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Stability of planar singularly perturbed switched systems

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Abstract— **This paper is concerned with the stability analysis of planar linear singularly perturbed switched systems. We show that this class of switched systems has always a stability behavior common to all switched systems corresponding to small values of the singular perturbation parameter. Moreover, we propose necessary and sufficient conditions for the asymptotic stability.**

Index Terms— **Singular perturbation - switched system - asymptotic stability**

I. INTRODUCTION

In practice, many systems involve dynamics operating on different time scales. In this case, standard control techniques might lead to ill-conditioning problems and singular perturbation methods may be used in order to avoid such a numerical phenomenon [9], [15]. They consist in decomposing the system into several subsystems, one for each time scale. Thus, a different controller is designed for each of them. Singular perturbation techniques also allow to neglect high-frequency dynamics and then reduce the controller order [10]. As far as a linear time invariant model is considered, this time scale separation makes these two subsystems independent of each other and thereby simplify the control design problem and avoid ill-conditioning problems.

The situation is complex when switched systems are considered [11], [18]. It has been shown that even if the slow and the fast subsystems can be computed, they cannot be considered separately [13]. Stability of these two subsystems independently does not imply stability of the original switched system for small values of the singular perturbation parameter meaning that the Tikhonov theorem which allows in the classical LTI case to consider the fast dynamics and the slow dynamics separately for stability analysis and control design does not necessarily hold. To our knowledge, there are only few contributions in the context of hybrid systems and singular perturbations. In [8], singular perturbation in piecewise-linear systems are considered. A technique that allows decoupling of such systems into fast and slow subsystems is proposed. In [7], it is shown how an approximate optimal control law can be constructed from the solution of the limit control problem for a particular class of singularly pertrubed hybrid systems: the fast mode of the system is represented by deterministic state equations whereas the slow mode of the system corresponds to a jump disturbance process. In [16], considering the effect of unmodeled sensor/actuator dynamics in the closed loop, it is proved that stability is robust to a class of singular perturbations. Here, we consider continuous time switched linear systems in the singular perturbation form. Our objective is to provide necessary and sufficient conditions for stability analysis in the planar case.

The stability of linear switched system on the plane has been actively studied in the past. A first result that should be mentioned has been obtained by Shorten and Narendra in [17], where the authors give a characterization of planar switched systems admitting a common quadratic Lyapunov function. It is well known that, even in dimension two, the existence of a quadratic Lyapunov is a sufficient but not necessary condition for global uniform asymptotic stability (a two-dimensional example illustrating this fact can be found, for instance, in [6]). Boscain, in collaboration with Balde and Mason, in a series of papers ([3], [1], [2]) provided a complete characterization of stability for linear planar switched system. The novelty of their approach, based on the concept of *worst trajectory*, is that, instead of being based on Lyapunov functions, it exploits the invariants of the system and its dynamical properties.

Here, we focus on second order systems switching between two singularly perturbed dynamics and give a complete stability characterization as the perturbation parameter goes to zero. In particular we show that a singularly perturbed planar switched system has always an asymptotic stability behavior, i.e., a stability behavior common to all switched systems corresponding to $\epsilon > 0$ small enough. It is an open question whether this property is still true for higher dimensional singularly perturbed switched systems.

The paper is organized as follows. Section II is dedicated to preliminaries, tools and problem formulation. The characterization of stability for planar switched linear systems in the singular perturbation form is presented in section III.

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The proof of the main result is given in section IV. We end the paper by a conclusion.

II. Tools

In this section, we introduce the relevant stability notions for singularly perturbed switched systems. We recall in details the invariant quantities associated in [2] with a planar switched system and the corresponding stability characterization.

A. Notation

For every positive natural number d, denote by $\mathbb{M}_d(\mathbb{R})$ the space of $d \times d$ real-valued matrices. For any $X \in M_d(\mathbb{R})$, let $\mathrm{tr}(X)$ and $\det(X)$ denote the trace and the determinant of X. Id_x denotes the identity matrix with dimension x. A continuous function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to be of *class* $\mathcal{KL}$ if, for every $r \geq 0$, $\beta(r, \cdot)$ is nonincreasing, $\beta(\cdot, r)$ is nondecreasing, and $\beta(0, r) = \lim_{s\to+\infty} \beta(r, s) = 0$. A function $f : (0, \infty) \to R$ is said to be k-homogeneous if $f(\alpha) = \alpha^k f(1)$, for all $\alpha \in (0, \infty)$.

B. Stability notions

Let us recall some asymptotic stability notion for linear switched systems. In particular, this paper focuses on uniform stability with respect to all switching signals.

