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Efficient Swing-up of the Acrobot Using Continuous Torque and Impulsive Braking

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***Abstract*— An efficient swing-up algorithm for the acrobot is proposed. The algorithm is based on using a combination of continuous torque and impulsive braking torque. The continuous torque is derived from a positive definite Lyapunov-like function, proposed earlier in the literature. The impulsive braking torques are applied at specific instants of time which result in discrete jumps in certain states of the system and a discrete reduction in the value of the Lyapunov-like function. We prove asymptotic stability of the closed-loop impulsive dynamical system and use simulation results to show that the proposed algorithm is more efficient than the algorithms using only continuous torques derived from the same Lyapunov-like function.**

NOMENCLATURE

For the nomenclature below, $k \in \{1, 2\}$ and $j \in \{1, 5\}$

l_k	length of the k-th link (m)
d_k	distance between the k-th joint and center-of-mass of the k-th link (m)
m_k	mass of the k-th link (kg)
I_k	mass moment of inertia of the k-th link about its center-of-mass $(kg.m^2)$
g	acceleration due to gravity, $(9.81\ m.s^{-2})$
q_j	constants, whose values depend on kinematic and dynamics parameters of the acrobot
θ_k	angular displacement of the k-th link, as defined in Fig.1 (rad)
$\dot{\theta}_k$	angular velocity of the k-th link (rad/s)
$\dot{\theta}_k^+$	angular velocity of the k-th link, immediately after application of the impulsive torque (rad/s)
$\dot{\theta}_k^-$	angular velocity of the k-th link, immediately prior to application of the impulsive torque (rad/s)
v_k	velocity of the center-of-mass of the k-th link (m/s)
$\vec{v_k}^-$	velocity of the center-of-mass of the k-th link, immediately prior to application of the impulsive torque (m/s)
$\vec{v_k}^+$	velocity of the center-of-mass of the k-th link, immediately after application of the impulsive torque (m/s)
E	total energy of the acrobot (J)
E_{des}	total energy of the acrobot when $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (\pi/2, 0, 0, 0)$ (J)
K_k	kinetic energy of the k-th link (J)
τ	external torque applied on the second link $(N.m)$
τ_b	external torque required to instantaneously stop the second joint $(N.m)$
$\vec{F}_k^{imp}$	impulsive force acting at the center-of-mass of the k-th link (N)
$\vec{F}_s^{imp}$	impulsive force acting at the shoulder joint (N)
$\vec{M}_k^{imp}$	impulsive moment acting at the center-of-mass of the k-th link $(N.m)$
S_k	$\sin\theta_k$
C_k	$\cos\theta_k$
S_{12}	$\sin(\theta_1+\theta_2)$
C_{12}	$\cos(\theta_1+\theta_2)$

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I. INTRODUCTION

The acrobot is a two-link robot in the vertical plane with an actuator at the elbow joint and a passive shoulder joint. It is an underactuated system and its control problem has similarities with that of the pendubot [1]. For the acrobot, the control problem requires swing-up from an arbitrary initial configuration to its configuration with maximum potential energy, followed by stabilization. The stabilization problem has seen many solutions, such as linear quadratic regulator [2], [3], and robust control [4] based designs. The swing-up control problem is more challenging and requires the system trajectory to be driven to the neighborhood of one equilibrium configuration from any point in the configuration space that contains four equilibria. Spong [2] proposed a method based on partial feedback linearization but this approach is very sensitive to the values of the control gains. Boone [5] proposed bang-bang control for near-optimal swing-up trajectories. The algorithm switches at finite time intervals and becomes computationally expensive for a large number of intervals. Xin and Kaneda [6] and Mahindrakar and Banavar [7] used a single Lyapunov function to design swing-up controllers. The control design by Xin and Kaneda [6] requires an initial perturbation and a strong condition to be imposed on controller parameters to guarantee convergence. The design by Mahindrakar and Banavar [7], on the other hand, results in relatively large continuous control inputs. Lai, et al. [8] used non-smooth Lyapunov functions with fuzzy logic to remove the constraint on the control parameter in the design by Xin and Kaneda [6].

