incurring into prohibitive computation efforts, as in multi-stage stochastic programming appraches that typically limit $M = 2$ or 3.

By optimizing the sample variance of $w(t+1)-p(t+1)$, in the absence of transaction costs problem (6) can be rewritten as the following *least squares* problem

$$\min_{u(t)} \quad \sum_{j=1}^{M}\pi^j\left(w^j(t+1) - p^j(t+1) - \left(\frac{1}{M} \cdot \sum_{i=1}^{M} w^i(t+1) - p^i(t+1)\right)\right)^2 \tag{7}$$

where $w^j(t+1) = (1+r)w(t) + \sum_{i=0}^n b_i^j(t)u_i(t)$ are the future wealths of the portfolio for each scenario $j = 1,\dots,M$, and π^j is the corresponding probability, $\pi^j \geq 0$, $\sum_{i=1}^M \pi^j = 1$.

An option pricing engine is needed to compute the future option prices $p^1(t+1)$, ..., $p^M(t+1)$ over the generated scenarios. This is the most time-consuming operation of the entire algorithm when simple analytical formulas for determining the option prices do not exist. In fact, numerical pricing engines must be used, based on either Monte Carlo simulation, or on other approximate methods such as the method described in [9]. See [6], [7] for a comparison of different pricing methods. In particular, [7] showed that SMPC is superior to delta hedging when dealing with exotic options and quite robust also to market modeling errors.

III. Transaction costs

One often suffers transaction costs when trading assets [10]. The investor pays a quantity $h_i(t)$ of wealth to change the number of assets in the portfolio from $u_i(t-1)$ at time $t-1$ to $u(t)$ at time t, for each asset i. Such wealth $h_i(t)$ is taken away from the overall wealth $w(t)$ of the portfolio, so that (4) becomes (cf. [11])

$$w(t+1) = (1+r)\left(w(t) - \sum_{i=1}^n h_i(t)\right) + \sum_{i=1}^n b_i(t)u_i(t) \quad (8)$$

Proposition 1: The variance of the hedging error $e(t) = w(t) - p(t)$ is not affected by transaction costs.

Proof: Let $\omega(t) = \sum_{i=1}^n h_i(t)$ be the total transaction cost paid at time t. As $\omega(t)$ is a deterministic function that only depends on $u(t)$ (it does not depend on $s(t)$), we get $E[w(t+1) - p(t+1)] = E[(1+r)w(t) + \sum_{i=1}^n b_i(t)u_i(t) - p(t+1) - (1+r)\omega(t)] = E[w_0(t+1) - p(t+1)] - (1+r)\omega(t)$, where $w_0(t+1)$ is the wealth at time $t+1$ in the absence of transaction costs. Therefore, while the expectation $E[e(t+1)]$ of the hedging error $e(t+1)$ is affected by $\omega(t)$, its variance $\mathrm{Var}[e(t+1)]$ is clearly not, as $\mathrm{Var}[e(t+1)] = E[(e(t+1) - E[e(t+1)])^2] = E[(w_0(t+1) - p(t+1) - (1+r)\omega(t) - E[w_0(t+1) - p(t+1)] + (1+r)\omega(t))^2] = \mathrm{Var}[w_0(t+1) - p(t+1)]$. ∎

Proposition 1 has shown that the minimum variance criterion (5) is therefore inadequate to handle transaction costs.

In the simplest case, transaction costs $h_i(t)$ are proportional to the traded quantity of stock $|u_i(t) - u_i(t-1)|$

$$h_i(u_i) = \epsilon_i |u_i(t) - u_i(t-1)| s_i(t) \quad (9)$$

where the fixed quantity ϵ_i depends on commissions on trading asset i, $i = 1,\dots,n$ (we assume no costs are applied on transacting the risk-free asset). Note that, from a system theoretical viewpoint, transaction costs introduce the additional state variable $u(t-1) \in \mathbb{R}^n$, whose dynamics is simply a unit delay.