Let us consider the following switched system

$$\dot{x} = \sigma(t)A_1x(t) + (1 - \sigma(t))A_2x(t) \tag{1}$$

with $A_1, A_2 \in \mathbb{M}_d(\mathbb{R})$ and $\sigma : [0, +\infty) \to \{0, 1\}$ measurable.

Definition 1: We say that the switched system is unbounded if there exists a trajectory (solution of (1)) that goes to infinity. We say that the switched system (1) is *globally uniformly asymptotically stable* (GUAS, for short) if there exists a class $\mathcal{KL}$ function β such that, for every switching signal σ and every initial condition $x(0)$, the solution of (1) satisfies the inequality

$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall t \geq 0.$$

A particular case of global uniform asymptotic stability for (1) is the so-called *quadratic stability*, which can be expressed in terms of common quadratic Lyapunov functions.

Definition 2: If there exists a common positive definite matrix P satisfying

$$A_i^T P + PA_i < 0, \qquad i = 1, 2, \tag{2}$$

then $V(x) = x^T Px$ is called a *common quadratic Lyapunov function* (CQLF, for short) for (1).

A standard stability criterion for switched systems is the following: If the switched system (1) admits a CQLF, then it is GUAS.

The main objective of the paper is the study of the stability of *singularly perturbed switched systems* (SPSS, for short) of the form

$$\dot{x} = \sigma(t)A_1^\epsilon x(t) + (1 - \sigma(t))A_2^\epsilon x(t) \tag{3}$$

with $\sigma : [0, +\infty) \to \{0, 1\}$ measurable,

$$A_i^\epsilon = \begin{pmatrix} \frac{1}{\epsilon}\mathrm{Id}_{d_1} & 0 \\ 0 & \mathrm{Id}_{d_2} \end{pmatrix} M_i, \qquad i \in \{1, 2\} \tag{4}$$

and $M_1, M_2 \in \mathbb{M}_d(\mathbb{R})$, $d_1 + d_2 = d$. The above definition reads as follows.

Definition 3: We say that the SPSS (3) is *GUAS (respectively, quadratically stable/unbounded) as* $\epsilon \to 0^+$ if there exists ϵ_0 such that for all ϵ in $(0, \epsilon_0)$, the switched system described by (3) (with ϵ fixed) is GUAS (respectively, quadratically stable/unbounded).

C. Stability of planar switched systems

With two matrices $X, Y \in \mathbb{M}_2(\mathbb{R})$, we can associate the following parameters independent of a common change of coordinates [2]:

$$\delta(X) = \mathrm{tr}(X)^2 - 4\det(X),$$

$$\Gamma(X, Y) = \frac{1}{2}(\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY)),$$

$$\tau(X, Y) = \begin{cases} \frac{\mathrm{tr}(X)}{\sqrt{|\delta(X)|}} & \text{if } \delta(X) \neq 0, \\ \frac{\mathrm{tr}(X)}{\sqrt{|\delta(Y)|}} & \text{if } \delta(X) = 0 \text{ and } \delta(Y) \neq 0, \\ \frac{\mathrm{tr}(X)}{2} & \text{if } \delta(X) = \delta(Y) = 0, \end{cases}$$

$$k(X, Y) = \frac{2\tau(X, Y)\tau(Y, X)}{\mathrm{tr}(X)\mathrm{tr}(Y)} \left(\mathrm{tr}(XY) - \frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y) \right),$$

$$\Delta(X, Y) = 4(\Gamma(X, Y)^2 - \det(X)\det(Y)),$$

$$t(X, Y) = \begin{cases} \frac{\pi}{2} - \arctan \frac{\mathrm{tr}(X)\mathrm{tr}(Y)(k(X,Y)\tau(X,Y)+\tau(Y,X))}{2\tau(X,Y)\tau(Y,X)\sqrt{\Delta(X,Y)}}, \\ \frac{2\sqrt{\Delta(X,Y)}}{\tau(X,Y)\left(\mathrm{tr}(XY)-\frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y)\right)}, \\ \mathrm{arctanh} \frac{2\tau(X,Y)\tau(Y,X)\sqrt{\Delta(X,Y)}}{\mathrm{tr}(X)\mathrm{tr}(Y)(k(X,Y)\tau(X,Y)-\tau(Y,X))}, \end{cases}$$

for respectively $\delta(X) < 0$, $\delta(X) = 0$ and $\delta(X) > 0$

$$\mathcal{R}(X, Y) = \frac{2\Gamma(X, Y) + \sqrt{\Delta(X, Y)}}{2\sqrt{\det(X)\det(Y)}} e^{\tau(X,Y)t(X,Y)+\tau(Y,X)t(Y,X)}.$$

Notice that the definitions of $\Gamma(.,.)$, $k(.,.)$, $\Delta(.,.)$ and $\mathcal{R}(.,.)$ are symmetric with respect to their two arguments, while those of $\tau(.,.)$ and $t(.,.)$ are not.