The main objective of this research is to explore new control methods for underactuated systems by enlarging the set of admissible control inputs to include impulsive forces. It has been pointed out that conventional actuators can apply impulse-like forces and inclusion of such inputs in the set of admissible inputs can result in efficient swing-up of the pendubot [1]. The pendubot and the acrobot are benchmark problems in underactuated systems and this paper presents an impulsive control method for swing-up of the acrobot.

There has been a fair amount of theoretical research on impulsive control of dynamical systems and credit for some of the early works goes to Pavlidis [9], Gilbert and Harasty [10], Menaldi [11] and Lakshmikantham [12]. In recent years, researchers have studied the problems of stability, controllability and observability, optimality (see [13], [14], [15], for example), and diverse application problems, including underactuated systems [16], [17], [1]. In this paper, we develop an impulse-momentum approach for swing-up of the acrobot. We review the dynamic model of the acrobot in section II. In section III, the expressions for jump in velocities and the change in energy due to the impulsive torque is derived. The control design is developed in section IV and the numerical simulations comparing the results with the results in [6], [7] are presented in section V. Concluding remarks are provided in section VI.

II. System Dynamics

A. Equations of Motion

Consider the acrobot in Fig.1. Assuming no friction in the joints, the equation of motion can be obtained using the Lagrangian formulation as follows [2]

$$A(\theta)\ddot{\theta} + B(\theta,\dot{\theta})\dot{\theta} + G(\theta) = T \tag{1}$$

where

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 \\ \tau \end{pmatrix} \tag{2}$$

and $A(\theta)$, $B(\theta,\dot{\theta})$, and $G(\theta)$, given by the expressions

$$A(\theta) = \begin{bmatrix} q_1 + q_2 + 2q_3C_2 & q_2 + q_3C_2 \\ q_2 + q_3C_2 & q_2 \end{bmatrix} \tag{3}$$

$$B(\theta,\dot{\theta}) = q_3S_2 \begin{bmatrix} -\dot{\theta}_2 & -(\dot{\theta}_1 + \dot{\theta}_2) \\ \dot{\theta}_1 & 0 \end{bmatrix} \tag{4}$$

$$G(\theta) = g \begin{bmatrix} q_4C_1 + q_5C_{12} \\ q_5C_{12} \end{bmatrix} \tag{5}$$

are the inertia matrix, matrix containing the Coriolis and centrifugal forces, and vector of gravity forces, respectively. In (3), (4), and (5), q_j, $j = 1, 2, \cdots, 5$ are positive constants obtained in literature, (see [8], for example).

B. Braking Torque

We consider the action that results in exponential decay of the second link velocity, $\dot{\theta}_2$, to zero. Therefore, we assume

$$\ddot{\theta}_2 = -k_1\dot{\theta}_2, \quad k_1 > 0 \tag{6}$$

where k_1 is a positive constant that will determine the rate of decay of $\dot{\theta}_2$. To compute the torque required for this action, we multiply (1) with the inverse of the inertia matrix to obtain

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \frac{1}{q_1q_2 - q_3^2C_2^2} \begin{bmatrix} -(q_2 + q_3C_2)\tau + h_1 \\ (q_1 + q_2 + 2q_3C_2)\tau + h_2 \end{bmatrix} \tag{7}$$

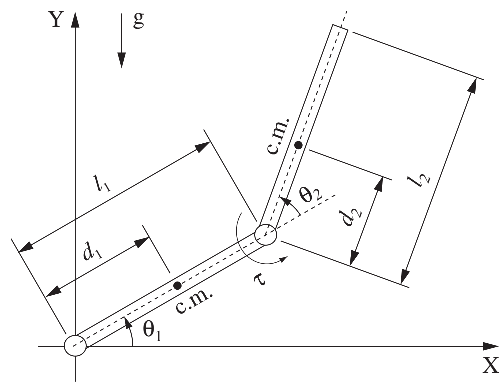

Fig. 1. The acrobot in an arbitrary configuration: the joint angles θ_1 and θ_2 are measured counter-clockwise with respect to the horizontal axis.

