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New Stabilization Method for Linear Systems with Time-varying Input Delay

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Abstract— **This paper presents new results on delay-dependent stability and stabilization for linear systems with time-varying delays in a given range. With an appropriate Lyapunov functional, some delay-dependent criteria for determining the stability of the time-delay systems are obtained. In this paper, we propose a new state transformation technology to facilitate controller designing efficiently. The method is also applicable to the existing stability conditions reported by now, while the existing technologies may fail to derive computational control procedures from the stability conditions. Finally, some numerical examples well illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

During the last decade, there has been a growing interest to analysis and synthesis of time-delay systems, which widely exist in various engineering systems such as chemical processes, neural networks and long transmission lines in pneumatic systems [2], [3], [5], [10]. In the literatures, Lyapunov-Razumikhin functional and Lyapunov-Krasovskii functional are widely used approaches for time-delay systems to obtain a delay-independent or a delay-dependent stability condition [4]. Basically, delay-dependent conditions may issue less conservative result than delay-independent ones especially when the delay is small. Therefor, more attention is paid on delay-dependent conditions. Currently, the results of delay-dependent stability mainly focus on time-varying delay with range zero to an upper bound. In practice, the range of delay may vary in a range for which the lower bound is not restricted to be zero [7]. For this reason, the stability of the systems with such interval time-varying delays has attracted considerable attention. For example, in [6], a discretized Lyapunov functional approach is employed to obtain stability criteria for linear uncertain systems with interval time-varying delays. By using free-weighting matrix, [7] presents some less conservative stability conditions. This result is improved by [11] where a new Lyapunov functional with fewer matrix variables is constructed and the convex analysis method is applied. Recently, these results are further extended in [12] by proposing a new type of augmented Lyapunov functional containing some triple-integral terms. Nevertheless there still exists some room for further improvement, which is one of the motivation of this paper.

On the other hand, control problems for time-delay systems are important issues. In [14], an integral-inequality method is proposed for the delay-dependent stabilization problem of linear systems with time-varying state and input delays. For constant but unknown time-delay, by introducing a state-transformation to discribe the delay-dependence dynamics, some control design schemes based on quadratic $\mathcal{H}_2$ performance, $\mathcal{H}_\infty$ criteria and simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis are established in [9]. For a stabilizable and detectable linear system with an arbitrarily large delay in the input channel, by explicit construction of stabilizing feedback laws, [8] shows that the system can be asymptotically stabilized by either linear state or output feedback as long as all the open-loop poles are on the closed left-half plane. However, to the best of the authors' knowledge, few results are reported in the existing literature on the control designing problem for linear system with interval time-varying input delay. It is not easy to apply the results in [6], [7], [11], [12] to obtain control procedures because of the cross terms of matrix variables [13] involved. This is the other motivation for the study of this paper.

In the paper, we propose new results on delay-dependent stability and stabilization for time-delay systems. Using a new Lyapunov functional, some less conservative delay-dependent stability conditions are derived for the linear systems with time-varying delay in a range. As pointed above, the existing stability results may be difficultly to obtain computational controller design procedure. For linear system with time-varying input delay, by carefully selecting the matrix variables, and choosing a nonsingular transformation matrix T which transforms the state x of origin system to a new auxiliary state $\bar{x} = Tx$, the cross terms coming from the stability condition can be removed. Assisted by this technology, a less conservative designing procedure is obtained from the stability conditions. The technology is also applicable to the existing stability conditions reported. Finally, some numerical examples well demonstrate the effectiveness of the proposed method.

Notation: Throughout the paper, $\mathbf{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices, I is the identity matrix with appropriate dimensions, and the superscripts "-1" and "T" stand for the inverse and transpose of a matrix, respectively. The notation $X > 0$ (respectively, $X \geq 0$), for $X \in \mathbf{R}^{n\times n}$ means that the matrix X is symmetric and positive definite (respectively, positive semi-definite).

II. PROBLEM FORMULATION

Consider the following time-delay systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - d(t)) \\ x(\theta) &= \phi(\theta),\ \theta \in [-d_2, 0] \end{aligned} \tag{1}$$

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where $x(t) \in \mathbf{R}^n$ denotes the state vector, $u(t) \in \mathbf{R}^m$ denotes the control input. A and B are constant real matrices and the pair (A, B) is assumed to be stabilizable. $\phi(\theta)$ is a continuous vector-valued initial function on $[-d_2, 0]$. $d(t)$ denotes the time-varying delay and satisfies

$$d_1 \leq \quad d(t) \quad \leq d_2, \tag{2}$$

$$\dot{d}(t) \quad \leq \mu, \tag{3}$$

where $0 \leq d_1 < d_2$ and $0 \leq \mu$ are constants. The controller takes following form

$$u(t) = Fx(t), \tag{4}$$

where F is controller gain matrix with appropriate dimensions. Substituting (4) into (1), we obtain the following closed-loop system

$$\begin{array}{l} \dot{x}(t) = Ax(t) + A_d x(t - d(t)) \\ x(\theta) = \phi(\theta),\ \theta \in [-d_2, 0] \end{array} \tag{5}$$

where $A_d = BF$. Then the problems considered in this paper can be formulated as: (i) to set up a delay-dependent stability for the time-delay system (5) with the given A and A_d; (ii) to design state feedback controller (4) such that system (1) asymptotically stable.

At the end of this section, we introduce the following lemma which is useful in the derivation of our results.