Using the above definitions it is possible to characterize GUAS planar switched systems as follows.

Theorem 1 ([2]): Let $A_1, A_2 \in \mathbb{M}_2(\mathbb{R})$ be Hurwitz. Then the stability of the switched system (1) is determined by the following four statements:

(S1) System (1) is quadratically stable if and only if $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ and $\mathrm{tr}(A_1A_2) > -2\sqrt{\det(A_1)\det(A_2)}$;

(S2) If $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$, then (1) is unbounded;

(S3) If $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$, then (1) is uniformly stable but not GUAS;

(S4) $\Gamma(A_1, A_2) > \sqrt{\det(A_1)\det(A_2)}$ and $\mathrm{tr}(A_1A_2) \leq -2\sqrt{\det(A_1)\det(A_2)}$, then (1) is GUAS, uniformly stable, or unbounded respectively if $\mathcal{R}(A_1, A_2) < 1$, $\mathcal{R}(A_1, A_2) = 1$ or $\mathcal{R}(A_1, A_2) > 1$.

Notice that the theorem classifies the stability of planar switched systems along six classes of systems (condition (S4) actually splits in three distinct sub-cases).

D. Stability of planar SPSSs: notations and preliminary remarks

The rest of the paper is concerned with the stability of planar SPSSs of the form (3). Indeed, in the switched context, singular perturbations with $d_1 = d_2 = 1$ are nontrivial.

Let us write

$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \qquad i = 1, 2. \tag{5}$$

A first necessary condition for the stability of (3) is that A_i^ϵ are Hurwitz matrices for all $\epsilon > 0$ small enough and for $i = 1, 2$. Hence,

$$\mathrm{tr}(A_i^\epsilon) = \frac{a_i}{\epsilon} + d_i \qquad \text{and} \qquad -\det(A_i^\epsilon) = \frac{-\det(M_i)}{\epsilon}$$

must be negative. We can therefore restrict our attention to the case in which M_1 and M_2 belong to the set

$$\Lambda = \Big\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(M) > 0 \text{ and } (a < 0 \text{ or } (a = 0 \text{ and } d < 0)) \Big\}$$

We already noticed that $\det(A_i^\epsilon)$ is -1-homogeneous with respect to ϵ. The same happens for $\Gamma(A_1^\epsilon, A_2^\epsilon)$. Indeed, we have

$$\det(A_i^\epsilon) = \frac{\det(M_i)}{\epsilon} = \frac{a_i d_i - b_i c_i}{\epsilon}, \tag{6}$$

$$\begin{aligned} \Gamma(A_1^\epsilon, A_2^\epsilon) &= \frac{\Gamma(M_1, M_2)}{\epsilon} \\ &= \frac{a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1}{2\epsilon}. \end{aligned} \tag{7}$$

Notice that

$$\delta(A_i^\epsilon) = \Big(\frac{a_i}{\epsilon} - d_i\Big)^2 + \frac{4 b_i c_i}{\epsilon}.$$

Hence, if $a_i \neq 0$ then there exists $\epsilon_0 > 0$ such that $\delta(A_i^\epsilon) > 0$ for every $\epsilon \in (0, \epsilon_0)$. When $a_i = 0$, on the other hand, $b_i c_i = -\det(M_i) < 0$ and therefore there exists $\epsilon_0 > 0$ such that $\delta(A_i^\epsilon) < 0$ for every $\epsilon \in (0, \epsilon_0)$. In particular, up to taking ϵ small enough, we can always assume that $\delta(A_i^\epsilon)$ is different from zero. This simplifies the definitions of $\tau(A_1^\epsilon, A_2^\epsilon)$ and $t(A_1^\epsilon, A_2^\epsilon)$ introduced in the previous section.

We write in the following

$$\begin{aligned} \delta_i^\epsilon &= \delta(A_i^\epsilon), & \tau_i^\epsilon &= \tau(A_i^\epsilon, A_{3-i}^\epsilon), \\ k^\epsilon &= k(A_1^\epsilon, A_2^\epsilon), & \Delta^\epsilon &= \Delta(A_1^\epsilon, A_2^\epsilon), \\ t_i^\epsilon &= t(A_i^\epsilon, A_{3-i}^\epsilon), & \mathcal{R}^\epsilon &= \mathcal{R}(A_1^\epsilon, A_2^\epsilon), \end{aligned}$$

with $i = 1, 2$.