where h_1 and h_2 are given by the expressions

$$\begin{aligned} h_1 &= q_2q_3(\dot{\theta}_1 + \dot{\theta}_2)^2S_2 + q_3^2\dot{\theta}_1^2S_2C_2 \\ &\quad + g(q_3q_5C_2C_{12} - q_2q_4C_1) \end{aligned} \tag{8}$$

$$\begin{aligned} h_2 &= -(\dot{\theta}_1 + \dot{\theta}_2)^2(q_2q_3 + q_3^2C_2)S_2 \\ &\quad -(q_1 + q_3C_2)q_3\dot{\theta}_1^2S_2 - g\,\{q_3q_5C_2C_{12} \\ &\quad -(q_2 + q_3C_2)q_4C_1 + q_1q_5C_{12}\} \end{aligned} \tag{9}$$

Substituting (6) into the second equation in (7) results in

$$\tau_b = \frac{-1}{q_1 + q_2 + 2q_3C_2}\left[k_1\dot{\theta}_2(q_1q_2 - q_3^2C_2^2) + h_2\right] \tag{10}$$

If the gain k_1 is chosen very large, the torque expression in (10) will act like an impulsive brake stopping the second link in a very short period of time.

III. Effect of Impulsive Braking

A. Velocity Change

An impulsive braking torque will result in impulsive forces and moments acting on both links of the acrobot. From the free-body of the second link in Fig.2(a), we can write

$$\vec{F}_2^{imp}\,\Delta t = m_2\,(\vec{v_2}^{+} - \vec{v_2}^{-}) \tag{11}$$

$$\vec{M}_2^{imp}\,\Delta t = I_2\,(\dot{\theta}_1^+ + \dot{\theta}_2^+) - I_2\,(\dot{\theta}_1^- + \dot{\theta}_2^-) \tag{12}$$

where Δt is the short interval of time over which the impulsive force and impulsive moment act, and $\vec{v_2}^{+}$ and $\vec{v_2}^{-}$ are given by the expressions

$$\begin{aligned} \vec{v_2}^{+} =& -\left[l_1\dot{\theta}_1^+S_1 + d_2\,(\dot{\theta}_1^+ + \dot{\theta}_2^+)S_{12}\right]\vec{i} \\ &+ \left[l_1\dot{\theta}_1^+C_1 + d_2\,(\dot{\theta}_1^+ + \dot{\theta}_2^+)C_{12}\right]\vec{j} \\ \vec{v_2}^{-} =& -\left[l_1\,\dot{\theta}_1^-S_1 + d_2(\dot{\theta}_1^- + \dot{\theta}_2^-)S_{12}\right]\vec{i} \\ &+ \left[l_1\,\dot{\theta}_1^-C_1 + d_2(\dot{\theta}_1^- + \dot{\theta}_2^-)C_{12}\right]\vec{j} \end{aligned} \tag{13}$$

From the free-body diagram in Fig.2(d), we can write

$$\vec{F}_1^{imp}\,\Delta t = m_1(\vec{v_1}^{+} - \vec{v_1}^{-}) \tag{14}$$

$$\vec{M}_1^{imp}\,\Delta t = I_1(\dot{\theta}_1^+ - \dot{\theta}_1^-) \tag{15}$$

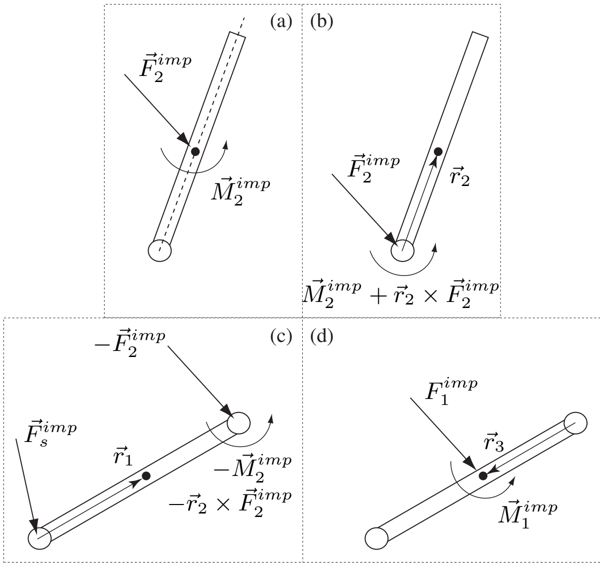

Fig. 2. Free-body diagrams showing impulsive forces and moments acting on the second link - (a) and (b), and first link - (c) and (d).