Piecewise affine transaction costs as in (9) make (8) a *hybrid dynamics*, which can be expressed in piecewise affine form [12], or in mixed logical dynamical (MLD) form [13]. To this end, introduce auxiliary variables $\delta_i(t) \in \{0,1\}$

$$[\delta_i(t) = 1] \leftrightarrow [u_i(t) - u_i(t-1) \geq 0] \quad (10)$$

and $q_i(t) \in \mathbb{R}$

$$q_i(t) = \begin{cases} u_i(t) - u_i(t-1) & \text{if } \delta_i(t) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

By using the so-called "big-M" technique, (10) can be translated into the mixed-integer linear inequalities

$$\begin{aligned} u_i(t) - u_i(t-1) &\geq -M_i(1 - \delta_i(t)) && (12a)\\ u_i(t) - u_i(t-1) &\leq M_i\delta_i(t) - \epsilon && (12b) \end{aligned}$$

and (11) into

$$\begin{aligned} q_i(t) &\leq u_i(t) - u_i(t-1) + M_i(1-\delta_i(t)) && (13a)\\ q_i(t) &\geq u_i(t) - u_i(t-1) - M_i(1-\delta_i(t)) && (13b)\\ q_i(t) &\leq M_i\delta_i(t) && (13c)\\ q_i(t) &\geq -M_i\delta_i(t) && (13d) \end{aligned}$$

where M_i is an upperbound on $|u_i(t) - u_i(t-1)|$, which is the maximum allowed asset reallocation, and $\epsilon > 0$ is a small scalar (e.g., the machine precision). Eq. (8) can be therefore rewritten in the following MLD form [13]

$$\begin{aligned} w(t+1) &= (1+r)(u_0(t) + \\ &\quad - \sum_{i=1}^n (q_i(t) - 2(u_i(t) - u_i(t-1)))) \\ &\quad + \sum_{i=1}^n s_i(t+1)u_i(t) && (14)\\ u(t-1+1) &= u(t) && (15)\\ \text{s.t.} &\quad (12), (13) && (16) \end{aligned}$$

with states $w(t), u(t-1)$, input $u(t)$, auxiliary vector $\delta(t) = [\delta_1(t)\ \dots\ \delta_n(t)]' \in \{0,1\}^n$ of binary variables, and auxiliary vector $q(t) = [q_1(t)\ \dots\ q_n(t)]' \in \mathbb{R}^n$ of continuous variables. By using the stochastic hybrid dynamical model (16), the minimum variance problem (7) becomes a mixed-integer quadratic programming (MIQP) problem, for which very efficient solvers are available. When preparing the final version of this paper, a related approach just appeared in [14].

Note that for options involving a single stock the number n of assets is usually very small ($n = 1$ or $n = 2$), so that the minimum variance problem with transaction costs may be also solved by enumerating the possible 2^n instances of vector $\delta(t)$ and by solving the corresponding quadratic programs (QP) (7) subject to $u_i(t) \geq u_i(t-1)$ if the corresponding $\delta_i(t) = 1$, or $u_i(t) \leq u_i(t-1)$ if $\delta_i(t) = 0$, for all $i = 1,\dots,n$. In the next section, we propose a more efficient method that completely avoids introducing integer variables.

A. Minimization of variance and expectation (QP-Var)

Let $x(t), y(t) \in \mathbb{R}^n$ two vectors whose i-th components are nonnegative and defined as

$$\begin{aligned} x_i(t) - y_i(t) = u_i(t) - u_i(t-1) \quad (17)\\ x_i(t) \geq 0, y_i(t) \geq 0,\ \forall t = 0,\dots,T \end{aligned}$$

Accordingly, the cost $h_i(t)$ for trading a quantity $u_i(t) - u_i(t-1)$ of the i-th asset is $h^i(t) = \epsilon^i(u_i(t) - u_i(t -$