Lemma 1[3] For any constant matrix $W > 0$, scalars $a < b$ and vector function $\omega(s) : [a, b] \to \mathbf{R}^n$ such that the following integrations are well defined, then

$$(b-a)\int_a^b \omega^T(s)W\omega(s)ds \geq \int_a^b \omega^T(s)dsW\int_a^b \omega(s)ds;$$

$$\begin{aligned} &\frac{(b-a)^2}{2}\int_a^b\int_\theta^b \omega^T(s)W\omega(s)dsd\theta \\ &\geq \int_a^b\int_\theta^b \omega^T(s)dsd\theta W\int_a^b\int_\theta^b \omega(s)dsd\theta; \end{aligned}$$

$$\begin{aligned} &\frac{(b-a)^2}{2}\int_a^b\int_a^\theta \omega^T(s)W\omega(s)dsd\theta \\ &\geq \int_a^b\int_a^\theta \omega^T(s)dsd\theta W\int_a^b\int_a^\theta \omega(s)dsd\theta. \end{aligned}$$

III. Stability Analysis

In this section, we consider asymptotic stability of time-delay system (5). Using a new Lyapunov functional, some delay-dependent stability criteria are obtained with the consideration of range for the time-varying delay.

Theorem 1 For the given scalars $0 \leq d_1 < d_2$ and μ, system (5) with $d(t)$ satisfying (2) and (3) is asymptotically stable if there exist matrices $P = [P_{ij}]_{5\times 5} > 0$, $Q > 0$, $R_i > 0$, $S_i > 0$, $Z_i > 0$, $i = 1, \ldots, 4$, and Y_j, $j = 1, 2, 3$ with appropriate dimensions such that

$$\begin{aligned} \Xi_0 &= \Gamma_0 P\Upsilon^T + \Upsilon P\Gamma_0^T + \Psi_0 \\ &\quad +\Lambda + Y_e A_e + A_e^T Y_e^T < 0 \end{aligned} \tag{6}$$

$$\begin{aligned} \Xi_1 &= \Gamma_1 P\Upsilon^T + \Upsilon P\Gamma_1^T + \Psi_1 \\ &\quad +\Lambda + Y_e A_e + A_e^T Y_e^T < 0 \end{aligned} \tag{7}$$

where $d_{12} = d_2 - d_1$,

$$\begin{aligned}
\Gamma_0 &= \begin{bmatrix} e_1 & e_4 & e_6 & d_1 e_8 & d_{12}e_{10} \end{bmatrix} \\
\Gamma_1 &= \begin{bmatrix} e_1 & e_4 & e_6 & d_1 e_8 & d_{12}e_9 \end{bmatrix} \\
\Upsilon &= \begin{bmatrix} e_2 & e_5 & e_7 & e_1 - e_4 & e_4 - e_6 \end{bmatrix} \\
\Psi_0 &= -d_{12}^2 e_{10}S_1 e_{10}^T - (e_4 - e_3)(2S_2 + Z_1)(e_4^T - e_3^T) \\
&\quad -(e_3 - e_6)S_2(e_3^T - e_6^T) \\
\Psi_1 &= -d_{12}^2 e_9 S_1 e_9^T - (e_4 - e_3)S_2(e_4^T - e_3^T) \\
&\quad -(e_3 - e_6)(2S_2 + Z_2)(e_3^T - e_6^T) \\
\Lambda &= diag\Big\{R_3 + d_1^2 S_3,\ R_4 + d_1^2 S_4 + \frac{d_1^2}{2}Z_3 + \frac{d_1^2}{2}Z_4, \\
&\quad -(1-\mu)Q,\ Q + R_1 - R_3 + d_{12}^2 S_1, \\
&\quad R_2 - R_4 + d_{12}^2 S_2 + \frac{d_{12}^2}{2}Z_1 + \frac{d_{12}^2}{2}Z_2,\ -R_1, \\
&\quad -R_2,\ -d_1^2 S_3,\ 0,\ 0\Big\} - (e_1 - e_4)S_4(e_1^T - e_4^T) \\
&\quad -2(e_1 - e_8)Z_3(e_1^T - e_8^T) \\
&\quad -2(e_8 - e_4)Z_4(e_8^T - e_4^T) \\
&\quad -2(e_4 - e_9)Z_1(e_4^T - e_9^T) \\
&\quad -2(e_3 - e_{10})Z_1(e_3^T - e_{10}^T) \\
&\quad -2(e_9 - e_3)Z_2(e_9^T - e_3^T) \\
&\quad -2(e_{10} - e_6)Z_2(e_{10}^T - e_6^T) \\
Y_e &= e_1 Y_1 + e_2 Y_2 + e_3 Y_3 \\
A_e &= Ae_1^T - e_2^T + A_d e_3^T
\end{aligned}$$

and $e_i \in \mathbf{R}^{10n\times n}, i = 1, \ldots, 10$ are block entry matrices, for example, $e_3^T = [0\ 0\ I\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$.

Proof: Construct a Lyapunov functional candidate as

$$V(t) = \sum_{i=0}^{3} V_i(t)$$

where

$$\begin{aligned}
V_0(t) &= \zeta^T(t)P\zeta(t) \\
V_1(t) &= \int_{t-d(t)}^{t-d_1} x^T(s)Qx(s)ds + \int_{t-d_2}^{t-d_1} x^T(s)R_1 x(s)ds \\
&\quad + \int_{t-d_2}^{t-d_1} \dot{x}^T(s)R_2\dot{x}(s)ds \\
&\quad + \int_{t-d_1}^{t} x^T(s)R_3 x(s)ds \\
&\quad + \int_{t-d_1}^{t} \dot{x}^T(s)R_4\dot{x}(s)ds \\
V_2(t) &= d_{12}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t-d_1} x^T(s)S_1 x(s)dsd\theta \\
&\quad +d_{12}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t-d_1} \dot{x}^T(s)S_2\dot{x}(s)dsd\theta \\
&\quad +d_1\int_{-d_1}^{0}\int_{t+\theta}^{t} x^T(s)S_3 x(s)dsd\theta \\
&\quad +d_1\int_{-d_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_4\dot{x}(s)dsd\theta
\end{aligned}$$

