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On the $PD^{\alpha}-$Type Iterative Learning Control for the Fractional-Order Nonlinear Systems

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Abstract— **In this paper, we discuss in time domain the convergence of the iterative process for fractional-order nonlinear systems. The $PD^{\alpha}-$type iterative learning updating laws are considered. Most of the classical fractional-order cases for linear or nonlinear systems fall into the scheme of this paper. A number of numerical simulations are illustrated to validate the concepts.**

Index Terms— **Iterative learning control, Fractional calculus, Nonlinear system.**

I. Introduction

Iterative learning control (ILC), which belongs to the intelligent control methodology, is an approach for improving the transient performance of systems that operate repetitively over a fixed time interval. In details, apply a fixed-length input signal to a system. After the complete input has been applied, the system is returned to the same initial state and the output trajectory that resulted from the applied input is compared with the desired reference. The error is used to construct a new input signal of the same length that is applied to the next iteration. The aim of the ILC algorithm is to properly refine the input sequence from one trial to the next trial so that as more and more trials are executed the output will approach the desired trajectory [1], [2]. The advantages of the ILC algorithm are shown in its applications to the nonlinear systems and the systems with uncertainty or unknown structure information, etc [1], [2], [3], [4], [5], [6]. For the theoretical works on the integer-order ILC schemes, a number of papers by Professor E. Rogers are cited [7], [8], [9], [10], [11]. Some other interesting conclusions and surveys can be found in [12], [13], [14], [15], [16], [17]. Moreover, in the past three years, the applications of the ILC technique to medical treatments and engineering are getting more and more popular [5], [6], [10], [11]. Particularly, some early papers regarding the applications of ILC to physiotherapy include but not limit to [18], [19].

The definition of fractional calculus was proposed more than 300 years ago. However, the applications of fractional calculus was started at $1980's$, in which the representative work is its applications to viscoelastic materials [20], [21]. In the past 30 years, the fractional calculus had been applied to various domains, such as material, physics, mechanics, biology, system and control, etc [22]. Particularly, the application of fractional calculus in dynamic systems is a meaningful and up to date work in modern science [22]. Moreover, the heredity is the typical property of fractional order operators. And it describes the intermediate processes in physics and mechanics [23]. Meanwhile, the fact that computation becomes faster and memory becomes cheaper makes the application of fractional calculus, in reality, possible and affordable [24]. For example, in [25], the authors studied the fractional calculus applications in control systems. In [26], the author introduced the concept of fractional PID controller. A fundamental idea of the fractional control strategy was presented in [27].

The combination of ILC and fractional calculus was first proposed in 2001. In the following ten years, many fractional-order ILC problems were presented aiming at enhancing the performance of ILC scheme for linear or nonlinear systems [28], [29], [30], [31]. The authors in [32] were the first to propose the $D^{\alpha}-$type iterative learning control algorithm and the convergence was proved in frequency domain. The $PD^{\alpha}-$type iterative learning control to LTI systems was investigated in [30]. The time domain analysis of fractional-order ILC is shown in [29], [31]. In recent years, the application of ILC to the fractional-order system becomes a popular topic. The development of new fractional-order ILC algorithms, which belongs to a branch of fractional-order control [22], [33], [34], [35], [36], is urgently needed.

In our earlier works [29], it was shown that the optimal iterative learning controller for the $\alpha_{\rm th}-$order linear system is also a $\alpha_{\rm th}-$order one as well. Many numerical simulations are provided to validate this conclusion. Therefore, motivated by the search for new iterative learning control algorithms and applying iterative process to the tracking problem of fractional-order nonlinear systems, in this paper, the fractional-order ILC scheme is shown as

$$u_{k+1}(t) = u_k(t) + K_p(t)e_k(t) + K_d(t)e_k^{(\alpha)}(t),$$

and the fractional-order nonlinear system is

$$y^{(\alpha)}(t) = f(t, y(t), u(t)),$$

where all the variables and coefficients are defined in the main text. The convergence condition is derived in time-domain and most of the previous conclusions are special cases of this one.