III. Characterization of the Stability of a Planar SPSS

In this section, using the invariants introduced above, we give necessary and sufficient conditions for the stability of the SPSS (3). The following theorem is the main result of the paper.

Theorem 2: Let $M_1, M_2 \in \Lambda$ be given as in (5). The stability of the singularly perturbed switched system (3) is described by the following five cases:

(SP1) System (3) is quadratically stable as $\epsilon \to 0^+$ if and only if $\Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)}$ and one the following conditions is satisfied
   1) $\Gamma(M_1, M_2) \leq \sqrt{\det(M_1)\det(M_2)}$,
   2) $a_1 a_2 \neq 0$,
   3) $a_1 a_2 = 0$ and $b_1 c_2 + b_2 c_1 \geq -2\sqrt{\det(M_1)\det(M_2)}$.

(SP2) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$, $a_1 a_2 = 0$ with $a_1^2 + a_2^2 \neq 0$, and $b_1 c_2 + b_2 c_1 < -2\sqrt{\det(M_1)\det(M_2)}$, then (3) is GUAS as $\epsilon \to 0^+$.

(SP3) If $\Gamma(M_1, M_2) = -\sqrt{\det(M_1)\det(M_2)}$, then for all $\epsilon > 0$ (3) is uniformly stable but not GUAS.

(SP4) If $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$, $a_1 = a_2 = 0$, and $b_1 c_2 + b_2 c_1 < -2\sqrt{\det(M_1)\det(M_2)}$, then (3) is unbounded as $\epsilon \to 0^+$.

(SP5) If $\Gamma(M_1, M_2) < -\sqrt{\det(M_1)\det(M_2)}$, then for all $\epsilon > 0$ (3) is unbounded.

Remark 1: Consider the following example introduced in [12], [13], where the SPSS is described by (3) and (4) with

$$M_1 = \begin{pmatrix} -1 & \alpha \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 0 \\ \alpha & -1 \end{pmatrix}. \tag{8}$$

In [12] and [13], sufficient conditions based on linear matrix inequalities (LMIs, for short, [4]) have been given for the asymptotic stability of the SPSS under an arbitrary switching law. Using these conditions, system described by (8) has been found to be quadratically stable for $-1 < \alpha < 1$.

Using (SP1), which provides necessary and sufficient conditions for the quadratic stability as $\epsilon \to 0^+$, we obtain that system (3) admits a CQLF as $\epsilon \to 0^+$ for $-2 < \alpha < 2$. Indeed, condition $\Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)}$ reads $\frac{2-\alpha^2}{2} > -1$ which is equivalent to $4 - \alpha^2 > 0$. Hence, the system admits a CQLF as $\epsilon \to 0^+$ if and only if $\alpha \in (-2, 2)$. Thus, the conditions based on LMIs given in [12] and [13] turn out to be just sufficient (and not necessary) for the quadratic stability.

Remark 2: The case (S4) in [2], recalled in Theorem 1, gives rise to the two cases (SP2) and (SP4) in Theorem 2. Notice that in case (S4) the system can be asymptotically stable, stable or unbounded, depending on the value of $\mathcal{R}$. Cases (SP2) and (SP4) correspond to the cases where $\mathcal{R}^\epsilon$ converges to 1 from below and has limit larger than 1, respectively. The case $\mathcal{R} = 1$ does not give rise to any subcase in Theorem 2, since it turns out to be impossible that $\mathcal{R}^\epsilon$ is identically equal to 1 as ϵ varies in a right neighborhood of 0.

Remark 3: The classification given in Theorem 2 guarantees that for ϵ in a right neighborhood of 0, system (3) belongs to one of the classes identified by Theorem 1. In particular, it cannot happen that, as ϵ converges to 0, the stability of (3) changes infinitely many times, with the values triggering the stability change clustering at 0.

It is nevertheless possible that, as $\epsilon > 0$ increases beyond the right neighborhood of 0 whose existence is guaranteed by Theorem 2, (3) changes its stability behavior, passing to a different chart of the atlas given in Theorem 1. (This makes sense only as long as A_1^ϵ and A_2^ϵ stay Hurwitz.) However, only some transitions are possible. First of all, by homogeneity reasons, transitions can happen only between cases (S1) and (S4) and is triggered by changes of sign of

$$\eta(\epsilon) = \mathrm{tr}(A_1^\epsilon A_2^\epsilon) + 2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)} = \frac{a_1 a_2}{\epsilon^2} + \frac{b_1 c_2 + b_2 c_1 + 2\sqrt{\det(M_1)\det(M_2)}}{\epsilon} + d_1 d_2.$$

If $a_1 a_2 \neq 0$ then $\eta(\epsilon)$ can change sign zero, one, or two times as ϵ varies in $(0, +\infty)$. Hence, (3) can pass form case (S1) to (eventually) case (S4) and then (eventually) back to case (S1) as ϵ increases.