$$
\begin{aligned}
V_3(t) &= \int_{-d_2}^{-d_1}\int_{\eta}^{-d_1}\int_{t+\theta}^{t-d_1}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta d\eta \\
&+\int_{-d_2}^{-d_1}\int_{-d_2}^{\eta}\int_{t+\theta}^{t-d_1}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta d\eta \\
&+\int_{-d_1}^{0}\int_{\eta}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_3\dot{x}(s)dsd\theta d\eta \\
&+\int_{-d_1}^{0}\int_{-d_1}^{\eta}\int_{t+\theta}^{t}\dot{x}^T(s)Z_4\dot{x}(s)dsd\theta d\eta
\end{aligned}
$$

and $\zeta(t) = col\left\{x(t), x(t-d_1), x(t-d_2), \int_{t-d_1}^{t} x(s)ds, \int_{t-d_2}^{t-d_1} x(s)ds\right\}$. Doing the time derivative of $V(t)$ along the trajectory of system (5), we have

$$
\begin{aligned}
\dot{V}(t) &= 2\zeta^T(t)P\dot{\zeta}(t) + x^T(t)(R_3+d_1^2S_3)x(t) \\
&+\dot{x}^T(t)(R_4+d_1^2S_4+\frac{d_1^2}{2}Z_3+\frac{d_1^2}{2}Z_4)\dot{x}^T(t) \\
&-(1-\dot{d}(t))x^T(t-d(t))Qx(t-d(t)) \\
&+x^T(t-d_1)(Q+R_1-R_3+d_{12}^2S_1)x(t-d_1) \\
&+\dot{x}^T(t-d_1)\Big(R_2-R_4+d_{12}^2S_2+\frac{d_{12}^2}{2}Z_1 \\
&+\frac{d_{12}^2}{2}Z_2\Big)\dot{x}(t-d_1) - x^T(t-d_2)R_1x(t-d_2) \\
&-\dot{x}^T(t-d_2)R_2\dot{x}(t-d_2) \\
&-d_{12}\int_{t-d_2}^{t-d_1}(x^T(s)S_1x(s)+\dot{x}^T(s)S_2\dot{x}(s))ds \\
&-d_1\int_{t-d_1}^{t}(x^T(s)S_3x(s)ds+\dot{x}^T(s)S_4\dot{x}(s))ds \\
&-\int_{-d_2}^{-d_1}\int_{t+\theta}^{t-d_1}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\
&-\int_{-d_2}^{-d_1}\int_{t-d_2}^{t+\theta}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
&-\int_{-d_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_3\dot{x}(s)dsd\theta \\
&-\int_{-d_1}^{0}\int_{t-d_1}^{t+\theta}\dot{x}^T(s)Z_4\dot{x}(s)dsd\theta. \qquad (8)
\end{aligned}
$$

Setting $\xi(t) = col\Big\{x(t), \dot{x}(t), x(t-d(t)), x(t-d_1), \dot{x}(t-d_1), x(t-d_2), \dot{x}(t-d_2), \frac{1}{d_1}\int_{t-d_1}^{t} x(s)ds, \frac{1}{d(t)-d_1}\int_{t-d(t)}^{t-d_1} x(s)ds, \frac{1}{d_2-d(t)}\int_{t-d_2}^{t-d(t)} x(s)ds\Big\}$ and $\alpha = \frac{d(t)-d_1}{d_{12}}$, we have

$$
\zeta^T(t) = \xi^T(t)\Gamma(\alpha),\ \dot{\zeta}^T(t) = \xi^T(t)\Upsilon,
$$

where $\Gamma(\alpha) = [e_1\ e_4\ e_6\ d_1e_8\ \alpha d_{12}e_9 + (1-\alpha)d_{12}e_{10}]$. According to the expression of system (5), we note that for any matrices $Y_i, i = 1, 2, 3$ with appropriate dimensions, the following equality holds

$$
\begin{aligned}
&2[x^T(t)Y_1 + \dot{x}^T(t)Y_2 + x^T(t-d(t))Y_3] \\
&\times[Ax(t) - \dot{x}(t) + A_dx(t-d(t))] = 0.
\end{aligned}
$$