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The rest of this paper is organized as follows. Some preliminaries are introduced in Section II. The $PD^{\alpha}-$type iterative learning control and its applications to fractional-order nonlinear systems are discussed in Section III, which is the main theoretical part of this paper. In Section IV, a number of numerical simulations are provided to validate the theories. Conclusions and future works are shown in Section V.

II. PRELIMINARIES

In this section, some basic definitions and properties are introduced, which will be used in the following part of this paper.

A. Laplace Transform

The Laplace transform of $f(t)$ is defined as

$$f(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \mathrm{d}t,$$

where $f(t)$ is piecewise continuous on every finite interval in $[0,\infty)$ satisfying $|f(t)| \leq Me^{at}$ for all $t \in [0,\infty)$, $s > a \geq 0$ and sufficient large constant $M > 0$.

B. Convolution

The convolution to be used in this paper is defined as

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau = \int_0^t f(\tau) g(t-\tau) \mathrm{d}\tau,$$

where $f(t)$ and $g(t)$ are integrable functions on $[0,t]$.

C. Fractional Calculus

Fractional calculus plays an important role in modern science [35], [36], [22]. In this paper, we use both Riemann-Liouville and Caputo fractional operators as our main tools. The unified formula of a fractional-order integral (Riemann-Liouville fractional-order integral) with order $\alpha \in (0,1)$ is defined as

$$ {}_{t_0}^{RL}\mathscr{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \tag{1}$$

where $f(t)$ is an arbitrary integrable function, ${}_{t_0}\mathscr{D}_t^{-\alpha}$ is the fractional integral of order α on $[t_0,t]$, and $\Gamma(\cdot)$ denotes the Gamma function. Especially, when $t_0 = 0$,

$$ {}_{0}^{RL}\mathscr{D}_t^{-\alpha} f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau. \tag{2}$$

For an arbitrary real number p, the Riemann-Liouville and Caputo fractional derivatives are defined respectively as

$$ {}_{t_0}^{RL}\mathscr{D}_t^{p} f(t) = \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} [\ {}_{t_0}\mathscr{D}_t^{-([p]-p+1)} f(t)] \tag{3}$$

and

$$ {}_{t_0}^{C}\mathscr{D}_t^{p} f(t) = \ {}_{t_0}\mathscr{D}_t^{-([p]-p+1)} [\frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} f(t)], \tag{4}$$

where $[p]$ stands for the integer part of p, ${}^{RL}\mathscr{D}$ and ${}^{C}\mathscr{D}$ are Riemann-Liouville and Caputo fractional derivatives, respectively. It can be proved that, if $f(0) = 0$,

$$ {}_{0}^{RL}\mathscr{D}_t^{-\alpha}\ {}_{0}^{C}\mathscr{D}_t^{\alpha} f(t) = f(t). \tag{5}$$

D. The $\lambda-$norm, maximum norm and induced norm

It is necessary to introduce the $\lambda-$norm in the ILC problems. For a r-vector-valued function $e(t)$ defined on $[0,T]$, the $\lambda-$norm is defined as:

$$\|e(t)\|_{\lambda} = \sup_{0 \leq t \leq T} \{e^{-\lambda t} \|e(t)\|_{\infty}\},$$

where $\|e(t)\|_{\infty} = \max_{1 \leq i \leq r} \{|e_i(t)|\}$ denotes the maximum norm of $e(t)$. Moreover, the induced norm of a matrix A is defined as:

$$\|A\| = \sup \left\{ \frac{\|Av\|}{\|v\|} : v \in V \text{ with } \|v\| \neq 0 \right\},$$

where $\|\cdot\|$ denotes an arbitrary vector norm. Especially when $\|\cdot\| = \|\cdot\|_{\infty}$,

$$\|Av\|_{\infty} \leq \|A\|_{\infty} \|v\|_{\infty}, \tag{6}$$

where $\|A\|_{\infty}$ denotes the maximum value of matrix A. Some other useful results can be found in [37].