$1))s_i(t) = \gamma_i(t)(x_i(t) - y_i(t))$, where $\gamma_i(t) \triangleq \epsilon^i s^i(t)$, $i = 1, \dots, n$. We can therefore replace $u(t)$ with the new vector $v(t) = [x'(t)\ y'(t)] \in \mathbb{R}^{2n}$ of decision variables

$$u(t) = [I \ \ -I]v(t) + u(t-1) \tag{18}$$

By (8), we express the vector of future hedging errors as

$$\begin{aligned}\begin{bmatrix} w^1(t+1) - p^1(t+1) \\ \vdots \\ w^M(t+1) - p^M(t+1) \end{bmatrix} &= B(t)u + (1+r) \\ &\cdot \left(\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} \gamma'(t) \\ \vdots \\ \gamma'(t) \end{bmatrix} (x-y) \right) - \begin{bmatrix} p^1(t+1) \\ \vdots \\ p^M(t+1) \end{bmatrix} \\ &= B(t)u + \Gamma(t)(x-y) + D(t) \end{aligned} \tag{19}$$

where $\gamma(t) = [\gamma_1(t) \dots \gamma_n(t)]'$, and the definition of $B(t)$, $\Gamma(t)$, $D(t)$ is obvious from (19). By substituting $u(t)$ as in (18), we get

$$\begin{bmatrix} w^1(t+1) - p^1(t+1) \\ \vdots \\ w^M(t+1) - p^M(t+1) \end{bmatrix} = A_v(t)v(t) + B_v(t)$$

where $B_v(t) \triangleq B(t)u(t-1) + D(t)$ and $A_v(t) \triangleq \begin{bmatrix} B(t) + \Gamma(t) & -B(t) - \Gamma(t) \end{bmatrix}$. The expected value of the hedging error is therefore $E[e(t+1)] = \pi A_v(t)v(t) + \pi B_v(t)$, where $\pi = [\pi_1 \ \dots \pi_M]$, and its variance

$$\mathrm{Var}[e(t+1)] = v'(t)H(t)v(t) + C'(t)v(t) + D(t) \tag{20}$$

where $H(t)$, $C(t)$, $D(t)$ depend on π, $A_v(t)$, $B_v(t)$ and it is easy to verify that they do not depend on γ, in accordance with Proposition 1.

In order to minimize both the variance and the expected value of the one-step ahead hedging error, we optimize

$$\begin{aligned} \min \quad & \mathrm{Var}[e(t+1)] + \alpha(E[e(t+1)])^2 \\ \text{s.t.} \quad & v(t) \geq 0 \end{aligned} \tag{21}$$

where $\alpha \geq 0$ is a fixed scalar. Problem (21) is a QP problem with $2n$ variables subject to nonnegativity constraints.

B. Minimization of conditional value at risk (LP-CVaR)

A drawback of the QP formulation (21) is that it requires the calibration of the scalar α that best trades off between variance (=risk) and expectation (=lack of hedging accuracy due to transaction costs). *Conditional Value at Risk* (CVaR) can be used as an alternative performance measure to penalize the hedging error, and is defined as follows. Let $f(u,s) : \mathbb{R}^{n+k} \to \mathbb{R}$ be a loss function associated with the decision vector $u \in \mathbb{R}^n$ and with the random vector $s \in \mathbb{R}^k$. In our case $u = u(t)$, $s = m(t+1)$, $f(u,s) = |e(t+1)|$ (in case super-replication of the payoff is not penalized, $f(u,s) = -e(t+1)$). Let $p(s)$ be the probability density function of s. With respect to a given probability β, $0 \leq \beta \leq 1$, the β-VaR (Value at Risk) is defined as the lowest value ℓ, such that, with probability β, the loss will not exceed ℓ. The number β is a fixed value, typically $\beta = 90\%$, 95%, or 99%. The main drawback of VaR is that the amount of loss occurring with probability $(1-\beta)$ is not taken into account directly. To avoid this, β-CVaR was introduced, that is the conditional expectation of the loss function above ℓ, quantifying what the *average loss* is when one loses *more than* ℓ, with probability $1-\beta$ [15].