Furthermore, it is equivalent to

$$
\xi^T(t)(Y_eA_e + A_e^TY_e^T)\xi(t) = 0. \qquad (9)
$$

Using Lemma 1, one can obtain

$$
\begin{aligned}
&d_1\int_{t-d_1}^{t} x^T(s)S_3x(s)ds \geq d_1^2\xi^T(t)e_8S_3e_8^T\xi(t); \qquad (10) \\
&d_1\int_{t-d_1}^{t}\dot{x}^T(s)S_4\dot{x}(s)ds \\
&\geq \xi^T(t)(e_1-e_4)S_4(e_1^T-e_4^T)\xi(t); \qquad (11) \\
&\int_{-d_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_3\dot{x}(s)dsd\theta \\
&\geq 2\xi^T(t)(e_1-e_8)Z_3(e_1^T-e_8^T)\xi(t); \qquad (12) \\
&\int_{-d_1}^{0}\int_{t-d_1}^{t+\theta}\dot{x}^T(s)Z_4\dot{x}(s)dsd\theta \\
&\geq 2\xi^T(t)(e_8-e_4)Z_4(e_8^T-e_4^T)\xi(t). \qquad (13)
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\tfrac{d_{12}}{d(t)-d_1} \geq \tfrac{d_2-d(t)}{d_{12}} + 1 = 2-\alpha, \\
&\tfrac{d_{12}}{d_2-d(t)} \geq \tfrac{d(t)-d_1}{d_{12}} + 1 = 1+\alpha,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_{t-d_2}^{t-d_1} x^T(s)S_1x(s)ds \\
=\ &\int_{t-d(t)}^{t-d_1} x^T(s)S_1x(s)ds + \int_{t-d_2}^{t-d(t)} x^T(s)S_1x(s)ds \\
\geq\ &\alpha d_{12}\xi^T(t)e_9S_1e_9^T\xi(t) \\
&+(1-\alpha)d_{12}\xi^T(t)e_{10}S_1e_{10}^T\xi(t) \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
&d_{12}\int_{t-d_2}^{t-d_1}\dot{x}^T(s)S_2\dot{x}(s)ds \\
=\ &d_{12}\int_{t-d(t)}^{t-d_1}\dot{x}^T(s)S_2\dot{x}(s)ds \\
&+d_{12}\int_{t-d_2}^{t-d(t)}\dot{x}^T(s)S_2\dot{x}(s)ds \\
\geq\ &(2-\alpha)\xi^T(t)(e_4-e_3)S_2(e_4^T-e_3^T)\xi(t) \\
&+(1+\alpha)\xi^T(t)(e_3-e_6)S_2(e_3^T-e_6^T)\xi(t). \qquad (15)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{-d_2}^{-d_1}\int_{t+\theta}^{t-d_1}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\
=\ &\int_{-d(t)}^{-d_1}\int_{t+\theta}^{t-d_1}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\
&+\int_{-d_2}^{-d(t)}\int_{t+\theta}^{t-d(t)}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\
&+(d_2-d(t))\int_{t-d(t)}^{t-d_1}\dot{x}^T(s)Z_1\dot{x}(s)ds \\
\geq\ &2\xi^T(t)(e_4-e_9)Z_1(e_4^T-e_9^T)\xi(t) \\
&+2\xi^T(t)(e_3-e_{10})Z_1(e_3^T-e_{10}^T)\xi(t) \\
&+(1-\alpha)\xi^T(t)(e_4-e_3)Z_1(e_4^T-e_3^T)\xi(t) \qquad (16)
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{-d_2}^{-d_1}\int_{t-d_2}^{t+\theta}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
= &\int_{-d(t)}^{-d_1}\int_{t-d(t)}^{t+\theta}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
&+\int_{-d_2}^{-d(t)}\int_{t-d_2}^{t+\theta}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \\
&+(d(t)-d_1)\int_{t-d_2}^{t-d(t)}\dot{x}^T(s)Z_2\dot{x}(s)ds \\
\geq &\ 2\xi^T(t)(e_9-e_3)Z_2(e_9^T-e_3^T)\xi(t) \\
&+2\xi^T(t)(e_{10}-e_6)Z_2(e_{10}^T-e_6^T)\xi(t) \\
&+\alpha\xi^T(t)(e_3-e_6)Z_2(e_3^T-e_6^T)\xi(t). \qquad (17)
\end{aligned}$$

Applying (3) and adding the left-hand side of (9) into the right-hand side of (8), from (10)-(17) we obtain

$$\dot{V}(t) \leq \xi^T(t)\Xi(\alpha)\xi(t),$$

where $\Xi(\alpha)=(1-\alpha)\Xi_0+\alpha\Xi_1$. We note that $\Xi(\alpha)$ is convex in $\alpha\in[0,1]$, thus, it is negative definite only if its vertices are, i.e. $\Xi_0<0$ and $\Xi_1<0$. One can see that if (6) and (7) are satisfied, then $\dot{V}(t)\leq-\epsilon\|x(t)\|^2$ for a sufficiently small $\epsilon>0$, from which we conclude that system (5) is asymptotically stable according to Lyapunov stability theory [5]. This ends the proof. ■

When $d_1=0$, Theorem 1 reduces to the following delay-dependent stability criterion.

Corollary 1 Given scalars $d_2>0$, $d_1=0$ and μ, system (5) with time-varying delay $d(t)$ satisfying (2) and (3) is asymptotically stable if there exist matrices $\tilde{P}=[\tilde{P}_{ij}]_{3\times3}>0$, $Q>0$, $R_i>0$, $S_i>0$, $Z_i>0$, $i=1,2$, and Y_j, $j=1,2,3$ such that