III. THE $PD^{\alpha}-$TYPE ILC SCHEME FOR FRACTIONAL-ORDER NONLINEAR SYSTEMS

In this section, the $PD^{\alpha}-$type ILC scheme is applied to the fractional-order nonlinear systems. A sufficient condition is derived to guarantee the convergence of the discussed algorithm. Some classical ILC cases fall into the scheme of this fractional-order one.

A. The fractional-order nonlinear system and the $PD^{\alpha}-$type ILC scheme

The fractional-order nonlinear system can be written as

$$y^{(\alpha)}(t) = f(t,y,u), \tag{7}$$

where $\alpha \in (0,1)$, $y(0) \in \mathbb{R}^{n\times 1}, u \in \mathbb{R}^{m\times 1}$, $\cdot^{(\alpha)}$ denotes the α_{th}-order Caputo derivative with respect to t, and the continuous differentiable function f satisfies

$$\left\| \frac{\partial f}{\partial y} \right\|_{\infty} \leq c \|y\|_{\infty}, \tag{8}$$

where $c > 0$ and $\|\text{matrix}\|_{\infty}$ and $\|\text{vector}\|_{\infty}$ denote respectively the maximum norm of a matrix and a vector.

Remark 3.1: It follows from (8) and the uniqueness and existence theorem of the fractional-order differential equations [35] that, for the fixed $y(0)$ and $u(t)$, there exists an unique solution of system (7).

Moreover, it has been proved and verified in fractional-order linear system case that the convergent speed is the fastest when the system and iterative learning scheme have the same order [29]. Besides, motivated by the previous references on fractional-order ILC schemes [32], [30], [29], [31], the fractional-order $PD^{\alpha}-$type ILC scheme to be used in this paper is presented below.

Let the reference be $y_d(t)$, where $y_d(0) = y(0)$, and the fractional-order ILC updating law be

$$u_{k+1}(t) = u_k(t) + K_p(t) e_k(t) + K_d(t) e_k^{(\alpha)}(t), \tag{9}$$

where $\alpha \in (0,1)$, $K_p(t)$ and $K_d(t)$ are gain functions, $k = 0,1,2,\cdots$, $t \in [0,T]$, $y_k(0) = y_d(0) = y(0)$,

$$\left\{ \begin{array}{l} y_k^{(\alpha)}(t) = f(t, y_k, u_k), \\ y_d^{(\alpha)}(t) = f(t, y_d, u_d), \\ e_k(t) = y_d(t) - y_k(t), \end{array} \right. \tag{10}$$

and $u_d(t)$ and $y_d(t)$ denote the desired control effort and system output, respectively.

B. The convergence condition

In this subsection, we derive the convergence condition of the PD^{α}-type ILC scheme for the fractional-order nonlinear systems which is the main theoretical part of this paper.

Based on the fractional-order nonlinear system (7) and the fractional-order ILC updating law (9), the following lemmas are introduced.

Lemma 3.1: For the fractional-order nonlinear system (7), it follows from (10) that

$$f_d - f_k = \left\{ \frac{\partial f_i}{\partial u_{kj}} \right\}_{\xi_{ij}(t)} \delta u_k(t) + \left\{ \frac{\partial f_i}{\partial y_l} \right\}_{\eta_{il}(t)} e_k(t),$$

where f is a continuous differentiable function, $f_d = f(t, y_d, u_d)$, $f_k = f(t, y_k, u_k)$, $\delta u_k = u_d - u_k$, and $\xi_{ij}(t)$ and $\eta_{il}(t)$, where $i, l \in \{1,2,\cdots,n\}$ and $j \in \{1,2,\cdots,m\}$, are defined in the following proof.