The probability of $f(u,s)$ not exceeding the threshold ℓ is

$$\psi(u,\ell) = \int_{f(u,s)\leq\ell} p(s)ds \tag{22}$$

The β-VaR and the β-CVaR are defined, respectively, as

$$\ell_\beta(u) = \min\{\ell \in \mathbb{R} : \ \psi(u,\ell) \geq \beta\} \tag{23}$$

and

$$\phi_\beta(u) = (1-\beta)^{-1} \int_{f(u,s)\geq\ell_\beta(u)} f(u,s)p(s)ds \tag{24}$$

In [15] the authors show that the β-CVaR of the loss associated with any u can be determined by the formula

$$\phi_\beta(u) = \min_{\ell\in\mathbb{R}} F_\beta(u,\ell) \tag{25}$$

where

$$F_\beta(u,\ell) = \ell + (1-\beta)^{-1} \int_{s\in\mathbb{R}^m} [f(u,s) - \ell]^+ p(s)ds \tag{26}$$

and $[\cdot]^+$ denote the positive part of its argument, $[f]^+ = \max\{f, 0\}$. The integral in (26) can be approximated by sampling the distribution of s, according to the density function $p(s)$. If the sampling generates a collection of M vectors s^1, $\dots$, s^M, each of which has probability π_j of occurring, $j = 1, \dots, M$, then the corresponding approximation $\tilde{F}_\beta(u,\ell)$ is

$$\tilde{F}_\beta(u,\ell) = \ell + \frac{1}{(1-\beta)} \sum_{j=1}^{M} \pi_j [f(u,s^j) - \ell]^+ \tag{27}$$

Finally, we use CVaR to formulate the SMPC problem for dynamic hedging:

$$\min_{u,\ell,\{v_j\}_{j=1}^M} \quad \ell + \frac{1}{1-\beta} \sum_{j=1}^{M} \pi_j v_j \tag{28a}$$

$$\text{s.t.} \quad v_j \geq w^j(t+1) - p^j(t+1) - \ell \tag{28b}$$

$$v_j \geq -w^j(t+1) + p^j(t+1) - \ell \tag{28c}$$

$$v_j \geq 0, \ j = 1, \dots, M \tag{28d}$$

where β is a fixed value, typically $\beta = 90\%$, 95%, or 99%. Problem (28) is an LP problem with $M + n + 1$ variables and $3M$ constraints. Note that by removing constraint (28b) onedoes not penalize super-replication of the payoff, as the loss function becomes $\max\{-e(t+1), 0\}$.

C. Minimization of worst-case error (LP-MinMax)

A simpler approach than CVaR is to penalize the worst-case loss over the set of M generated scenarios, that is the largest absolute value $|e(t+1)|$ of the hedging error. The resulting formulation is the linear program

$$\min_{u,\ell} \quad \ell \tag{29a}$$

$$\text{s.t.} \quad \ell \geq w^j(t+1) - p^j(t+1) \tag{29b}$$

$$\ell \geq -w^j(t+1) + p^j(t+1) \tag{29c}$$

$$\ell \geq 0 \tag{29d}$$

Note that the LP (29) is simpler than (28) as it only involves $n+1$ variables and $2M+1$ constraints.

IV. Simulation results

We test the SMPC formulations for dynamic hedging of Section II on a European plain vanilla call option and on a barrier option. All simulations were run on a MacBook Pro 2.66 GHz Intel Core 2 Duo processor and 4 Gb RAM running MATLAB R2009b. The QP solver QUADPROG of the Optimization Toolbox was used to solve QP problems, while the solver GLPK [16] was used to solve LP problems.

