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Iterative solutions for general coupled matrix equations with real coefficients

Li Xie, Huizhong Yang, Yanjun Liu, Feng Ding

Abstract— **This paper applies the hierarchical identification principle and the gradient search method to study iterative solutions for a class of general coupled matrix equations with real coefficients. As long as the convergence factors are appropriately chosen, the proposed algorithms for any initial values can provide iterative solutions that are arbitrarily close to the unique solutions of the equations. Two numerical examples are given to demonstrate the effectiveness of the proposed algorithms.**

Index Terms: Coupled matrix equations; hierarchical identification; gradient search; iterative algorithm; estimation.

I. Introduction

Throughout the paper, the transpose and the trace of matrix X are denoted by X^{T} and $\mathrm{tr}[X]$, respectively, and the norm is defined as $\|X\|^2 = \mathrm{tr}[X^{\mathrm{T}}X]$. The symbol I_n stands for an identity matrix of size $n\times n$, $\mathbf{0}$ is an zero matrix with appropriate dimension. For two matrices A and B, $A\otimes B$ is their Kronecker product; and for matrix $X=[x_1,x_2,\cdots,x_n]\in\mathbb{R}^{m\times n}$, $x_i\in\mathbb{R}^m$, the vector operator is defined as $\mathrm{col}[X]=[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\cdots,x_n^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{mn}$. Furthermore, we have $\mathrm{col}[X^{\mathrm{T}}]=P_{mn}\mathrm{col}[X]$, where P_{mn} is a square $mn\times mn$ matrix partitioned into $m\times n$ submatrices such that the (i,j)th submatrix has a 1 in its (j,i)th position and zeros elsewhere.

Matrix equations have received much attention because of their important applications in control theory, signal processing, filtering and many other fields [1]–[4]. For example, the Sylvester matrix equations can be used for pole assignment, feedback design and fault detection [5], [6]; and the coupled Lyapunov matrix equations are encountered in stability analysis of linear jump systems with Markovian transitions [7]. There exist numerous methods for solving matrix equations, in which the iterative ones are especially computational efficient [8]–[10]. In this literature, Dehghan and Hajarian introduced three iterative algorithms to obtain the reflexive and anti-reflexive solutions for the matrix equations $A_1X_1B_1+A_2X_2B_2=C$ [11]; Zhou et al. developed a gradient based iterative algorithm to find the weighted least squares solutions for general coupled Sylvester matrix equations [12]; Wu et al. considered iterative solutions for a class of general complex matrix equations with the conjugate and transpose of the unknowns, and demonstrated that the proposed algorithm can converge to the exact solutions within finite iteration steps [13].

Recently, by applying the hierarchical identification principle, Ding, Liu and Ding presented a gradient based and a least squares based iterative algorithms for solving the generalized Sylvester matrix equations, which regarded the unknown matrices as the parameter matrices to be identified [14]. Such methods are also developed to solve the extended Sylvester-conjugate matrix equations [15], general linear matrix equations $\sum_{i=1}^{p}A_iXB_i=F$ [16] and $\sum_{i=1}^{p}A_iXB_i+\sum_{i=1}^{q}C_iX^{\mathrm{T}}D_i=F$ [17], and general coupled matrix equations $\sum_{j=1}^{p}A_{ij}X_jB_{ij}=F_i,\ i=1,2,\cdots,p$ [18], [19]. Based on the work mentioned above, solutions for more general coupled matrix equations with real coefficients are considered in this paper, and the gradient based iterative algorithms are derived by using the hierarchical identification principle and the gradient search method.

The iterative algorithms are related to recursive estimation algorithms in system identification, e.g., the multi-innovation identification methods [20]–[32] and the iterative identification methods [33]–[37].

The rest of this paper is organized as follows. Section II presents a gradient based iterative algorithm to solve a simple form of coupled matrix equations with real coefficients; and Section III develops the algorithm for more general cases. Section IV provides two examples to demonstrate the effectiveness of the proposed algorithms. Finally, we end the paper with some concluding remarks in Section V.

II. Simple Coupled Matrix Equations

This section concentrates on solutions for the coupled matrix equations

$$\begin{cases} A_1X+X^{\mathrm{T}}B_1+C_1Y+Y^{\mathrm{T}}D_1=F_1 \\ A_2X+X^{\mathrm{T}}B_2+C_2Y+Y^{\mathrm{T}}D_2=F_2 \end{cases}, \tag{1}$$

where $A_1,\ A_2,\ C_1,\ C_2\in\mathbb{R}^{n\times m}$, $B_1,\ B_2,\ D_1,\ D_2\in\mathbb{R}^{m\times n}$, and $F_1,\ F_2\in\mathbb{R}^{n\times n}$ are given constant matrices, $X,\ Y\in\mathbb{R}^{m\times n}$ are two unknown matrices to be solved.