where

$$\begin{aligned} \vec{v_1}^{+} &= d_1\dot{\theta}_1^{+}(-S_1\vec{i} + C_1\vec{j}) \\ \vec{v_1}^{-} &= d_1\dot{\theta}_1^{-}(-S_1\vec{i} + C_1\vec{j}) \end{aligned} \tag{16}$$

Using the force and moments diagrams in Figs.2(b) and (c), we can express $\vec{F}_1^{imp}$ and $\vec{M}_1^{imp}$ as follows

$$\vec{F}_1^{imp} = \vec{F}_s^{imp} - \vec{F}_2^{imp} \tag{17}$$

$$\vec{M}_1^{imp} = -\vec{M}_2^{imp} - (\vec{r}_2 - \vec{r}_3) \times \vec{F}_2^{imp} - \vec{r}_1 \times \vec{F}_s^{imp} \tag{18}$$

where $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, shown in Fig.2, have the expressions

$$\begin{aligned} \vec{r}_1 &= d_1(C_1\vec{i} + S_1\vec{j}) \\ \vec{r}_2 &= d_2(C_{12}\vec{i} + S_{12}\vec{j}) \\ \vec{r}_3 &= (d_1 - l_1)(C_1\vec{i} + S_1\vec{j}) \end{aligned} \tag{19}$$

Substituting (18) into (15) and simplifying using (11), (12), (14), (17) and (19), we get

$$[q_1 + q_2 + 2q_3C_2](\dot{\theta}_1^{+} - \dot{\theta}_1^{-}) = [q_2 + q_3C_2](\dot{\theta}_2^{+} - \dot{\theta}_2^{-}) \tag{20}$$

Since impulsive braking results in $\dot{\theta}_2^{+} = 0$, (20) can be used to obtain the velocity of the first link after braking:

$$\dot{\theta}_1^{+} = \dot{\theta}_1^{-} - \left[\frac{q_2 + q_3C_2}{q_1 + q_2 + 2q_3C_2}\right]\dot{\theta}_2^{-} \tag{21}$$

B. Energy Change

The configuration of the acrobot will not change over the small period of time, Δt, when the impulse is applied. Therefore, the change in total energy of the system is only due to the change in the kinetic energy which can be expressed as

$$\begin{aligned} \Delta E =& \Delta K \\ =& \frac{1}{2}\, q_1[(\dot{\theta}_1^{+})^2 - (\dot{\theta}_1^{-})^2] + \frac{1}{2}\, q_2[(\dot{\theta}_2^{+})^2 - (\dot{\theta}_2^{-})^2] \\ &+ q_3C_2[(\dot{\theta}_1^{+} + \dot{\theta}_2^{+}) - (\dot{\theta}_1^{-} + \dot{\theta}_2^{-})] \end{aligned} \tag{22}$$

Substituting (20) into (22) and having $\dot{\theta}_2^{+} = 0$, the change in total energy of the system due to impulsive braking is resulted as:

$$\Delta E = -\frac{q_1q_2 - q_3^2C_2^2}{2(q_1 + q_2 + 2q_3C_2)}\,(\dot{\theta}_2^{-})^2 \tag{23}$$

which implies energy loss.

IV. Swing-up Controller

A. Preliminaries

Consider the following impulsive system [14],

$$\begin{aligned} \dot{x}(t) &= f(t,x), \quad t \neq \eta_i(x) \\ \Delta x(t) &= H_i(x), \quad t = \eta_i(x),\ i = 1,2,... \end{aligned} \tag{24}$$

and let Ω be a domain in R^n containing the origin and $x(t_0) \in \Omega$ where t_0 is the initial time. Then assume the following conditions hold for the system in (24):