We have tested the proposed three SMPC algorithms under different scenario generation settings: $M = 100$ and $M = 1000$ with Monte Carlo simulation ($\pi_i = \frac{1}{M}$, $\forall i = 1, \ldots, M$), and $M = 5$ with π_i obtained by discretizing a Gaussian distribution of $s(t+1)$ as described in [7]. The prediction horizon is $N = 1$, and $\Delta_T = 1$ week is the time interval between two consecutive trading dates. The option expires after $T = 24$ intervals, and $r_a = 4\%$ is the annualized continuously compounded interest rate so that $r = e^{0.04 \frac{1}{54}} - 1 = 0.00074102$ is the return of the risk free investment over Δ_T.

We consider a single stock $s_1(t)$ with initial spot price $s_1(0) = 100$ €. For European call options (2), we consider the strike price $K = 100$ €, while for barrier options, we consider an UP-AND-OUT option with barrier $x_u = 120$ €, where the barrier level is checked only at trading instants. The number of traded assets is $n = 1$ when only the underlying stock is traded (when hedging the call option), or $n = 2$ when also a European call option with expiration at time $T\Delta T$ and strike price $s_1(t)(1+r)^{T-t}$ is also traded in the portfolio (when hedging the barrier option).

We consider the log-normal stock price model (1) with $\mu = r_a$, $dz_1^x \sim \mathcal{N}(0,1)$ and volatility $\sigma = 0.5$, and consider only the nominal case, that is we assume the real market generates prices according to the same model (see [7] for hedging in the presence of market modeling errors, in the absence of transaction costs).

A. European call option

We first test the SMPC algorithm on a European call option, only trading the underlying stock and the risk free asset ($n = 1$). The transaction cost to trade the underlying stock is $\epsilon_1 = 2.5\%$. The strategy is tested over $N_s = 100$ simulations.

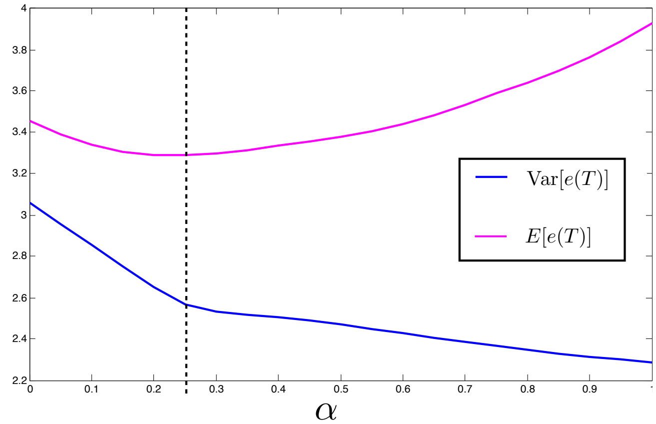

Fig. 1. Final variance $\text{Var}[e(T)]$ and expectation $E[e(T)]$ of the hedging error, for calibration of parameter α in (21)

1) QP-Var formulation: Consider the method based on QP described in Section III-A, where problem (21) is solved instead of (7). We first need to calibrate the relative weight α between variance and expectation of the hedging error. To this end, we run a set of N_s simulations with three different values of M (predicted scenarios): $M = 100$ ($\pi_j = \frac{1}{100}$), $M = 1000$ ($\pi_j = \frac{1}{1000}$), and $M = 5$ (π_i is obtained by sampling the Gaussian function). Different values of α are tested to analyze the variance and expected absolute value of the final hedging error $e(T)$. It is apparent from Figure 1 that the best trade-off between variance and expected absolute hedging error is obtained for $\alpha \simeq 0.25$, which is the value selected to carry out the simulations. The results for this method are shown in the first row of Table I. We can see that by increasing the number of scenarios in the prediction step, we obtain a reduction of the maximum hedging error. The discretization leads to some minor savings of CPU time, but the remaining performance get worse.