By using the vector operator, equation (1) can be converted into the following equivalent form,

$$S\ \mathrm{col}[X,Y]=\mathrm{col}[F_1,F_2],$$

$$S:=\begin{bmatrix} I_n\otimes A_1+(B_1^{\mathrm{T}}\otimes I_n)P_{nm} & I_n\otimes C_1+(D_1^{\mathrm{T}}\otimes I_n)P_{nm} \\ I_n\otimes A_2+(B_2^{\mathrm{T}}\otimes I_n)P_{nm} & I_n\otimes C_2+(D_2^{\mathrm{T}}\otimes I_n)P_{nm} \end{bmatrix},$$

$$P_{nm}=P_{mn}^{-1}=P_{mn}^{\mathrm{T}},$$

$$\mathrm{col}[X,Y]:=\begin{bmatrix}\mathrm{col}[X]\\ \mathrm{col}[Y]\end{bmatrix},\quad \mathrm{col}[F_1,F_2]:=\begin{bmatrix}\mathrm{col}[F_1]\\ \mathrm{col}[F_2]\end{bmatrix}.$$

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The authors are with the Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), and the Control Science and Engineering Research Center, Jiangnan University, Wuxi 214122, P.R. China. xieli2412@126.com (L. Xie); yhz@jiangnan.edu.cn (H.Z. Yang); yanjunliu_1983@126.com (Y.J. Liu); fding@jiangnan.edu.cn (F. Ding).

Then the exact solutions of equation (1) can be given by the following lemma.

Lemma 1: Equation (1) has unique solutions if and only if $\mathrm{rank}\{S, \mathrm{col}[F_1, F_2]\} = \mathrm{rank}[S] = mn$ (i.e., S has a full column rank). In this case, the unique solutions can be given by

$$\mathrm{col}[X,Y] = (S^{\mathrm{T}}S)^{-1}S^{\mathrm{T}}\mathrm{col}[F_1,F_2], \tag{2}$$

and the corresponding homogeneous coupled matrix equation in (1) with $F_1 = \mathbf{0}$, $F_2 = \mathbf{0}$ has unique solutions $X = Y = \mathbf{0}$.

However, when the dimensions of X and Y become large, computing the solutions in (2) requires excessive computer memory and the resulted computational cost is high. This motivates us to study the iterative algorithm to solve (1).

Based on the hierarchical identification principle, regarding the unknown matrices X and Y in (1) as the parameter matrices to be identified, we define the following matrices,

$$Q_1 := \begin{bmatrix} F_1 - X^{\mathrm{T}}B_1 - C_1Y - Y^{\mathrm{T}}D_1 \\ F_2 - X^{\mathrm{T}}B_2 - C_2Y - Y^{\mathrm{T}}D_2 \end{bmatrix}, \tag{3}$$

$$Q_2 := [F_1 - A_1X - C_1Y - Y^{\mathrm{T}}D_1,\ F_2 - A_2X - C_2Y - Y^{\mathrm{T}}D_2], \tag{4}$$

$$Q_3 := \begin{bmatrix} F_1 - A_1X - X^{\mathrm{T}}B_1 - Y^{\mathrm{T}}D_1 \\ F_2 - A_2X - X^{\mathrm{T}}B_2 - Y^{\mathrm{T}}D_2 \end{bmatrix}, \tag{5}$$

$$Q_4 := [F_1 - A_1X - X^{\mathrm{T}}B_1 - C_1Y,\ F_2 - A_2X - X^{\mathrm{T}}B_2 - C_2Y]. \tag{6}$$

Then, from (1), we obtain four fictitious subsystems,

$$\mathrm{Sub1}: \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X = Q_1, \quad \mathrm{Sub2}: \quad X^{\mathrm{T}}[B_1, B_2] = Q_2,$$

$$\mathrm{Sub3}: \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} Y = Q_3, \quad \mathrm{Sub4}: \quad Y^{\mathrm{T}}[D_1, D_2] = Q_4.$$