$$\begin{aligned}
\tilde{\Xi}_0 &= \tilde{\Gamma}_0\tilde{P}\tilde{\Upsilon}^T+\tilde{\Upsilon}\tilde{P}\tilde{\Gamma}_0^T+\tilde{\Psi}_0 \\
&\quad +\tilde{\Lambda}+Y_{\tilde{e}}A_{\tilde{e}}+A_{\tilde{e}}^TY_{\tilde{e}}^T<0 \qquad (18)\\
\tilde{\Xi}_1 &= \tilde{\Gamma}_1\tilde{P}\tilde{\Upsilon}^T+\tilde{\Upsilon}\tilde{P}\tilde{\Gamma}_1^T+\tilde{\Psi}_1 \\
&\quad +\tilde{\Lambda}+Y_{\tilde{e}}A_{\tilde{e}}+A_{\tilde{e}}^TY_{\tilde{e}}^T<0 \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Gamma}_0 &= \begin{bmatrix}\tilde{e}_1 & \tilde{e}_4 & d_2\tilde{e}_7\end{bmatrix}\\
\tilde{\Gamma}_1 &= \begin{bmatrix}\tilde{e}_1 & \tilde{e}_4 & d_2\tilde{e}_6\end{bmatrix}\\
\tilde{\Upsilon} &= \begin{bmatrix}\tilde{e}_2 & \tilde{e}_5 & \tilde{e}_1-\tilde{e}_4\end{bmatrix}\\
\tilde{\Psi}_0 &= -d_2^2\tilde{e}_7S_1\tilde{e}_7^T-(\tilde{e}_1-\tilde{e}_3)(2S_2+Z_1)(\tilde{e}_1^T-\tilde{e}_3^T)\\
&\quad -(\tilde{e}_3-\tilde{e}_4)S_2(\tilde{e}_3^T-\tilde{e}_4^T)\\
\tilde{\Psi}_1 &= -d_2^2\tilde{e}_6S_1\tilde{e}_6^T-(\tilde{e}_1-\tilde{e}_3)S_2(\tilde{e}_1^T-\tilde{e}_3^T)\\
&\quad -(\tilde{e}_3-\tilde{e}_4)(2S_2+Z_2)(\tilde{e}_3^T-\tilde{e}_4^T)\\
\tilde{\Lambda} &= diag\Big\{Q+R_1+d_2^2S_1, R_2+\frac{d_2^2}{2}(2S_2+Z_1+Z_2),\\
&\quad -(1-\mu)Q,\ -R_1,\ -R_2,\ 0,\ 0\Big\}\\
&\quad -2(\tilde{e}_1-\tilde{e}_6)Z_1(\tilde{e}_1^T-\tilde{e}_6^T)\\
&\quad -2(\tilde{e}_3-\tilde{e}_7)Z_1(\tilde{e}_3^T-\tilde{e}_7^T)\\
&\quad -2(\tilde{e}_6-\tilde{e}_3)Z_2(\tilde{e}_6^T-\tilde{e}_3^T)\\
&\quad -2(\tilde{e}_7-\tilde{e}_4)Z_2(\tilde{e}_7^T-\tilde{e}_4^T)\\
Y_{\tilde{e}} &= \tilde{e}_1Y_1+\tilde{e}_2Y_2+\tilde{e}_3Y_3\\
A_{\tilde{e}} &= A\tilde{e}_1^T-\tilde{e}_2^T+A_d\tilde{e}_3^T
\end{aligned}$$

and $\tilde{e}_i\in\mathbf{R}^{7n\times n}, i=1,\ldots,7$ are block entry matrices, for example, $\tilde{e}_3^T=[0\ 0\ I\ 0\ 0\ 0\ 0]$.

Remark 1 Theorem 1 and Corollary 1 give stability criteria of system (5) with $d(t)$ satisfying (2)-(3) for $d_1\geq0$ and $d_1=0$, respectively. They can be applied to both slow and fast time-varying delays only if μ is known. However, the information of delay rate may not be known in many cases, or $d(t)$ even not be differentiable, then Theorem 1 and Corollary 1 fail to work. Regarding these circumstance, rate-independent criterions for $d(t)$ only satisfying (2) with $d_1\geq0$ or $d_1=0$ can be derived by choosing $Q=0$ in Theorem 1 or Corollary 1, respectively.

IV. Matrix Transformation for Stabilization

In this section, we present a computational procedure for the gain matrix F of the controller (4) such that the system (1) stable. Noticing $A_d=BF$ in Theorem 1, the controller gain F appear in the term Y_eA_e and its symmetric one, that is, the inequalities (6) and (7) involve nonlinear terms Y_iBF in unknown matrix variables Y_i, for $i=1,2,3$, and F, which makes the inequalities listed in Theorem 1 be not computational for F.

For the sequel development, introduce a matrix transformation technique to present a computational procedure for the gain F. Suppose that the rank of the matrix B is r, i.e. $rank(B)=r$, satisfying $1\leq r\leq m$. Then there exists an invertible transformation matrix $T\in\mathbf{R}^{n\times n}$ satisfying

$$TB=\begin{bmatrix}0 & \bar{B}_0^T\end{bmatrix}^T, \qquad (20)$$

where $\bar{B}_0\in\mathbf{R}^{r\times m}$ and $rank(\bar{B}_0)=r$.

With the help of such matrix T, introducing a new state $\bar{x}(t)$, $\bar{x}(t):=Tx(t)$ for the closed-loop system (5), then (5) is equivalent to

$$\begin{aligned}
&\dot{\bar{x}}(t)=\bar{A}\bar{x}(t)+\bar{B}\bar{F}\bar{x}(t-d(t)),\\
&\bar{x}(\theta)=\bar{\phi}(\theta),\ \theta\in[-d_2,0]
\end{aligned} \qquad (21)$$

where $\bar{A}=TAT^{-1}$, $\bar{B}=TB$, $\bar{F}=FT^{-1}$, $\bar{\phi}(\theta)=T\phi(\theta)$. Then we have the following computational result for F.