Proof: Using (10) yields

$$\begin{aligned} & f_d - f_k = f(t, y_d, u_d) - f(t, y_k, u_k) \\ =& f(t, y_d, u_d) - f(t, y_d, u_k) + f(t, y_d, u_k) - f(t, y_k, u_k) \\ =& \left\{ \frac{\partial f_i}{\partial u_{kj}} \right\}_{\xi_{ij}(t)} \delta u_k(t) + \left\{ \frac{\partial f_i}{\partial y_l} \right\}_{\eta_{il}(t)} e_k(t) \\ =& \ {}_kA(t) e_k(t) + {}_kB(t) \delta u_k(t), \end{aligned}$$

where there exist functions $\xi_{ij}(t)$ and $\eta_{il}(t)$ satisfying

$$f_i(t, y_d, u_d) - f_i(t, y_d, u_k) = \sum_{j=1}^{m} \left. \frac{\partial f_i}{\partial u_{kj}} \right|_{u_{kj} = \xi_{ij}(t)} \delta u_j(t),$$

$$f_i(t, y_d, u_k) - f_i(t, y_k, u_k) = \sum_{l=1}^{n} \left. \frac{\partial f_i}{\partial y_l} \right|_{y_l = \eta_{il}(t)} \delta y_l(t).$$

■

Lemma 3.2: For the fractional-order nonlinear systems (10), suppose $\left\| \frac{\partial f}{\partial u} \right\|_{\infty} \leq \gamma \|u\|_{\infty}$ then there exists a sufficient large λ satisfying

$$\|e_k\|_{\lambda} \leq O(\lambda^{-1}) \|\delta u_k\|_{\lambda}.$$

Proof: Applying ${}_0D_t^{-\alpha}$ to both sides of equation (10), it follows from $y_k(0) = y_d(0)$, equations (2) and (5) and Lemma 3.1 that

$$\begin{aligned} \|e_k\|_{\lambda} &= \sup_{0 \leq t \leq T} \left\{ e^{-\lambda t} \left\| \frac{t^{\alpha-1}}{\Gamma(\alpha)} * [f_d - f_k] \right\|_{\infty} \right\} \\ \leq & \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|f_d - f_k\|_{\infty} \mathrm{d}\tau \\ \leq & \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} [c\|e_k(\tau)\|_{\infty} + \gamma \|\delta u_k(\tau)\|_{\infty}] \mathrm{d}\tau \\ \leq & \sup_{0 \leq t \leq T} \int_0^t \frac{e^{-\lambda(t-\tau)}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \sup_{0 \leq \tau \leq T} e^{-\lambda \tau} [c\|e_k(\tau)\|_{\infty} \\ & + \gamma \|\delta u_k(\tau)\|_{\infty}] \mathrm{d}\tau \\ \leq & [c\|e_k\|_{\lambda} + \gamma \|\delta u_k\|_{\lambda}] \cdot \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda(t-\tau)} \mathrm{d}\tau \\ & \cdot \sup_{0 \leq t \leq T} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{d}\tau \\ = & \frac{(1 - e^{-\lambda T}) T^{\alpha}}{\lambda \Gamma(\alpha+1)} [c\|e_k\|_{\lambda} + \gamma \|\delta u_k\|_{\lambda}]. \end{aligned}$$

It follows that $\|e_k\|_{\lambda} \leq O(\lambda^{-1}) \|\delta u_k\|_{\lambda}$, where λ is large enough that $\lambda \Gamma(\alpha+1) - c(1 - e^{-\lambda T}) T^{\alpha} > 0$, and

$$O(\lambda^{-1}) = \frac{\gamma (1 - e^{-\lambda T}) T^{\alpha}}{\lambda \Gamma(\alpha+1) - c(1 - e^{-\lambda T}) T^{\alpha}}.$$

■

Lemma 3.3: For the fractional-order nonlinear system (7) and the PD^{α}−type ILC scheme (9) and (10), suppose $\delta u_k = u_d(t) - u_k(t)$, $(k = 0,1,2,\cdots)$ and $\left\| \frac{\partial f}{\partial u} \right\|_{\infty} \leq \gamma \|u\|_{\infty}$, we have

$$\|\delta u_{k+1}\|_{\infty} \leq \rho \|\delta u_k\|_{\infty},$$

where ρ is defined in the following proof.