Applying the gradient search method [16], [19] to the above four subsystems leads to the following iterative equations:

$$X(k) = X(k-1) + \mu \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{\mathrm{T}} \left\{ Q_1 - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X(k-1) \right\}, \tag{7}$$

$$X(k) = X(k-1) + \mu[B_1, B_2]\{Q_2 - X^{\mathrm{T}}(k-1)[B_1, B_2]\}^{\mathrm{T}}, \tag{8}$$

$$Y(k) = Y(k-1) + \mu \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^{\mathrm{T}} \left\{ Q_3 - \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} Y(k-1) \right\}, \tag{9}$$

$$Y(k) = Y(k-1) + \mu[D_1, D_2]\{Q_4 - Y^{\mathrm{T}}(k-1)[D_1, D_2]\}^{\mathrm{T}}, \tag{10}$$

where $\mu > 0$ is the iterative step size or convergence factor to be given later. Substituting (3)-(6) into (7)-(10), respectively, and replacing the unknown matrices X and Y with their estimates $X(k-1)$ and $Y(k-1)$ yield

$$X(k) = X(k-1) + \mu \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} F_1 - X^{\mathrm{T}}(k-1)B_1 - C_1Y(k-1) \\ F_2 - X^{\mathrm{T}}(k-1)B_2 - C_2Y(k-1) \end{bmatrix} - \begin{bmatrix} Y^{\mathrm{T}}(k-1)D_1 + A_1X(k-1) \\ Y^{\mathrm{T}}(k-1)D_2 + A_2X(k-1) \end{bmatrix} \right\}, \tag{11}$$

$$\begin{aligned} X(k) = {} & X(k-1) + \mu[B_1, B_2] \\ & \times [F_1 - A_1X(k-1) - C_1Y(k-1) - Y^{\mathrm{T}}(k-1)D_1 - X^{\mathrm{T}}(k-1)B_1, \\ & \quad F_2 - A_2X(k-1) - C_2Y(k-1) - Y^{\mathrm{T}}(k-1)D_2 - X^{\mathrm{T}}(k-1)B_2]^{\mathrm{T}}, \end{aligned} \tag{12}$$

$$Y(k) = Y(k-1) + \mu \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} F_1 - A_1X(k-1) - X^{\mathrm{T}}(k-1)B_1 \\ F_2 - A_2X(k-1) - X^{\mathrm{T}}(k-1)B_2 \end{bmatrix} - \begin{bmatrix} Y^{\mathrm{T}}(k-1)D_1 + C_1Y(k-1) \\ Y^{\mathrm{T}}(k-1)D_2 + C_2Y(k-1) \end{bmatrix} \right\}, \tag{13}$$

$$\begin{aligned} Y(k) = {} & Y(k-1) + \mu[D_1, D_2] \\ & \times [F_1 - A_1X(k-1) - X^{\mathrm{T}}(k-1)B_1 - C_1Y(k-1) - Y^{\mathrm{T}}(k-1)D_1, \\ & \quad F_2 - A_2X(k-1) - X^{\mathrm{T}}(k-1)B_2 - C_2Y(k-1) - Y^{\mathrm{T}}(k-1)D_2]^{\mathrm{T}}. \end{aligned} \tag{14}$$

Taking the average of (11) and (12) as the iterative solution $X(k)$, and the average of (13) and (14) as the iterative solution $Y(k)$, we obtain a gradient based iterative algorithm for the solutions of (1):

$$X(k) = \frac{X_a(k) + X_b(k)}{2}, \tag{15}$$

$$X_a(k) = X(k-1) + \mu \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta F_1(k) \\ \Delta F_2(k) \end{bmatrix}, \tag{16}$$

$$X_b(k) = X(k-1) + \mu[B_1, B_2][\Delta F_1(k), \Delta F_2(k)]^{\mathrm{T}}, \tag{17}$$

$$Y(k) = \frac{Y_a(k) + Y_b(k)}{2}, \tag{18}$$

$$Y_a(k) = Y(k-1) + \mu \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta F_1(k) \\ \Delta F_2(k) \end{bmatrix}, \tag{19}$$

$$Y_b(k) = Y(k-1) + \mu[D_1, D_2][\Delta F_1(k), \Delta F_2(k)]^{\mathrm{T}}, \tag{20}$$

$$\begin{aligned} \Delta F_i(k) = {} & F_i - A_iX(k-1) - X^{\mathrm{T}}(k-1)B_i - C_iY(k-1) \\ & - Y^{\mathrm{T}}(k-1)D_i,\ i = 1, 2. \end{aligned} \tag{21}$$

The convergence factor μ can be simply taken to satisfy

$$\begin{aligned} 0 < \mu < 2\big\{ & \|A_1\|^2 + \|B_1\|^2 + \|C_1\|^2 + \|D_1\|^2 \\ & + \|A_2\|^2 + \|B_2\|^2 + \|C_2\|^2 + \|D_2\|^2 \big\}^{-1}. \end{aligned} \tag{22}$$

To initialize the algorithm, we take $X(0)$ and $Y(0)$ as some small real matrices, e.g., $X(0) = Y(0) = 10^{-6}\mathbf{1}_{m\times n}$ with $\mathbf{1}_{m\times n}$ being an $m \times n$ matrix whose elements are all 1.