(a) $f(t,x)$ is a bounded function that is continuous with respect to t and Lipschitz continuous with respect to x with $f(t,0) = 0$ for all $t \geq t_0$.
(b) $(x + H_i(x)) : \Omega \to \Omega$ and $H_i(0) = 0,\ i = 1,2,...$
(c) $\eta_i(x)$ satisfies $t_0 < \eta_1 < \eta_2 < ...$, $\eta_i(x) \to \infty$ as $i \to \infty$ uniformly on $x \in \Omega$, and $\eta_i(x) : \Omega \to (t_0,\infty)$ is a continuous function of x for $i = 1,2,....$
(d) If we consider the hypersurfaces

$$\sigma_i = \{(t,x) :\ t = \eta_i(x),\ x \in \Omega\}, \quad i = 1,2,... \tag{25}$$

then the integral curves of the system (24) meet successively each one of the hypersurfaces $\sigma_1, \sigma_2, ...$ exactly once.

The following theorem can be stated on the stability of the impulsive system in (24):

Theorem 1: [14] Consider the system in (24) and assume that:
(1) Conditions (a)-(d) hold.
(2) There exists a function $V(t,x) \in V_0$ such that $V(t,0) = 0$ for all $t > t_0$;

$$\beta(\|x\|) \leq V(t,x) \tag{26}$$

where β is a class K function and $(t,x) \in [t_0,\infty) \times \Omega$;

$$V(t^{+}, x + H_i(x)) \leq V(t,x) \tag{27}$$

for $(t,x) \in \sigma_i,\ i = 1,2,...$ and the following inequality is valid for $t \in [t_0,\infty)$, $x \in \Omega$:

$$D^{+}V(t,x)^1 \leq 0,\ t \neq \eta_i(x),\ i = 1,2,... \tag{28}$$

then the zero solution of the system (24) is stable.

[1] $D^{+}V$ is the upper right Dini Derivative with respect to time.

B. Controller Design

The control objective is to find a controller such that the following conditions corresponding to the second link configuration and the total energy are satisfied:

$$\theta_2 = 0 \quad , \quad \dot{\theta}_2 = 0 \quad , \quad E = E_{des} \tag{29}$$

With $\theta_2 = 0$ and $\dot{\theta}_2 = 0$, the system acts as a pendulum. Then, by choosing $E = E_{des}$, the system converges to a heteroclinic orbit with equilibrium points corresponding to the upright configuration.

Considering the control objective in (29), the states of the impulsive system in (24) can be defined as follows for this problem:

$$x(t) = \left[\ \theta_2\ ,\ \ \dot{\theta}_2\ ,\ \ E - E_{des}\ \right]^T \tag{30}$$

Using the results of *Theorem 1*, we proceed to prove stability of the equilibrium $x = 0$. We first satisfy the conditions of Theorem 1. Condition (a) is automatically satisfied by the choice of x in (30).

The times $t = \eta_i(x),\ i = 1, 2, ...$ are the instants when an impulsive braking torque is applied to the second link using the torque expression in (10) with large gain k_1. We choose η_i as follows:

$$\eta_i(x) = \{\ t \in [t_0, \infty)\ |\ x \in \rho_i(x)\} \tag{31}$$

where

$$\rho_i(x) = \{x \in R^3\ |\ \dot{\theta}_2\ddot{\theta}_2 < 0\} \tag{32}$$

Applying the impulsive brake on the second link will result in a jump in x, namely

$$\Delta x = H_i(x) \tag{33}$$

Since there is no jump in θ_2, $H_i(x)$ is given by

$$H_i(x) = \begin{bmatrix} 0 \\ -\dot{\theta}_2^- \\ -\frac{q_1 q_2 - q_3^2 C_2^2}{2(q_1+q_2+2q_3C_2)}(\dot{\theta}_2^-)^2 \end{bmatrix} \tag{34}$$

which satisfies the condition (b) in Theorem 1.