If $a_1 = a_2 = 0$ then $d_1 d_2 > 0$ and $\epsilon\eta(\epsilon)$ is affine with respect to ϵ, with a positive coefficient multiplying ϵ. Hence, we can have either (S1) for every $\epsilon > 0$ or a single transition form (S4) to (S1).

In the general case $a_1 a_2 = 0$, we can either have (S1) or (S4) for every $\epsilon > 0$, or a single transition from (S4) to (S1), or a single transition from (S1) to (S4).

Consider the planar SPSS characterized by the matrices

$$M_1 = \begin{pmatrix} -1 & 0.01 \\ -9 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 2 \\ -2 & -2 \end{pmatrix}.$$

The solutions of $\mathrm{tr}(A_1^\epsilon A_2^\epsilon) = -2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$ are $\epsilon_0 = 0.0784$ and $\epsilon_1 = 6.3742$, so that for all $\epsilon \in (0, \epsilon_0) \bigcup (\epsilon_1, \infty)$ the system is of the type (S1), while for $\epsilon \in (\epsilon_0, \epsilon_1)$ it is of type (S4).

Analyzing $\mathcal{R}^\epsilon$, we compute $\epsilon_2 = 0.32$ and $\epsilon_3 = 1.62$, solutions of $\mathcal{R}^\epsilon - 1 = 0$, so that for all $\epsilon \in (\epsilon_0, \epsilon_2) \bigcup (\epsilon_3, \epsilon_1)$ we have $\mathcal{R}^\epsilon - 1 < 0$ and for all $\epsilon \in (\epsilon_2, \epsilon_3)$ the same quantity is positive .Hence, as ϵ varies in (ϵ_0, ϵ_1) all three subcases of (S4) show up.

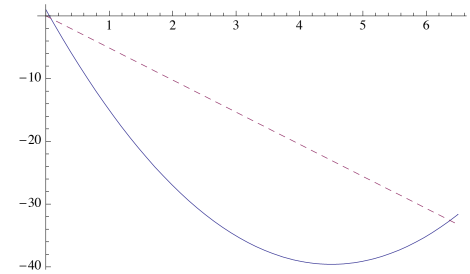

Fig. 1. The dashed and continuous line are the graph of $\epsilon \mapsto \epsilon\,\mathrm{tr}(A_1^\epsilon A_2^\epsilon)$ and $\epsilon \mapsto -2\epsilon\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$, respectively.

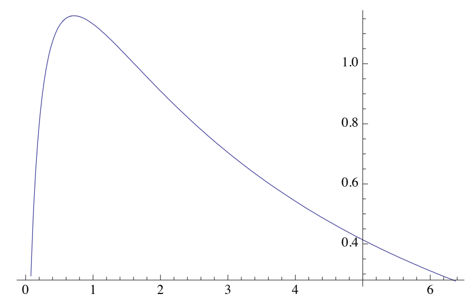

Fig. 2. Graph of $(\epsilon_0, \epsilon_1) \ni \epsilon \mapsto \mathcal{R}^\epsilon$.

IV. Proof of Theorem 2

The proofs of (SP3) and (SP5) simply follow from the statements (S2) and (S3) of Theorem 1, thanks to the homogeneity of $\epsilon \mapsto \det(A_i^\epsilon)$ and $\epsilon \mapsto \Gamma(A_1^\epsilon, A_2^\epsilon)$ noticed in (6) and (7).

As for (SP1), we should prove that, under the assumption $\Gamma(M_1, M_2) > -\sqrt{\det(M_1)\det(M_2)}$, if either $\Gamma(M_1, M_2) \leq \sqrt{\det(M_1)\det(M_2)}$ or $a_1 a_2 \neq 0$ or $a_1 a_2 = 0$ and $b_1 c_2 + b_2 c_1 \geq -2\sqrt{\det(M_1)\det(M_2)}$, then

$$\mathrm{tr}(A_1^\epsilon A_2^\epsilon) > -2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)} \tag{9}$$

for all $\epsilon > 0$ small enough. The conclusion then follows from (S1) in Theorem 1.