Theorem 1: If the equation in (1) has unique solutions X and Y, then for any initial values, the iterative solutions $X(k)$ and $Y(k)$ given by the algorithm in (15)-(22) converge to the true solutions X and Y, i.e.,

$$\lim_{k\to\infty} X(k) = X, \quad \lim_{k\to\infty} Y(k) = Y.$$

III. General Coupled Matrix Equations

Consider the following more general coupled matrix equations with transpose of p unknown matrices,

$$\begin{cases} \sum\limits_{j=1}^{p} [A_{1j}X_jB_{1j} + C_{1j}X_j^{\mathrm{T}}D_{1j}] = F_1, \\ \sum\limits_{j=1}^{p} [A_{2j}X_jB_{2j} + C_{2j}X_j^{\mathrm{T}}D_{2j}] = F_2, \\ \quad \vdots \\ \sum\limits_{j=1}^{p} [A_{pj}X_jB_{pj} + C_{pj}X_j^{\mathrm{T}}D_{pj}] = F_p, \end{cases} \tag{23}$$

where $A_{ij} \in \mathbb{R}^{r\times m}$, $B_{ij} \in \mathbb{R}^{n\times s}$, $C_{ij} \in \mathbb{R}^{r\times n}$, $D_{ij} \in \mathbb{R}^{m\times s}$, and $F_i \in \mathbb{R}^{r\times s}$ are given constant matrices, $X_j \in \mathbb{R}^{m\times n}$, $j = 1, 2, \cdots, p$, are the unknown matrices to be determined.

In order to simplify the representation of the gradient based iterative algorithm to be proposed later, we use the block-matrix star product, denoted by notation $\star$ in [18], [19].

Let

$$X=\begin{bmatrix}X_1\\X_2\\\vdots\\X_p\end{bmatrix},\ Y=\begin{bmatrix}Y_1\\Y_2\\\vdots\\Y_p\end{bmatrix},\ X_i,Y_i^{\mathrm{T}}\in\mathbb{R}^{m\times n},$$

$$S_A=[A_{ij}]_{p\times p}=\begin{bmatrix}A_{11}&A_{12}&\cdots&A_{1p}\\A_{21}&A_{22}&\cdots&A_{2p}\\\vdots&\vdots&&\vdots\\A_{p1}&A_{p2}&\cdots&A_{pp}\end{bmatrix},$$

$$S_{A^{\mathrm{T}}}=[A_{ij}^{\mathrm{T}}]_{p\times p}\begin{bmatrix}A_{11}^{\mathrm{T}}&A_{12}^{\mathrm{T}}&\cdots&A_{1p}^{\mathrm{T}}\\A_{21}^{\mathrm{T}}&A_{22}^{\mathrm{T}}&\cdots&A_{2p}^{\mathrm{T}}\\\vdots&\vdots&&\vdots\\A_{p1}^{\mathrm{T}}&A_{p2}^{\mathrm{T}}&\cdots&A_{pp}^{\mathrm{T}}\end{bmatrix},$$

$$S_B=[B_{ij}]_{p\times p},\quad S_{B^{\mathrm{T}}}=[B_{ij}^{\mathrm{T}}]_{p\times p},$$
$$S_C=[C_{ij}]_{p\times p},\quad S_{C^{\mathrm{T}}}=[C_{ij}^{\mathrm{T}}]_{p\times p},$$
$$S_D=[D_{ij}]_{p\times p},\quad S_{D^{\mathrm{T}}}=[D_{ij}^{\mathrm{T}}]_{p\times p},$$
$$S_p=[B_{ij}^{\mathrm{T}}\otimes A_{ij}+(D_{ij}^{\mathrm{T}}\otimes C_{ij})P_{nm}]_{p\times p}.$$