To satisfy conditions (c) and (d), we must ensure that the time between impulses is nonzero. To this end, let us first define the set $\rho_i^\perp(x)$ as

$$\rho_i^\perp(x) = \{x \in R^3\ |\ \dot{\theta}_2\ddot{\theta}_2 \geq 0\} \tag{35}$$

such that $x \in \rho_i^\perp(x)$ for all times $t \neq \eta_i(x)$. Impulsive braking results in $\dot{\theta}_2 = 0$ or $\dot{\theta}_2\ddot{\theta}_2 = 0$ which implies that x is on the boundary of $\rho_i^\perp(x)$. By taking the derivative of $\dot{\theta}_2\ddot{\theta}_2$ with respect to time immediately following the impulse, we get

$$D(\dot{\theta}_2\ddot{\theta}_2) = (\ddot{\theta}_2)^2 + \dot{\theta}_2\theta_2^{(3)} = (\ddot{\theta}_2)^2 \geq 0 \tag{36}$$

which means that $\dot{\theta}_2\ddot{\theta}_2$ is nondecreasing after the impulse. It follows from continuity that following an impulse, there exists a nonzero time interval $\epsilon > 0$ before x can enter the set $\rho_i(x)$ again, *i.e.*,

$$\eta_{i+1}(x) - \eta_i(x) \geq \epsilon \ \ \forall\ i = 1, 2, ... \tag{37}$$

Therefore, conditions (c) and (d) are satisfied.

Now, we consider a positive definite locally Lipschitz scalar function V as

$$V(x) = \frac{1}{2}\left[k_p(\theta_2)^2 + k_d(\dot{\theta}_2)^2 + k_e(E - E_{des})^2\right] \tag{38}$$

where k_p, k_d, k_e are positive constants. At times $t \neq \eta_i(x)$, the upper right Dini derivative of (38) is equivalent to taking the time derivative, and is equal to:

$$\dot{V} = [\ k_p\theta_2 + k_d\{a(\theta,\dot{\theta}) + b(\theta)\tau\} + k_e(E - E_{des})\tau]\ \dot{\theta}_2 \tag{39}$$

where we used $\dot{E} = \tau\ \dot{\theta}_2$ and

$$\ddot{\theta}_2 = a(\theta,\dot{\theta}) + b(\theta)\tau \tag{40}$$

The variables $a(\theta,\dot{\theta})$ and $b(\theta)$ can be obtained from (7) as

$$\begin{aligned} a(\theta,\dot{\theta}) &= \frac{1}{q_1q_2 - q_3^2C_2^2}h_2 \\ b(\theta) &= \frac{1}{q_1q_2 - q_3^2C_2^2}(q_1 + q_2 + 2q_3C_2) \end{aligned} \tag{41}$$

If the torque expression in (39) is chosen as

$$\tau \quad = \quad -\frac{k_p(\theta_2) + k_d\ a(\dot{\theta},\theta) + k_c\ddot{\theta}_2}{k_d b(\theta) + k_e(E - E_{des})} \tag{42}$$

then

$$\dot{V} = -k_c\ \dot{\theta}_2\ \ddot{\theta}_2 \tag{43}$$

where k_c is a positive constant. Since $x \in \rho_i^\perp(x)$ when $t \neq \eta_i(x)$, we have

$$D^+V = -k_c\dot{\theta}_2\ddot{\theta}_2 \leq 0, \ \ \forall\ t \neq \eta_i(x) \tag{44}$$

The above equation implies that the inequality condition in (28) is satisfied. By substituting (40) into (42) and solving for τ, we get

$$\tau \quad = \quad -\frac{k_p(\theta_2) + (k_d + k_c)\ a(\dot{\theta},\theta)}{(k_d + k_c)b(\theta) + k_e(E - E_{des})} \tag{45}$$

The torque expression in (45) will encounter a singularity if

$$(k_d + k_c)\ b(\theta) + k_e(E - E_{des}) = 0 \tag{46}$$

Since $b(\theta) > 0$ and $k_c, k_d, k_e > 0$, it is sufficient to choose the gains as follows:

$$\frac{k_d}{k_e}\min\{b(\theta)\} > E_{des} \tag{47}$$

to avoid the singularity.