Theorem 2 For given scalars $0\leq d_1<d_2$ and $0\leq\mu$, if there exist matrices $P=[P_{ij}]_{5\times5}>0$, $Q>0$, $R_i>0$, $S_i>0$, $Z_i>0$, $i=1,\ldots,4$, and matrices $\hat{F}=\begin{bmatrix}0 & \hat{F}_0^T\end{bmatrix}^T$ with $\hat{F}_0\in\mathbf{R}^{r\times m}$, and $\hat{Y}_j=\begin{bmatrix}\hat{Y}_{j,11} & 0\\ \hat{Y}_{j,21} & \hat{Y}_0\end{bmatrix}$ with $\hat{Y}_0\in\mathbf{R}^{r\times r}$, $j=1,2,3$ such that the following LMIs hold:

$$\begin{aligned}
\Omega_0 &= \Gamma_0P\Upsilon^T+\Upsilon P\Gamma_0^T+\Psi_0+\Lambda+\hat{Y}_e\hat{A}_e+\hat{A}_e^T\hat{Y}_e^T\\
&\quad +(e_1+e_2+e_3)\hat{F}e_3^T+e_3\hat{F}^T(e_1+e_2+e_3)^T<0 \qquad (22)\\
\Omega_1 &= \Gamma_1P\Upsilon^T+\Upsilon P\Gamma_1^T+\Psi_1+\Lambda+\hat{Y}_e\hat{A}_e+\hat{A}_e^T\hat{Y}_e^T\\
&\quad +(e_1+e_2+e_3)\hat{F}e_3^T+e_3\hat{F}^T(e_1+e_2+e_3)^T<0 \qquad (23)
\end{aligned}$$

where $\hat{Y}_e = e_1\hat{Y}_1 + e_2\hat{Y}_2 + e_3\hat{Y}_3$ and $\hat{A}_e = \bar{A}e_1^T - e_2^T$, then system (1) is asymptotically stabilized by controller (4) with

$$F = \bar{F}T, \tag{24}$$

where $\bar{F}$ is the matrix satisfying

$$\bar{B}_0\bar{F} = \hat{Y}_0^{-1}\hat{F}_0. \tag{25}$$

Proof: Similar to the proof of Theorem 1, we choose the same Lyapunov functional candidate with state $\bar{x}(t)$ for system (21). Considering the system (21) and the definition of $\hat{Y}_i$, we have

$$\hat{Y}_i\bar{B}\bar{F} = \left[\begin{array}{cc} 0 & (\hat{Y}_0\bar{B}_0\bar{F})^T \end{array}\right]^T, \ i = 1, 2, 3.$$

Setting $\hat{F}_0 = \hat{Y}_0\bar{B}_0\bar{F}$, from the definition of $\hat{F}$, we find that

$$\hat{F} = \hat{Y}_i\bar{B}\bar{F}, \ i = 1, 2, 3$$

which implies

$$\begin{aligned} &\hat{Y}_e\bar{B}\bar{F}e_3^T + e_3\bar{F}^T\bar{B}^T\hat{Y}_e^T \\ &= (e_1 + e_2 + e_3)\hat{F}e_3^T + e_3\hat{F}^T(e_1 + e_2 + e_3)^T. \end{aligned}$$

Following a similar proof procedure of Theorem 1, we know that system (21) with $\bar{F}$ satisfying (25) is asymptotically stable, if the inequalities (22) and (23) hold. Then equivalently, system (1) is asymptotically stabilized by (4) with F given by (24). This ends the proof. ■

For $d_1 = 0$, similar as the proof of Theorem 2, we have the following result.

Corollary 2 Given scalars $d_2 > 0$, $d_1 = 0$ and μ, if there exist matrices $\tilde{P} = [\tilde{P}_{ij}]_{3\times 3} > 0$, $Q > 0$, $R_i > 0$, $S_i > 0$, $Z_i > 0$, $i = 1, 2$, and matrices $\hat{F} = \left[\begin{array}{cc} 0 & \hat{F}_0 \end{array}\right]$ with $\hat{F}_0 \in \mathbf{R}^{r\times m}$, and $\hat{Y}_j = \left[\begin{array}{cc} \hat{Y}_{j,11} & 0 \\ \hat{Y}_{j,21} & \hat{Y}_0 \end{array}\right]$ with $\hat{Y}_0 \in \mathbf{R}^{r\times r}$, $j = 1, 2, 3$ such that the following LMIs hold:

$$\begin{aligned} \tilde{\Omega}_0 &= \tilde{\Gamma}\tilde{P}\tilde{\Upsilon}^T + \tilde{\Upsilon}\tilde{P}\tilde{\Gamma}^T + \tilde{\Psi}_0 + \tilde{\Lambda} + \hat{Y}_{\tilde{e}}\hat{A}_{\tilde{e}} + \hat{A}_{\tilde{e}}^T\hat{Y}_{\tilde{e}}^T \\ &\quad +(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3)\hat{F}\tilde{e}_3^T + \tilde{e}_3\hat{F}^T(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3)^T < 0 \\ \tilde{\Omega}_1 &= \tilde{\Gamma}\tilde{P}\tilde{\Upsilon}^T + \tilde{\Upsilon}\tilde{P}\tilde{\Gamma}^T + \tilde{\Psi}_1 + \tilde{\Lambda} + \hat{Y}_{\tilde{e}}\hat{A}_{\tilde{e}} + \hat{A}_{\tilde{e}}^T\hat{Y}_{\tilde{e}}^T \\ &\quad +(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3)\hat{F}\tilde{e}_3^T + \tilde{e}_3\hat{F}^T(\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3)^T < 0 \end{aligned}$$

where $\hat{Y}_{\tilde{e}} = \tilde{e}_1\hat{Y}_1 + \tilde{e}_2\hat{Y}_2 + \tilde{e}_3\hat{Y}_3$ and $\hat{A}_{\tilde{e}} = \bar{A}\tilde{e}_1^T - \tilde{e}_2^T$, then the system (1) is asymptotically stabilized by controller (4) with $F = \bar{F}T$, where $\bar{F}$ is the matrix satisfying $\bar{B}_0\bar{F} = \hat{Y}_0^{-1}\hat{F}_0$.