Then the block-matrix star product is defined as

$$X\star Y=\begin{bmatrix}X_1\\X_2\\\vdots\\X_p\end{bmatrix}\star\begin{bmatrix}Y_1\\Y_2\\\vdots\\Y_p\end{bmatrix}=\begin{bmatrix}X_1Y_1\\X_2Y_2\\\vdots\\X_pY_p\end{bmatrix},$$

$$S_A\star X=\begin{bmatrix}A_{11}X_1&A_{12}X_2&\cdots&A_{1p}X_p\\A_{21}X_1&A_{22}X_2&\cdots&A_{2p}X_p\\\vdots&\vdots&&\vdots\\A_{p1}X_1&A_{p2}X_2&\cdots&A_{pp}X_p\end{bmatrix},$$

$$S_A\star S_B=\begin{bmatrix}A_{11}B_{11}&A_{12}B_{12}&\cdots&A_{1p}B_{1p}\\A_{21}B_{21}&A_{22}B_{22}&\cdots&A_{2p}B_{2p}\\\vdots&\vdots&&\vdots\\A_{21}B_{p1}&A_{p2}B_{p2}&\cdots&A_{pp}B_{pp}\end{bmatrix}.$$

The star product is superior to matrix multiplication, thus $AB\star C=A(B\star C)\neq(AB)\star C$. Furthermore, the following properties exist:

$$\mathrm{tr}\left\{X_j^{\mathrm{T}}\begin{bmatrix}A_{1j}\\A_{2j}\\\vdots\\A_{pj}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{F}_1\\\tilde{F}_2\\\vdots\\\tilde{F}_p\end{bmatrix}\star\begin{bmatrix}B_{1j}^{\mathrm{T}}\\B_{2j}^{\mathrm{T}}\\\vdots\\B_{pj}^{\mathrm{T}}\end{bmatrix}\right\}=\mathrm{tr}\left\{\begin{bmatrix}A_{1j}X_jB_{1j}\\A_{2j}X_jB_{2j}\\\vdots\\A_{pj}X_jB_{pj}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{F}_1\\\tilde{F}_2\\\vdots\\\tilde{F}_p\end{bmatrix}\right\},$$

$$\left\|\begin{bmatrix}A_{1j}\\A_{2j}\\\vdots\\A_{pj}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{F}_1\\\tilde{F}_2\\\vdots\\\tilde{F}_p\end{bmatrix}\star\begin{bmatrix}B_{1j}^{\mathrm{T}}\\B_{2j}^{\mathrm{T}}\\\vdots\\B_{pj}^{\mathrm{T}}\end{bmatrix}\right\|^2\leq\sum_{i=1}^{p}\|A_{ij}\|^2\|B_{ij}\|^2\left\|\begin{bmatrix}\tilde{F}_1\\\tilde{F}_2\\\vdots\\\tilde{F}_p\end{bmatrix}\right\|^2.$$

Lemma 2: Provided that the matrix S_p is full column rank, equation (23) has unique solutions and can be given by

$$\mathrm{col}[X_1,X_2,\cdots,X_p]=(S_p^{\mathrm{T}}S_p)^{-1}\mathrm{col}[F_1,F_2,\cdots,F_p],\qquad(24)$$

the corresponding homogeneous matrix equation in (23) has unique solutions $X_j=\mathbf{0},\ j=1,2,\cdots,p$.

Although (24) can be used to obtain the exact solutions of the coupled matrix equations in (23), it requires excessive computer memory because of computing the inversion of the large matrix $S_p^{\mathrm{T}}S_p$ of size $(mnp)\times(mnp)$ as the dimension of X_j increases. Thus, we need to seek an alternative way to study the iterative solutions for (23), the details are as follows.

According to the hierarchical identification principle, regard the unknown matrices $X_j,\ j=1,2,\cdots,p$, as the parameter matrices to be identified and decompose equation (23) into p subsystems,

$$\begin{cases}A_{1j}X_jB_{1j}=F_1-\sum\limits_{i=1,i\neq j}^{p}[A_{1i}X_iB_{1i}+C_{1i}X_i^{\mathrm{T}}D_{1i}]-C_{1j}X_j^{\mathrm{T}}D_{1j},\\A_{2j}X_jB_{2j}=F_2-\sum\limits_{i=1,i\neq j}^{p}[A_{2i}X_iB_{2i}+C_{2i}X_i^{\mathrm{T}}D_{2i}]-C_{2j}X_j^{\mathrm{T}}D_{2j},\\\quad\vdots\\A_{pj}X_jB_{pj}=F_p-\sum\limits_{i=1,i\neq j}^{p}[A_{pi}X_iB_{pi}+C_{pi}X_i^{\mathrm{T}}D_{pi}]-C_{pj}X_j^{\mathrm{T}}D_{pj}.\end{cases}\qquad(25)$$

Let

$$\Delta F_i(k):=F_i-\sum_{j=1}^{p}[A_{ij}X_j(k-1)B_{ij}+C_{ij}X_j^{\mathrm{T}}(k-1)D_{ij}].\qquad(26)$$

Applying the gradient search method to solve (25) gives

$$X_j(k)=X_j(k-1)+\mu\begin{bmatrix}A_{1j}\\A_{2j}\\\vdots\\A_{pj}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Delta F_1(k)\\\Delta F_2(k)\\\vdots\\\Delta F_p(k)\end{bmatrix}\star[B_{1j},B_{2j},\cdots,B_{pj}]^{\mathrm{T}},\quad j=1,2,\cdots,p.\qquad(27)$$

where $\mu>0$ is the iterative step size or convergence factor to be given later.