Proof: It follows from (6) and Lemma 3.1 that

$$\begin{aligned} & \|\delta u_{k+1}\|_{\infty} = \|u_d - u_{k+1}\|_{\infty} = \|u_d - u_k - K_p\ e_k - K_d\ e_k^{(\alpha)}\|_{\infty} \\ =& \|\delta u_k - K_p\ e_k - K_d [f_d - f_k]\|_{\infty} \\ =& \|[I_m - K_d\ {}_kB(t)]\, \delta u_k - {}_kA(t) e_k\|_{\infty} \\ \leq & \|I_m - K_d\ {}_kB(t)\|_{\infty} \|\delta u_k\|_{\infty} + \left\|K_p - {}_kA(t)\right\|_{\infty} \|e_k\|_{\infty} \end{aligned}$$

Applying the λ−norm to the above equation yields

$$\begin{aligned} \|\delta u_{k+1}\|_{\lambda} \leq & \sup_{0 \leq t \leq T} \|\delta u_{k+1}\|_{\infty} \\ \leq & \sup_{0 \leq t \leq T} \{\|I_m - K_d\ {}_kB(t)\|_{\infty} \|\delta u_k\|_{\infty}\} \\ & + \sup_{0 \leq t \leq T} \left\{ \left\|K_p - {}_kA(t)\right\|_{\infty} \|e_k\|_{\infty} \right\} \\ =& \rho \|\delta u_k\|_{\lambda} + \mu \|e_k\|_{\lambda}, \end{aligned} \tag{11}$$

where

$$\rho = \sup_{0 \leq t \leq T} \|I_m - K_d\ {}_kB(t)\|_{\infty},$$

$$\mu = \sup_{0 \leq t \leq T} \left\|K_p - {}_kA(t)\right\|_{\infty}.$$

Using Lemma 3.2 and (11), we have

$$\|\delta u_{k+1}\|_{\lambda} \leq (\rho + \mu O(\lambda^{-1}) \|\delta u_k\|_{\lambda}$$

so that, let $\lambda \to \infty$, $\|\delta u_{k+1}\|_{\infty} \leq \rho \|\delta u_k\|_{\infty}$. ■

Theorem 3.2: For the fractional-order nonlinear system (7) with the $PD^{\alpha}-$type ILC scheme (9) and (10), where $t \in [0,T]$, suppose $f(t,y,u)$ satisfies (8), and

$$\left\| I_m - K_d \frac{\partial f}{\partial u} \right\|_{\infty} < 1, \tag{12}$$

we have $\lim_{k\to\infty} y_k(t) = y_d(t), \quad (t \in [0,T])$.

Proof: It follows from Lemma 3.3 and (8) that $\|\delta u_{k+1}\|_{\lambda} \leq [\rho + \mu O(\lambda^{-1})]\|\delta u_k\|_{\lambda}$. Using (12) yields $\rho < 1$, so that there exists a sufficient large $\tilde{\lambda}$ satisfying $\rho + \mu O(\tilde{\lambda}^{-1}) < 1$. Therefore,

$$\lim_{k\to\infty} \|\delta u_k\|_{\tilde{\lambda}} = \lim_{k\to\infty} \|u_d - u_k\|_{\tilde{\lambda}} = 0.$$

In other words, $\lim_{k\to\infty} u_k(t) = u_d(t)$, where $t \in [0,T]$. It then follows from the uniqueness and existence theorem for fractional-order differential equations [35] that $\lim_{k\to\infty} y_k(t) = y_d(t)$. ■

Remark 3.3: For the fractional/integer-order state space system

$$\begin{cases} x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases}$$

where $\alpha \in (0,1]$ and $t \in [0,T]$, with the $PD^{\alpha}-$type ILC scheme (9), it follows from $\frac{\partial[C(t)B(t)u(t)]}{\partial u(t)} = C(t)B(t)$ that the convergence condition is

$$\|I - K_d(t)C(t)B(t)\|_{\infty} < 1,$$

which is equivalent to the convergence conditions in [1], [30], [29].