Remark 2 It should be noticed that in the stabilization method proposed above different transformation matrix T may lead to different result of Theorem 2, which even makes the inequalities (22) and (23) unsolvable. A significant problem comes out: how to choose the transformation matrix T, making the stabilization problem solvable? In what follows, we will propose a method to find such T.

Suppose that the transformed matrix $\bar{A}$ in system (21) takes the form

$$\bar{A} = \left[\begin{array}{cc} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{array}\right],$$

where $\bar{A}_{11} \in \mathbf{R}^{(n-r)\times(n-r)}$ and $\bar{A}_{22} \in \mathbf{R}^{r\times r}$. Let $\lambda_{min}(A)$ denote the minimum real part of A's eigenvalues. Our purpose is to choose a nonsingular matrix T satisfying (20) such that $\bar{A}_{11}$ is Hurwitz matrix and $-\lambda_{min}(\bar{A}_{11})$ small enough. The following is a procedure to calculate such T.

Step 1: Find a nonsingular T_0 satisfying (20). Then we have the transformed matrices

$$\hat{A} = T_0AT_0^{-1} = \left[\begin{array}{cc} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{array}\right],$$

$$\bar{B} = T_0B = \left[\begin{array}{cc} 0 & \bar{B}_0^T \end{array}\right]^T,$$

where $\hat{A}_{11} \in \mathbf{R}^{(n-r)\times(n-r)}$, $\hat{A}_{22} \in \mathbf{R}^{r\times r}$ and $\bar{B}_0 \in \mathbf{R}^{r\times m}$.

Step 2: To construct $\bar{T}$ satisfying $\bar{T}\bar{B} = \bar{B}$ with the form

$$\bar{T} = \left[\begin{array}{cc} I & 0 \\ \bar{T}_{21} & I \end{array}\right], \ \bar{T}_{21} \in \mathbf{R}^{r\times(n-r)} \tag{26}$$

such that $\bar{A}_{11} = \hat{A}_{11} - \hat{A}_{12}\bar{T}_{21}$, where $\bar{A} = \bar{T}\hat{A}\bar{T}^{-1}$, is Hurwitz matrix and $-\lambda_{min}(\bar{A}_{11})$ small enough. It can be solved if there exist matrices $\Phi^{-1} > 0$ and $\bar{T}_{21}$ satisfy the following inequalities

$$\begin{aligned} &\Phi^{-1}\bar{A}_{11} + \bar{A}_{11}^T\Phi^{-1} < 0 \\ &\Phi^{-1}\bar{A}_{11} + \bar{A}_{11}^T\Phi^{-1} > -\delta\Phi^{-1}, \end{aligned} \tag{27}$$

for given scalar $\delta > 0$ which is chosen small enough. Multiplying Φ on the left and the right side of each inequality in (27) and replacing $\bar{A}_{11}$ with $\hat{A}_{11} - \hat{A}_{12}\bar{T}_{21}$, it is seen to be equivalent to the following LMIs by setting $\hat{T}_{21} = \bar{T}_{21}\Phi$

$$\begin{aligned} &\hat{A}_{11}\Phi + \Phi\hat{A}_{11}^T - \hat{A}_{12}\hat{T}_{21} - \hat{T}_{21}^T\hat{A}_{12}^T < 0 \\ &\hat{A}_{11}\Phi + \Phi\hat{A}_{11}^T - \hat{A}_{12}\hat{T}_{21} - \hat{T}_{21}^T\hat{A}_{12}^T > -\delta\Phi. \end{aligned} \tag{28}$$

Solving the LMIs (28) for matrix variables Φ and $\hat{T}_{21}$, we can get $\bar{T}_{21}$ in (26) by $\bar{T}_{21} = \hat{T}_{21}\Phi^{-1}$.

Step 3: Obtain the transformation matrix $T = \bar{T}T_0$, where $\bar{T}$ with $\bar{T}_{21}$ given by (26) in Step 2.

V. Numerical Examples

In this section, two examples are given to demonstrate the effectiveness of the method proposed in this paper.

Example 1 Consider the system (5) with

$$A = \left[\begin{array}{cc} -2 & 0 \\ 0 & -0.9 \end{array}\right], \quad A_d = \left[\begin{array}{cc} -1 & 0 \\ -1 & -1 \end{array}\right].$$

For various μ and unknown μ, the allowable upper bounds, d_2, which guarantee the asymptotic stability of system (5) for given lower bounds, d_1, are listed in Tables I and II, respectively. From Table I and Table II, it can be seen that the stability results obtained in the paper are less conservative than those in [12].

Example 2 Consider the following system:

$$\dot{x}(t) = \left[\begin{array}{cc} -0.8 & -0.01 \\ 1 & 0.1 \end{array}\right]x(t) + \left[\begin{array}{c} 0.4 \\ 0.1 \end{array}\right]u(t - d(t)), \tag{29}$$

where $d(t)$ satisfies (2) with $d_1 = 0$ and μ unknown. Choosing

$$T_0 = \left[\begin{array}{cc} 1 & -4 \\ 0 & 10 \end{array}\right]$$

TABLE I

ALLOWABLE UPPER BOUND d_2 WITH GIVEN d_1 FOR DIFFERENT μ

d_1	Methods	$\mu = 0.1$	$\mu = 0.3$	$\mu = 0.5$	$\mu = 0.9$
1	[12]	4.1945	3.0538	2.3058	1.9008
	Theorem 1	4.3923	3.1208	2.3418	2.0921
2	[12]	4.4932	3.0129	2.5663	2.5663
	Theorem 1	4.5705	3.0989	2.6987	2.6987
3	[12]	4.3979	3.3408	3.3408	3.3408
	Theorem 1	4.5400	3.4186	3.4186	3.4186
4	[12]	4.1978	4.1690	4.1690	4.1690
	Theorem 1	4.2305	4.2097	4.2097	4.2097
5	[12]	5.0275	5.0275	5.0275	5.0275
	Theorem 1	5.0440	5.0440	5.0440	5.0440