Similarly, From (23) we have

$$\begin{cases}C_{1j}X_j^{\mathrm{T}}D_{1j}=F_1-\sum\limits_{i=1,i\neq j}^{p}[A_{1i}X_iB_{1i}+C_{1i}X_i^{\mathrm{T}}D_{1i}]-A_{1j}X_jB_{1j},\\C_{2j}X_j^{\mathrm{T}}D_{2j}=F_2-\sum\limits_{i=1,i\neq j}^{p}[A_{2i}X_iB_{2i}+C_{2i}X_i^{\mathrm{T}}D_{2i}]-A_{2j}X_jB_{2j},\\\quad\vdots\\C_{pj}X_j^{\mathrm{T}}D_{pj}=F_p-\sum\limits_{i=1,i\neq j}^{p}[A_{pi}X_iB_{pi}+C_{pi}X_i^{\mathrm{T}}D_{pi}]-A_{pj}X_jB_{pj}.\end{cases}$$

Using the gradient search method gives

$$X_j(k)=X_j(k-1)+\mu\begin{bmatrix}D_{1j}^{\mathrm{T}}\\D_{2j}^{\mathrm{T}}\\\vdots\\D_{pj}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Delta F_1^{\mathrm{T}}(k)\\\Delta F_2^{\mathrm{T}}(k)\\\vdots\\\Delta F_p^{\mathrm{T}}(k)\end{bmatrix}\star[C_{1j}^{\mathrm{T}},C_{2j}^{\mathrm{T}},\cdots,C_{pj}^{\mathrm{T}}]^{\mathrm{T}},\quad j=1,2,\cdots,p.\qquad(28)$$

Taking the average of equations (27) and (28), we propose the following gradient based iterative algorithm to compute the solutions $X_j(k)$ for (23):

$$X_{ja}(k) = X_j(k-1) + \mu \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{pj} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta F_1(k) \\ \Delta F_2(k) \\ \vdots \\ \Delta F_p(k) \end{bmatrix} \star [B_{1j}, B_{2j}, \cdots, B_{pj}]^{\mathrm{T}}, \quad (29)$$

$$X_{jb}(k) = X_j(k-1) + \mu \begin{bmatrix} D_{1j}^{\mathrm{T}} \\ D_{2j}^{\mathrm{T}} \\ \vdots \\ D_{pj}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta F_1^{\mathrm{T}}(k) \\ \Delta F_2^{\mathrm{T}}(k) \\ \vdots \\ \Delta F_p^{\mathrm{T}}(k) \end{bmatrix} \star [C_{1j}^{\mathrm{T}}, C_{2j}^{\mathrm{T}}, \cdots, C_{pj}^{\mathrm{T}}]^{\mathrm{T}}, \quad (30)$$

$$X_j(k) = \frac{X_{ja}(k) + X_{jb}(k)}{2}, \quad (31)$$

$$0 < \mu < 2 \left\{ \sum_{i=1}^{p} \sum_{j=1}^{p} \|A_{ij}\|^2 \|B_{ij}\|^2 + \|C_{ij}\|^2 \|D_{ij}\|^2 \right\}^{-1}. \quad (32)$$

Theorem 2: If the coupled matrix equation in (23) has unique solutions X_j, $j = 1, 2, \cdots, p$, then the iterative solutions $X_j(k)$ given by the algorithm in (29)-(32) converge to the true solutions X_j for any initial values $X_j(0)$, i.e., $\lim_{k\to\infty} X_j(k) = X_j$; in other words, the error matrices $X_j(k) - X_j$ converge to zero when k is infinite.

Considering the space restrictions, the proofs of *Theorem 1* and *2* are omitted here, but they can be derived similarly as in [18], [19].