2) LP-CVaR and LP-MinMax formulations: The last two rows of Table I highlight the performance of the two proposed LP formulations, where either the LP (28) with $\beta = 0.99$ or the LP (29) is solved instead of (7).

One may first notice that the two LP formulations lead to the same results when $M = 100$ scenarios are generated by Monte Carlo simulation, or when $M = 5$ scenarios are selected by sampling the Gaussian distribution. The explanation for this is that calculating the CVaR as described in Section III-B) on a class of 100 samples with $\beta = 0.99$ exactly corresponds to considering the worst case within the entire group of scenarios, as in the minmax approach. When the number of scenarios is increased to $M = 1000$, improvements of LP-CVaR compared to LP-MinMax become more evident (in terms of both expected absolute value of the hedging error and variance), although CPU time of LP-CVaR is larger (as expected). In the last row of Table I the results obtained with delta hedging on the same option are shown. We can see that for plain vanilla options this last method outperforms the SMPC approach.

Model	**Monte Carlo** $M = 100$					**Monte Carlo** $M = 1000$					**discretized Gaussian** $M = 5$				
	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)
QP-Var	-2.30	2.73	-14.91	12.13	0.0247	-1.81	2.56	-12.30	10.81	0.0453	-1.64	2.69	-12.87	11.42	0.022
LP-CVaR	-1.34	2.58	-7.55	8.18	0.0067	-1.14	2.38	-5.99	7.13	0.6671	-1.31	2.65	-7.00	8.68	0.001
LP-MinMax	-2.67	3.83	-12.13	16.69	0.0067	-1.02	2.42	-6.49	7.93	0.31	-1.31	2.65	-7.00	8.68	0.001
Delta Hedging	-0.1312	1.77	-5.4	4.84	0.00012	-0.1312	1.77	-5.4	4.84	0.00012	-0.1312	1.77	-5.4	4.84	0.00012

TABLE I

SMPC RESULTS FOR THE EUROPEAN CALL OPTION

Model	**LS** $M = 100$					**LS** $M = 5$					**LS** $M = 1000$				
	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)	$E[e(T)]$	$E[\|e(T)\|]$	$\min(e(T))$	Var$[e(T)]$	CPU(s)
QP-Var	-2.19	6.85	-42.76	104.78	0.09	-1.06	3.82	-11.30	25.44	0.08	-1.65	10.82	-40.14	228.19	0.2001
LP-CVaR	-0.72	1.29	-12.16	7.14	0.38	-0.70	1.37	-13.63	8.55	0.08	-0.65	1.27	-11.73	6.85	0.1203
LP-MinMax	-0.72	1.29	-12.16	7.14	0.38	-0.70	1.37	-13.63	8.55	0.08	-0.72	1.33	-12.44	7.40	0.1223
Delta Hedging	-0.70	1.79	-16.14	13.61	0.0041	-0.70	1.79	-16.14	13.61	0.0041	-0.70	1.79	-16.14	13.61	0.0041

TABLE II

SMPC RESULTS FOR THE BARRIER OPTION

B. Barrier option

Since the value of a barrier option is much lower than the corresponding call option, we have decreased the transaction costs at 1.5% of the underlying price to test the SMPC algorithms. Pricing of future option values is made by using Longstaff-Schwartz's (LS) approximation [9], as well as by using Monte Carlo simulation. A number $M = 100$ of future scenarios is considered in both cases, and compared to the case of $M = 5$ scenarios obtained by sampling the Gaussian distribution. In the first case the approximation method of Longstaff and Schwartz is used to price future option values, in the latter Monte Carlo simulation is used.