It has been proven in [2] that

$$-\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)} < \Gamma(A_1^\epsilon, A_2^\epsilon) \leq \sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)}$$

(i.e., $-\sqrt{\det(M_1)\det(M_2)} < \Gamma(M_1, M_2) \leq \sqrt{\det(M_1)\det(M_2)}$) automatically implies that (9) holds true (for all $\epsilon > 0$), proving (SP1.1).

So we have to compare

$$\mathrm{tr}(A_1^\epsilon A_2^\epsilon) = \frac{a_1 a_2}{\epsilon^2} + \frac{b_1 c_2 + b_2 c_1}{\epsilon} + d_1 d_2$$

with

$$-2\sqrt{\det(A_1^\epsilon)\det(A_2^\epsilon)} = -2\frac{\sqrt{\det(M_1)\det(M_2)}}{\epsilon}$$

when either $a_1 a_2 \neq 0$ or $a_1 a_2 = 0$ and

$$b_1 c_2 + b_2 c_1 \geq -2\sqrt{\det(M_1)\det(M_2)}$$

Since M_1 and M_2 belong to Λ, if $a_1 a_2 \neq 0$, then necessarily $a_1, a_2 < 0$. Hence, there exists $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$,

$$\frac{a_1 a_2}{\epsilon^2} + \frac{b_1 c_2 + b_2 c_1}{\epsilon} + d_1 d_2 > -2\frac{\sqrt{\det(M_1)\det(M_2)}}{\epsilon},$$

proving (SP1.2).

If, now, $a_1 a_2 = 0$ and $b_1 c_2 + b_2 c_1 > -2\sqrt{\det(M_1)\det(M_2)}$ then again (9) holds true for $\epsilon > 0$ small enough.

In the case $a_1 = a_2 = 0$ the fact that M_1 and M_2 belong to Λ implies that $d_1 d_2 > 0$ and (9) is true for all $\epsilon > 0$.

In order to prove (SP1.3), we are left to consider the case in which

$$b_1 c_2 + b_2 c_1 = -2\sqrt{\det(M_1)\det(M_2)} \qquad (10)$$

and $a_1 a_2 = 0$, $a_1^2 + a_2^2 \neq 0$. Without loss of generality, $a_1 = 0$ and $a_2 \neq 0$. In particular, $b_1 c_1$ and a_2 are negative.

Equation (10) implies that $(b_1 c_2 - b_2 c_1)^2 = -4 b_1 c_1 a_2 d_2$. Thus, d_2 is nonpositive. We claim that $d_2 < 0$. Indeed, we would otherwise have $\det(M_2) = b_2 c_2 < 0$, yielding $(b_1 c_2)(b_2 c_1) > 0$. Thus, $b_1 c_2 - b_2 c_1$ would be different zero, leading to a contradiction.

Therefore, $d_1 d_2 > 0$ and (9) is true for all $\epsilon > 0$. This concludes the proof of (SP1.3) and, therefore, of (SP1).

We are left to prove (SP2) and (SP4). In both cases $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$, $a_1 a_2 = 0$, and $b_1 c_2 + b_2 c_1 < -2\sqrt{\det(M_1)\det(M_2)}$. For (SP2) we can assume that $a_2 \neq 0$, while for (SP4) $a_1 = a_2 = 0$.

Case (SP4): $a_1 = a_2 = 0$. As already noticed, since M_1 and M_2 belong to Λ, then $d_1, d_2 < 0$ and $b_1 c_1, b_2 c_2 < 0$. In particular,

$$\mathrm{sign}(b_i) = -\mathrm{sign}(c_i), \quad i = 1, 2.$$

Thus,

$$\mathrm{sign}(b_1 c_2) = \mathrm{sign}(b_2 c_1).$$

Since

$$b_1 c_2 + b_2 c_1 < -2\sqrt{\det(M_1)\det(M_2)}$$

moreover, we have $b_1 c_2, b_2 c_1 < 0$.

Notice that, for ϵ small enough,

$$b_1 c_2 + b_2 c_1 + \epsilon d_1 d_2 < -2\sqrt{\det(M_1)\det(M_2)}.$$

We are, therefore, in the case described by (S4) in Theorem 1 and the system stability depends on the sign of $\mathcal{R}^\epsilon - 1$.

Let

$$C_\Delta = 4(\Gamma(M_1, M_2)^2 - \det(M_1)\det(M_2)) \qquad (11)$$

so that $\sqrt{\Delta^\epsilon} = \sqrt{C_\Delta}/\epsilon$.