Let

$$X(k) := \begin{bmatrix} X_1(k) \\ X_2(k) \\ \vdots \\ X_p(k) \end{bmatrix}, \; F := \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_p \end{bmatrix}, \; \Delta F(k) := \begin{bmatrix} \Delta F_1(k) \\ \Delta F_2(k) \\ \vdots \\ \Delta F_p(k) \end{bmatrix},$$

$$X_a(k) := \begin{bmatrix} X_{1a}(k) \\ X_{2a}(k) \\ \vdots \\ X_{pa}(k) \end{bmatrix}, \; X_b(k) := \begin{bmatrix} X_{1b}(k) \\ X_{2b}(k) \\ \vdots \\ X_{pb}(k) \end{bmatrix},$$

$$X^{\mathrm{H}}(k) := [X_1(k), X_2(k), \cdots, X_p(k)]^{\mathrm{T}},$$
$$F^{\mathrm{H}} := [F_1, F_2, \cdots, F_p]^{\mathrm{T}},$$
$$\Delta F^{\mathrm{H}}(k) := [\Delta F_1(k), \Delta F_2(k), \cdots, \Delta F_p(k)]^{\mathrm{T}},$$
$$I_{np\times n} := [I_n, I_n, \cdots, I_n]^{\mathrm{T}}.$$

By using the star product properties, a more compact form of the gradient based iterative algorithm in (29)-(31) can be written as

$$\begin{aligned} X_a(k) &= X(k-1) + \mu S_A^{\mathrm{T}} \star \Big\{ F - [S_A \star X(k-1) \star S_B \\ &\quad + S_C \star X^{\mathrm{H}}(k-1) \star S_D] I_{np\times n} \Big\} \star S_B^{\mathrm{T}} I_{np\times n} \\ &= X(k-1) + \mu S_A^{\mathrm{T}} \star \Delta F(k) \star S_B^{\mathrm{T}} I_{np\times n}, \quad (33) \end{aligned}$$

$$\begin{aligned} X_b(k) &= X(k-1) + \mu S_{D^{\mathrm{T}}}^{\mathrm{T}} \star \Big\{ F^{\mathrm{H}} - (S_{B^{\mathrm{T}}} \star X^{\mathrm{H}}(k-1) \star S_{A^{\mathrm{T}}} \\ &\quad + S_{D^{\mathrm{T}}} \star X(k-1) \star S_{C^{\mathrm{T}}}) I_{np\times n} \Big\} \star S_{C^{\mathrm{T}}}^{\mathrm{T}} I_{np\times n} \\ &= X(k-1) + \mu S_{D^{\mathrm{T}}}^{\mathrm{T}} \star \Delta F^{\mathrm{H}}(k) \star S_{C^{\mathrm{T}}}^{\mathrm{T}} I_{np\times n}, \quad (34) \end{aligned}$$

$$X(k) = \frac{X_a(k) + X_b(k)}{2}. \quad (35)$$

IV. Numerical Examples

In this section, we provide two examples to validate the effectiveness of the proposed algorithms.

Example 1 Consider the coupled matrix equation in (1) with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 4 & 1 \\ -3 & 5 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 5 & 0 \\ -4 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 3 & -1 \\ 2 & -2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}, \; D_2 = \begin{bmatrix} 3 & 1 \\ -5 & -1 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 0 & 40 \\ -11 & 3 \end{bmatrix}, \; F_2 = \begin{bmatrix} 20 & 13 \\ 8 & -9 \end{bmatrix}.$$

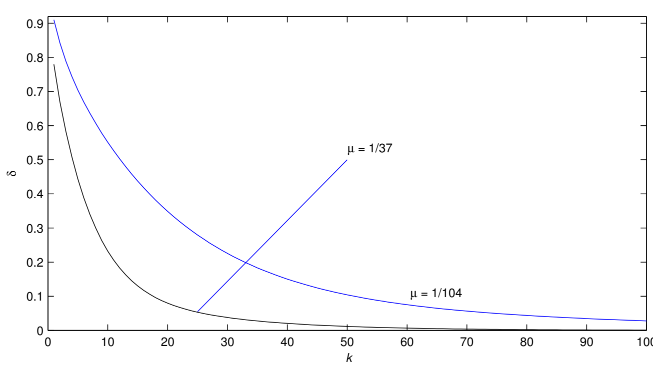
Using Lemma 2, the unique solutions of this equation can be given by

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 8 \\ -1 & 5 \end{bmatrix}.$$

Taking $X(0) = Y(0) = 10^{-6}\mathbf{1}_{2\times 2}$ and applying the gradient based iterative algorithm in (15)-(22) to compute $X(k)$ and $Y(k)$, the iterative errors $\delta(k) := \sqrt{(\|X(k)-X\|^2 + \|Y(k)-Y\|^2)/(\|X\|^2 + \|Y\|^2)}$ versus k with different convergence factor μ is illustrated in Fig. 1. Specially, when $\mu = 1/37$ and $k = 100$, the iterative results are

$$X(100) = \begin{bmatrix} 0.99991 & 1.99988 \\ 3.00406 & 4.00039 \end{bmatrix},$$
$$Y(100) = \begin{bmatrix} 2.00154 & 7.99689 \\ -0.99511 & 4.99945 \end{bmatrix},$$

and $\delta(100) = 6.53519 \times 10^{-4}$.