Remark 3.4: For the fractional-order nonlinear affine system

$$y^{(\alpha)}(t) = f(t, y(t)) + \gamma u(t),$$

where $\alpha \in (0,1)$ and $t \in [0,T]$. Using the $PD^{\alpha}-$type ILC scheme (9), it follows from $\frac{\partial[\gamma u(t)]}{\partial u(t)} = \gamma$ that the convergence condition is

$$\|I - \gamma K_d(t)\|_{\infty} < 1.$$

IV. Simulations

Suppose the fractional-order nonlinear system is

$$y^{\left(\frac{9}{10}\right)}(t) = y^{\frac{3}{5}}(t) + u(t), \tag{13}$$

and the $PD^{\alpha}-$type ILC updating law is

$$u_{k+1}(t) = u_k(t) + \frac{1}{2}e_k(t) + \frac{9}{10}e_k^{\left(\frac{9}{10}\right)}(t). \tag{14}$$

In this case, it can be easily seen that $\rho = |1 - 9/10| = 1/10 < 1$. Therefore, $y_k(t)$ is tending to $y_d(t)$ as $k \to \infty$, where $t \in [0,T]$. Moreover, let the initial control input be $u_0(t) = 0$ and the reference be $y_d(t) = 12t^2(1-t)$ so that $y(0) = 0$ and $T = 1$, the simulation results are shown in Figures 1 and 2. It can be seen from these two figures that the tracking errors are very small after the fifth iteration ($\|y_d(t) - y_4(t)\|_2 = 0.0454$).

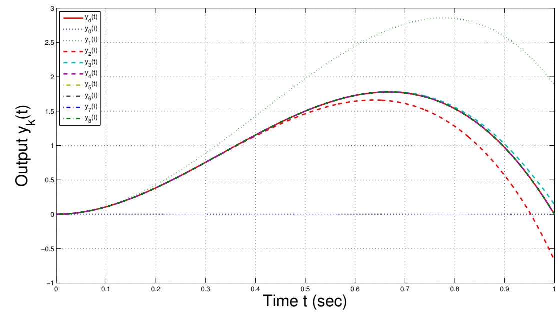

Fig. 1. The system outputs of (13) with ILC updating law (14), where $k = 0,1,2,\cdots,8$ and the initial control input is $u_0(t) = 0$, the reference is $y_d(t) = 12t^2(1-t)$ and $T = 1$.

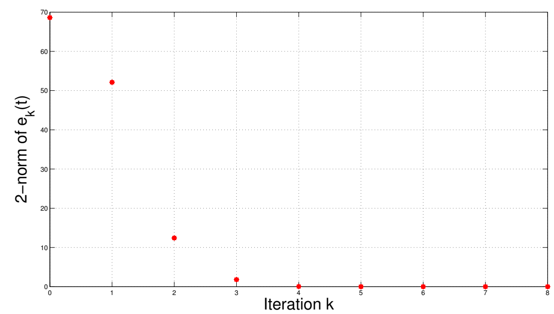

Fig. 2. For the fractional-order nonlinear system (13) and the ILC updating law (14), the two norms of $y_d(t) - y_k(t)$ for different iteration k are shown in this figure, where $k = 0,1,2,\cdots,8$ and the reference is $y_d(t) = 12t^2(1-t)$.

For the fractional-order system (13) and the fractional-order iterative learning scheme (14), replace the reference by the squarewave

$$r(t) = \begin{cases} -1, & (t \in (0, \frac{1}{2})), \\ 1 & (t \in [\frac{1}{2}, 1)), \end{cases}$$

the simulation results are shown in Figures 3 and 4. It can be seen that there are some overshoots at the non-smooth points $t = \{0, \frac{1}{2}, 1\}$. However, the performance is still good in other domain.