Recall that

$$\mathcal{R}^\epsilon = \frac{\Gamma(M_1, M_2) + \sqrt{C_\Delta}}{\sqrt{\det(M_1)\det(M_2)}} e^{\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon}$$

where, as it follows from the identity $a_1 = a_2 = 0$ and the definitions of $t(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ given in Section II-C, for $\epsilon > 0$ small enough,

$$t_i^\epsilon = \frac{\pi}{2} - \arctan \frac{d_i \mathrm{tr}(A_{3-i}^\epsilon)(k^\epsilon \tau_i^\epsilon + \tau_{3-i}^\epsilon)}{2\tau_1^\epsilon \tau_2^\epsilon \sqrt{\Delta^\epsilon}},$$
$$\tau_i^\epsilon = \frac{\mathrm{tr}(A_i^\epsilon)}{\sqrt{-\delta_i^\epsilon}} = \frac{d_i}{\sqrt{-d_i^2 - \frac{4 b_i c_i}{\epsilon}}}.$$

In particular, $t_1^\epsilon, t_2^\epsilon \in (0, \pi)$ for ϵ small enough. As $\epsilon \to 0^+$, τ_i^ϵ has the Taylor expansion

$$\tau_i^\epsilon = C_{\tau_i}\sqrt{\epsilon} + O(\epsilon^{3/2}), \quad \text{with} \quad C_{\tau_i} = \frac{d_i}{2\sqrt{-b_i c_i}}. \qquad (12)$$

Thus, $\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon \underset{\epsilon \to 0^+}{\longrightarrow} 0$ which implies

$$\mathrm{e}^{\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon} \underset{\epsilon \to 0^+}{\longrightarrow} 1. \qquad (13)$$

From the assumption $\Gamma(M_1, M_2) > \sqrt{\det(M_1)\det(M_2)}$, we have $(\Gamma(M_1, M_2) + \sqrt{C_\Delta})/\sqrt{\det(M_1)\det(M_2)} > 1$, and we conclude that, as $\epsilon \to 0^+$, $\mathcal{R}^\epsilon$ tends to a constant larger than one. Hence, the system is unstable for all $\epsilon > 0$ small enough.

Case (SP2): $a_1 = 0$, $a_2 \neq 0$.

For $\epsilon > 0$ small enough we can assume $\delta_1^\epsilon < 0$ and $\delta_2^\epsilon > 0$, leading to the expressions

$$t_1^\epsilon = \frac{\pi}{2} - \arctan \frac{d_1 \mathrm{tr}(A_2^\epsilon)(k^\epsilon \tau_1^\epsilon + \tau_2^\epsilon)}{2\tau_1^\epsilon \tau_2^\epsilon \sqrt{\Delta^\epsilon}}$$
$$t_2^\epsilon = \mathrm{arctanh} \frac{2\tau_1^\epsilon \tau_2^\epsilon \sqrt{\Delta^\epsilon}}{\mathrm{tr}(A_1^\epsilon)\mathrm{tr}(A_2^\epsilon)(k^\epsilon \tau_2^\epsilon - \tau_1^\epsilon)}.$$

As above, our aim is to study the asymptotic sign of $\mathcal{R}^\epsilon - 1$ as $\epsilon \to 0^+$. In order to characterize $\mathcal{R}^\epsilon$, we use the expansion for τ_1^ϵ as $\epsilon \to 0^+$ obtained in (12), i.e.,

$$\tau_1^\epsilon = C_{\tau_1}\sqrt{\epsilon} + O(\epsilon^{3/2}), \quad \text{with} \quad C_{\tau_1} = \frac{d_1}{2\sqrt{-b_1 c_1}}, \qquad (14)$$

and the following expansions for τ_2^ϵ,

$$\tau_2^\epsilon = \frac{\frac{a_2}{\epsilon} + d_2}{\sqrt{(\frac{a_2}{\epsilon} - d_2)^2 + \frac{4 b_2 c_2}{\epsilon}}} = -1 + O(\epsilon).$$

Using these expansions in the expressions of t_1^ϵ and t_2^ϵ, we get

$$t_1^\epsilon = \pi + C_{t_1}\sqrt{\epsilon} + O(\epsilon), \qquad \text{with } C_{t_1} = \frac{\sqrt{C_\Delta}}{a_2 b_1 c_1},$$
$$t_2^\epsilon = \mathrm{arctanh}\left(\frac{\sqrt{C_\Delta}}{2\Gamma(M_1, M_2)}\right) + O(\epsilon).$$