$$2\{\|A_1\|^2 + \|B_1\|^2 + \|C_1\|^2 + \|D_1\|^2 + \|A_2\|^2 + \|B_2\|^2 + \|C_2\|^2 + \|D_2\|^2\}^{-1} = 1/104$$

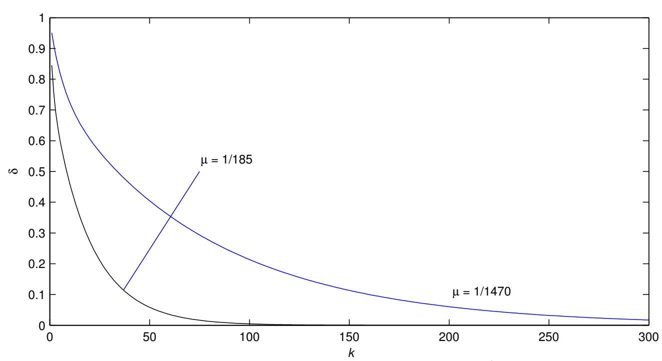
Fig. 1. The errors $\delta(k)$ versus k of Example 1

As depicted in Fig. 1, the error $\delta(k)$ decreases and converges to zero as k increases, which verifies the effectiveness of the proposed algorithm. In addition, the convergence performance associated with $\mu = 1/37$ is better than that associated with $\mu = 1/104$. This indicates that the sufficient condition given in (22) to ensure the convergence of the algorithm is very conservative, and how to choose an optimal convergence factor is the focus of our future work.

Example 2 Consider a general coupled matrix equation with the form of (23), where the coefficient matrices A_{ij}, B_{ij}, C_{ij}, D_{ij}, F_i and the unique solution X_j with $i = 1,2$ and $j = 1,2$ are given by

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},\ B_{11} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix},\ C_{11} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix},$$
$$D_{11} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix},\ B_{12} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix},$$
$$C_{12} = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix},\ D_{12} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix},\ C_{21} = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix},\ D_{21} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix},\ B_{22} = \begin{bmatrix} -2 & 3 \\ -1 & 3 \end{bmatrix},\ C_{22} = \begin{bmatrix} -1 & 1 \\ -7 & 0 \end{bmatrix},$$
$$D_{22} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},\ F_1 = \begin{bmatrix} -63 & 23 \\ 61 & 9 \end{bmatrix},\ F_2 = \begin{bmatrix} 4 & -7 \\ 23 & -24 \end{bmatrix},$$
$$X_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ X_2 = \begin{bmatrix} 2 & 8 \\ -1 & 5 \end{bmatrix}.$$

The gradient based iterative algorithm in (29)-(32) for initial values $X_1(0) = X_2(0) = 10^{-6}\mathbf{1}_{2\times 2}$ is applied to solve this coupled matrix equation. For different convergence factor $\mu = 1/1470$ and $1/185$, the simulation results are shown in Fig. 2.



$$\left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} \|A_{ij}\|^2 \|B_{ij}\|^2 + \|C_{ij}\|^2 \|D_{ij}\|^2 \right\}^{-1} = 1/1470$$

Fig. 2. The errors $\delta(k)$ versus k of Example 2

From Fig. 2, we can see that the convergence rate of the algorithm depends on the convergence factor μ, and a larger value leads to a faster rate. For $\mu = 1/185$, we have

$$X(300) = \begin{bmatrix} 1.00000 & 1.99994 \\ 3.00024 & 3.99986 \end{bmatrix},$$
$$Y(300) = \begin{bmatrix} 2.00000 & 8.00001 \\ -1.00005 & 4.99996 \end{bmatrix},$$

and $\delta(300) = 2.62284 \times 10^{-5}$.

V. Conclusions

The gradient based iterative algorithms are developed to solve the general coupled matrix equations with real coefficients by applying the hierarchical identification principle and the gradient search method. The given simulation results well demonstrate that the proposed algorithms have good convergence properties and high accuracy. Moreover, the problem studied in this paper is quite general, thus the proposed algorithms are also applicable to solving its special cases, such as ones in [18], [19].

The basic idea of the proposed algorithm can be applied to study identification problems of time-varying systems [38], nonlinear systems [39]–[43], dual-rate/multirate systems [44]–[67], as well as to design filters [68].

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