Moreover, suppose the fractional-order nonlinear system is [1]

$$y^{\left(\frac{9}{10}\right)}(t) = y^{\frac{3}{5}}(t) + u^{\frac{8}{9}}(t), \tag{15}$$

and the $PD^{\alpha}-$type ILC updating law is

$$u_{k+1}(t) = u_k(t) + 5e_k(t) + \frac{9}{10}e_k^{\left(\frac{9}{10}\right)}, \tag{16}$$

where the $\frac{1}{2}$ in (14) is replaced by 5. Let the initial control input be $u_0(t) = 0$ and the reference be $y_d(t) = 12t^2(1-t)$ so that $y(0) = 0$ and $t \in [0,1]$, the simulation results are shown in Figures 5 and 6. It can be seen from these two figures that the tracking errors are increasing in the first three iterations. However, it decreases to very small values for $k \geq 8$ ($\|y_d(t) -$

[1] Compare (13) with (15), the only difference is that the $u(t)$ is replaced by $u^{\frac{8}{9}}(t)$.

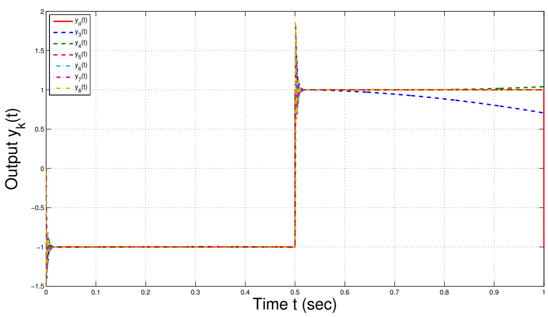

Fig. 3. The system outputs of (13) with ILC updating law (14), where $k = 3,4,5,\cdots,8$ and the initial control input is $u_0(t) = 0$, the reference is $y_d(t) = r(t)$ and $T = 1$.

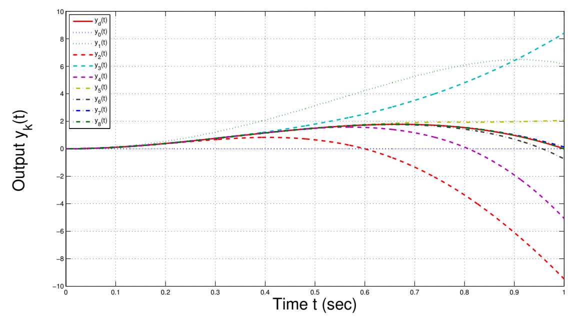

Fig. 5. The system outputs of (15) with ILC updating law (16), where $k = 0,1,2,\cdots,8$ and the initial control input is $u_0(t) = 0$, the reference is $y_d(t) = 12t^2(1-t)$ and $T = 1$.

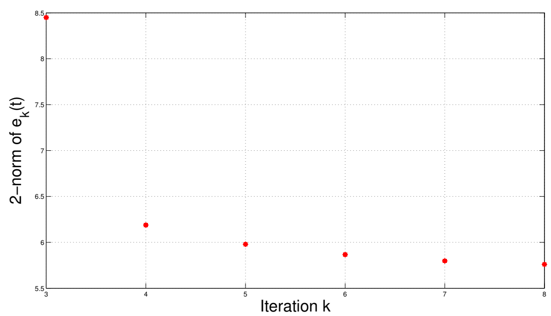

Fig. 4. For the fractional-order nonlinear system (13) and the ILC updating law (14), the two norms of $y_d(t) - y_k(t)$ for different iteration k are shown in this figure, where $k = 3,4,5,\cdots,8$ and the reference is $y_d(t) = r(t)$.

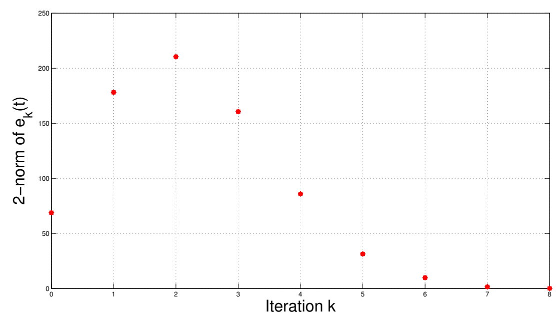

Fig. 6. For the fractional-order nonlinear system (15) and the ILC updating law (16), the two norms of $y_d(t) - y_k(t)$ for different iteration k are shown in this figure, where $k = 0,1,2,\cdots,8$, the reference is $y_d(t) = 12t^2(1-t)$ and $T = 1$.