TABLE II

ALLOWABLE UPPER BOUND d_2 WITH GIVEN d_1 FOR UNKNOW μ

Methods	d_1	1	2	3	4	5
[12]	d_2	1.9008	2.5663	3.3408	4.1690	5.0275
Remark 1	d_2	2.0921	2.6987	3.4186	4.2097	5.0440

in Step 1, for different δ, the maximum upper bounds d_2 and corresponding matrices F, such that system (29) with controller (4) is asymptotically stable, are listed in Table III. As shown in the table, it can be seen that the value of δ has remarkable influence on the allowable upper bounds and $\lambda_{max}(A + BF)$, which reveal the fact that there exists a trade-off between upper bounds and $\lambda_{max}(A + BF)$ when designing the stabilizing controller.

TABLE III

ALLOWABLE UPPER BOUND d_2 AND F FOR DIFFERENT δ

δ	d_2	F	$\lambda_{max}(A + BF)$
0.01	1.6937	$[-1.2988 \ \ -0.2619]$	-0.0020
0.1	1.6873	$[-1.2431 \ \ -0.2864]$	-0.0136
1	1.6102	$[-1.0149 \ \ -0.3544]$	-0.0538
2	1.4596	$[-0.9078 \ \ -0.5088]$	-0.1408
3	1.2566	$[-0.9174 \ \ -0.7710]$	-0.3175
4	1.0614	$[-0.9662 \ \ -1.0738]$	-0.5969
5	0.9190	$[-1.4069 \ \ -1.1046]$	-0.4241
7	0.7495	$[-1.9782 \ \ -1.2715]$	-0.3669
10	0.6099	$[-2.4651 \ \ -1.5993]$	-0.4177

For $\delta = 4.3$ and $d_2 = 1$, we obtain $F = \begin{bmatrix} -1.1049 & -1.0532 \end{bmatrix}$. Taking $d(t) = 0.5sin(t) + 0.5$ and initial state $x(0) = [1 \ 1]^T$, the simulation result of the resulting closed-loop system is given in Fig.1, which well show the feasibility of the design procedure proposed above.

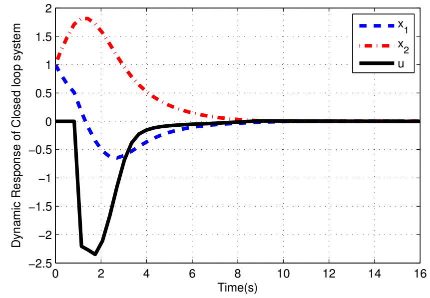

Fig. 1. Response of the closed-loop system

VI. CONCLUSION

We have developed new results for delay-dependent stability and stabilization for time-delay systems. The delay considered in this paper may vary in a range for which the lower bound is not restricted to be zero. With a different Lyapunov functional defined, some new delay-dependent stability criteria have been derived. By using a new state transformation method, the cross terms in stability criteria can be dealt with. Therefor, the criteria can be used to solve the problem of stabilizing the linear systems with time-varying input delay. All the developed results are formulated as LMIs. Numerical examples well illustrate the design procedure and the criterion is less conservative than existing ones.

REFERENCES

[1] S. Boyd, L. EL Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelpha: SIAM, 1994.

[2] L. Dugard and E. I. Verriest, *Stability and Control of Time-delay Systems*, London: Springer, 1998.

[3] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of time-delay systems*, Boston: Birkhauser, 2003.

[4] K. Gu and S. I. Niculescu, "Survey on recent results in the stability and control of time-delay systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 124, pp. 158-165, 2003.

[5] J. K. Hale and S. M. Verduyn Lunel, *Introduction of functional differential equations*, New York: Springer, 1993.

[6] Q. L. Han and K. Gu, "Stability of linear systems with time-varying delay: A generalized discretized Lyapunov functional approach," *Asian Journal of Control*, vol. 3, pp. 170-180, 2001.

[7] Y. He, Q. Wang, C. Lin, and M. Wu, "Delay-range-dependent stability for systems with time-varying delay," *Automatica*, vol. 43, pp. 371-376, 2007.

[8] Z. Lin and H. Fang, "On asymptotic stabilizability of linear systems with delayed input," *IEEE Transactions Automatic Control*, vol. 52, no. 6, pp. 998-1013, 2007.

[9] M. S. Mahmoud and A. Ismail, "New results on delay-dependent control of time-delay systems," *IEEE Transactions Automatic Control*, vol. 50, no. 1, pp. 95-100, 2005.

[10] J. P. Richard, "Time-delay systems:an overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667-1694, 2003.

[11] H. Y. Shao, "New delay-dependent stability criteria for systems with interval delay," *Automatica*, vol. 45, pp. 744-749, 2009.

[12] J. Sun, G. P. Liu, J. Chen, and D. Reers, "Improved delay-range-dependent stability criteria for linear systems with time-varying delays," *Automatica*, vol. 46, pp. 466-470, 2010.

[13] S. Xu and J. Lam, "A survey of linear matrix inequality techniques in stability analysis of delay systems," *International Journal of Systems Science*, vol. 39, no. 12, pp. 1095-1113, 2008.

[14] X. M. Zhang, M. Wu, J. H. She, and Y. He, "Delay-dependent stabilization of linear systems with time-varying state and input delays," *Automatica*, vol. 41, pp. 1405-1412, 2005.