We have run $N_s = 50$ simulations for each setting. Since we have carried simulation only with 100 or 5 scenarios, the LP-MinMax and LP-CVaR approaches provide the same results. In the case of discretized Monte Carlo, each one of the three models exhibits an identical performance, although CPU time is different. Longstaff-Schwartz's method with 100 scenarios leads to a lower hedging error with respect to the approaches based on $M = 5$ scenarios weighted by the corresponding discretized probabilities, as reported in Table II. We can see that for exotic options delta hedging (whose performances are reported in the last row of Table II) provides worse results than the proposed SMPC algorithm. In conclusion, Longstaff-Schwartz's option pricing method with $M = 100$ scenarios is the best approach in terms of expected absolute hedging error and variance, and in particular the LP-MinMax approach, showing a comparable performances but a lower computational effort. The largest hedging errors happen when the stock price gets close to the barrier without overpassing it, as hedging becomes particularly difficult because of the discontinuity of the payoff function.

V. CONCLUSIONS

SMPC is a suitable trading strategy for replicating financial options in the presence of transaction costs. Each one of the three proposed approaches (QP-Var, LP-CVaR, and LP-MinMax), SMPC shows good hedging performance, but only outperforms the traditional delta hedging technique when applied on exotic options. When CPU time is a concern, LP-MinMax is probably the best candidate formulation for SMPC, as it provides acceptable performance while involving only a small number of variables, and without requiring the calibration of the tradeoff parameter α as in QP-Var.

REFERENCES

[1] F. Black and M. Scholes, "Pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[2] J. Primbs, "Dynamic hedging of basket options under proportional transaction costs using receding horizon control," *Int. J. Control*, vol. 82, no. 10, pp. 1841–1855, 2009.

[3] S. Fedotov and S. Mikhailov, "Option pricing for incomplete markets via stochastic optimization: transaction costs, adaptive control and forecast," *International Journal of Theoretical and Applied Finance*, vol. 4, no. 1, pp. 179–195, 1999.

[4] J. Gondzio, R. Kouwenbergb, and T. Vorst, "Hedging options under transaction costs and stochastic volatility," *Journal of Economic Dynamics & Control*, vol. 27, pp. 1045–1068, 2003.

[5] J. Primbs, "LQR and receding horizon approaches to multi-dimensional option hedging under transaction costs," in *Proc. American Contr. Conf.*, 2010, pp. 6891–6896.

[6] A. Bemporad, L. Bellucci, and T. Gabbriellini, "Dynamic option hedging via stochastic model predictive control based on scenario simulation," *Quantitative Finance*, 2009, conditionally accepted for publication.

[7] A. Bemporad, T. Gabbriellini, L. Puglia, and L. Bellucci, "Scenario-based stochastic model predictive control for dynamic option hedging," in *Proc. 49th IEEE Conf. on Decision and Control*, Atlanta, GA, USA, 2010.

[8] S. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies*, vol. 6, pp. 327–343, 1993.

[9] F. Longstaff and E. Schwartz, "Valuing american options by simulation: A simple least-squares approach," *Review of Financial Studies*, vol. 114, no. 1, pp. 113–147, 2001.

[10] C. Edirisinghe, V. Naik, and R. Uppal, "Optimal replication of options with transactions costs and trading restrictions," *Journal of Financial and Quantitative Analysis*, vol. 28, no. 1, pp. 117–138, 1993.

[11] J. Primbs and Y. Yamada, "A new computational tool for analyzing dynamic hedging under transaction costs," *Quantitative Finance*, vol. 8, no. 4, pp. 405–413, 2008.

[12] A. Bemporad, "Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form," *IEEE Trans. Automatic Control*, vol. 49, no. 5, pp. 832–838, 2004.

[13] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[14] J. Glen, "Mean-variance portfolio rebalancing with transaction costs and funding changes," *Journal of the Operational Research Society*, vol. 62, pp. 667–676, 2011.

[15] R. Rockafellar and S. Uryasev, "Optimization of conditional value at risk," *Journal of Risk*, vol. 2, pp. 21–42, 2000.

[16] A. Makhorin, *GLPK (GNU Linear Programming Kit) User's Guide*, 2004. [Online]. Available: http://www.gnu.org/software/glpk/glpk.html