$y_8(t)\|_2 = 0.1401$). In other words, K_d influences the tracking speed of the ILC scheme. But, if the convergence condition is satisfied, K_d does not the change the fact of convergence.

Lastly, the use of $\lambda-$norm permits us to discuss the perturbed cases. For the perturbed system

$$y^{\left(\frac{9}{10}\right)}(t) = y^{\frac{3}{5}}(t) + u^{\frac{8}{9}}(t) + n(t), \tag{17}$$

with ILC strategy (14), where $n(t)$ is a white noise and its power spectral density and sample time are equal to 0.1, the simulation results are shown in Figures 7 and 8, where $\|y_d(t) - y_4(t)\|_2 = 0.0704$. It can be seen that the tracking process is working well for $k \geq 4$. The reason of robustness can be summarized as the fractional-order integral itself is a kind of filter [28], which reduces the influence of $n(t)$.

V. Conclusions and Future Works

For the fractional-order nonlinear systems, it was proved that the $PD^{\alpha}-$type ILC scheme guarantees that

$$\lim_{k\to\infty} y_k(t) = y_d(t), \quad \text{for all } t \in [0,T],$$

where $\|\frac{\partial f}{\partial y}\|_\infty \leq c\|y\|_\infty$ and $\|\frac{\partial f}{\partial u}\|_\infty \leq \gamma\|u\|_\infty$ were required. This was a sufficient condition to derive the uniqueness and existence of the system equation and was an important index for the convergence conditions. Most of the classical fractional-order ILC schemes fell into the scheme of this paper. It was shown in the Matlab/Simulink that the PD^{α} algorithm discussed in this paper can be applied to both continuous and discontinuous references and perturbed systems. For the discontinuous one, although there were some overshoots at the truncated points, the tracking performance on other domains were still good. The tracking speed could be improved by the terms of $u(t)$ in the system equation and the ILC scheme as well.

Our future works include the fractional-order and generalized fractional-order ILC schemes to the nonlinear systems and their applications. Moreover, the fractional-order robust and adaptive ILC schemes will also be included in our future works.

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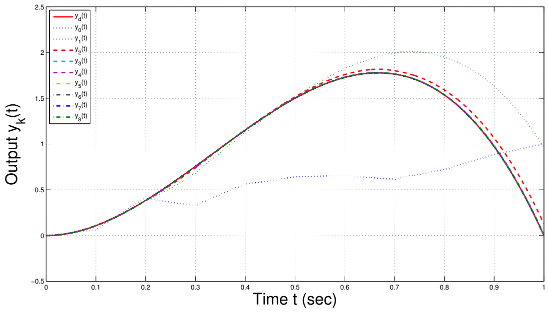

Fig. 7. The outputs of the perturbed system (17) with ILC updating law (14), where $k = 0, 1, 2, \cdots, 8$ and the initial control input is $u_0(t) = 0$, the reference is $y_d(t) = 12t^2(1-t)$ and $T = 1$.

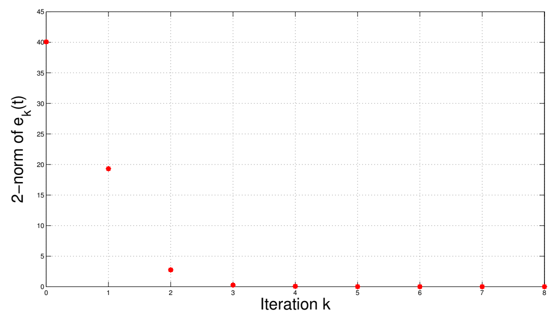

Fig. 8. For the fractional-order nonlinear perturbed system (17) and the ILC updating law (14), the two norms of $y_d(t) - y_k(t)$ for different iteration k are shown in this figure, where $k = 0, 1, 2, \cdots, 8$, the reference is $y_d(t) = 12t^2(1-t)$ and $T = 1$.

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