Thanks to the above expansions, we get

$$\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon = -\mathrm{arctanh}\left(\frac{\sqrt{C_\Delta}}{2\Gamma(M_1, M_2)}\right) + \pi C_{\tau_1}\sqrt{\epsilon} + O(\epsilon)$$

yielding

$$\begin{aligned} e^{\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon} &= e^{-\mathrm{arctanh}\frac{\sqrt{C_\Delta}}{2\Gamma(M_1,M_2)}} e^{\pi C_{\tau_1}\sqrt{\epsilon}} + O(\epsilon) \\ &= e^{-\mathrm{arctanh}\frac{\sqrt{C_\Delta}}{2\Gamma(M_1,M_2)}} (1 + \pi C_{\tau_1}\sqrt{\epsilon}) + O(\epsilon). \end{aligned}$$

Using the identity $\mathrm{arctanh}(x) = \log\left(\sqrt{\frac{1+x}{1-x}}\right)$ we get, as $\epsilon \to 0^+$,

$$e^{\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon} = \sqrt{\frac{1 - \frac{\sqrt{C_\Delta}}{2\Gamma(M_1,M_2)}}{1 + \frac{\sqrt{C_\Delta}}{2\Gamma(M_1,M_2)}}}(1 + \pi C_{\tau_1}\sqrt{\epsilon}) + O(\epsilon).$$

Replacing C_Δ by its expression (11), we obtain

$$\begin{aligned} e^{\tau_1^\epsilon t_1^\epsilon + \tau_2^\epsilon t_2^\epsilon} = &\sqrt{\frac{1 - \sqrt{1 - \frac{\det(M_1)\det(M_2)}{\Gamma(M_1,M_2)^2}}}{1 + \sqrt{1 - \frac{\det(M_1)\det(M_2)}{\Gamma(M_1,M_2)^2}}}}(1 + \pi C_{\tau_1}\sqrt{\epsilon}) \\ &+ O(\epsilon). \end{aligned}$$

Let

$$\xi = \frac{\det(M_1)\det(M_2)}{\Gamma(M_1, M_2)^2}.$$

Since

$$\frac{\Gamma(M_1,M_2) + \sqrt{\Gamma(M_1,M_2)^2 - \det(M_1)\det(M_2)}}{\sqrt{\det(M_1)\det(M_2)}} = \frac{1+\sqrt{1-\xi}}{\xi},$$

then

$$\begin{aligned} \mathcal{R}^\epsilon &= \frac{1+\sqrt{1-\xi}}{\xi}\sqrt{\frac{1-\sqrt{1-\xi}}{1+\sqrt{1-\xi}}}(1 + \pi C_{\tau_1}\sqrt{\epsilon}) + O(\epsilon) \\ &= \frac{\sqrt{1+\sqrt{1-\xi}}\sqrt{1-\sqrt{1-\xi}}}{\sqrt{\xi}}(1 + \pi C_{\tau_1}\sqrt{\epsilon}) + O(\epsilon) \\ &= 1 + \pi C_{\tau_1}\sqrt{\epsilon} + O(\epsilon). \end{aligned}$$

As C_{τ_1} is negative, we have that $\mathcal{R}^\epsilon$ is smaller than 1 for ϵ small. Consequently, the system is GUAS as $\epsilon \to 0^+$.

V. Conclusion

In this paper we give a complete classification of quadratically stable, GUAS, stable, and unbounded singularly perturbed planar switched systems. More precisely, we characterize the asymptotic stability behavior of such singularly perturbed switched systems as the perturbation parameter goes to zero. In particular we show that a singularly perturbed planar switched system has always an asymptotic stability behavior, ie, a stability behavior common to all switched systems corresponding to $\epsilon > 0$ small enough. It is an open question whether this is still true for higher dimensional singularly perturbed switched systems.

The characterization is based on analogous results for planar switched systems obtained in [2]. Whereas the results in [2] distinguish six cases, the perturbation procedure allows us to reduce them to five. The sixth one happens to be unstable with respect to the perturbation parameter (hence, it can occur only for isolated values of it).

The characterization of asymptotic quadratic stability allows us to single out the conservatism of previous conditions, based on LMIs, proposed in the literature ([12], [13]).

An important aspect in control problems of singularly perturbed systems, which is not raised in this article, concerns the evaluation of the maximum value of ϵ that guarantees the stability for any $\epsilon \in (0, \epsilon_{max}]$. Looking for the exact value of ϵ_{max} is a challenging and a difficult problem, known as the ϵ-bound problem, and some contributions in the literature have only succeeded in proposing sufficient and conservative upper bounds (see [5] and references therein). Also, it was shown that the classical time scale separation does not hold for singularly perturbed swtiched systems in discrete time case [14]. The stability characterization of planar singularly perturbed switched systems is an open problem.

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