Finally, consider an instant of time $t = \eta_i(x)$ when there is a jump in x. From (38) and using (23), (33) and (41), the change in $V(x(t))$ can be written as

$$\begin{aligned} &V(x(t^+)) - V(x(t^-)) \\ &= -\frac{1}{2}k_d(\dot{\theta}_2^-)^2 + \frac{1}{2}k_e\left[(E^+ - E_{des})^2 - (E^- - E_{des})^2\right] \\ &= -\frac{1}{2}k_d(\dot{\theta}_2^-)^2 + \frac{1}{2}k_e(E^+ + E^- - 2E_{des})\ \Delta E \\ &= -\frac{1}{2b(\theta)}(\dot{\theta}_2^-)^2[k_d b(\theta) + \frac{k_e}{4b(\theta)}(\dot{\theta}_2^-)^2 + k_e(E^+ - E_{des})] \end{aligned} \tag{48}$$

Using (47), it can be shown

$$V(x(t^+)) \leq V(x(t^-)), \quad \forall\, t = \eta_i(x), \quad i = 1, 2, ... \quad (49)$$

which satisfies the inequality (27).
With all conditions in *Theorem 1* satisfied, we conclude that the continuous torque in (45) and the braking torque in (10) applied at the times $t = \eta_i(x)$ defined in (31), guarantees stability of the equilibrium $x = 0$.

Having proved stability of the equilibrium $x = 0$, we investigate the largest invariant set in which the system trajectories converge. Since $D^+V \leq 0$ for all $t \neq \eta_i(x)$ and $V(x(t^+)) \leq V(x(t^-))$ for all $t = \eta_i(x)$, we use the Invariance Principle in [18]. In order to find the largest invariant set, we must first find a set Z which is the union of the sets where $\dot{V} = 0$ and the sets where the impulse results in no change in $V(x)$, namely $V(x(t^+)) = V(x(t^-))$. Therefore, using (43) and (48), the set Z can be found as,

$$\begin{aligned} Z =& \{x \in R^3 \mid \dot{\theta}_2\ddot{\theta}_2 = 0,\ x \in \rho_i^{\perp}(x)\} \cup \\ & \{\ x \in R^3 \mid \dot{\theta}_2 = 0,\ x \in \rho_i(x)\} \end{aligned} \quad (50)$$

Note that from (32), if $x \in \rho_i(x)$ then $\dot{\theta}_2 \neq 0$. Therefore Z may be simplified to

$$Z = \{x \in R^3 \mid \dot{\theta}_2\ddot{\theta}_2 = 0,\ x \in \rho_i^{\perp}(x)\} \quad (51)$$

The invariant set M in Z is obtained when $\dot{\theta}_2\ddot{\theta}_2 \equiv 0$. Since $\dot{\theta}_2 \equiv 0$ implies $\ddot{\theta}_2 \equiv 0$, we need only investigate $\ddot{\theta}_2 \equiv 0$. If $\ddot{\theta}_2 \equiv 0$ and $\dot{\theta}_2 \neq 0$, then we find that $\theta_2 = \dot{\theta}_2 t + c$ where c is a constant, implying that $V \to \infty$ as $t \to \infty$. This is a contradiction to $\dot{V} \equiv 0$ and V bounded. Therefore, $\dot{\theta}_2 \equiv 0$ forms the largest invariant set M. Using equation (40) and (42), the invariant set may be written as

$$M = \{\ x \in Z \mid \dot{\theta}_2 \equiv 0,\ k_e(E - E_{des})\tau + k_p(\theta_2) \equiv 0\ \} \quad (52)$$

The set M is the same as that obtained in [6] and [7] and therefore, there exists conditions on k_p, k_d, and k_e as well as initial conditions such that the system trajectories asymptotically converge to the equilibrium $x = 0$. For the sake of brevity, those conditions are not explicitly discussed here. As $x \to 0$, the acrobot behaves as a pendulum and reaches a neighborhood of its desired equilibrium configuration in finite time. The linear controller is then invoked to stabilize this equilibrium configuration $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (\pi/2, 0, 0, 0)$.

V. Numerical Simulations

We compare the efficacy of our controller with the controllers proposed in [6] and [7], which use the same positive definite function V as in (38). The gains k_p, k_d, and k_e and the initial conditions are chosen such that the conditions required for convergence, presented in [6] and [7], are satisfied. By satisfying these conditions, we ensure that the condition for singularity-free torque is also satisfied.

For the first simulation, the kinematic and dynamic parameters are taken from Mahindrakar and Banavar [7]:

$$\begin{aligned} &m_1 = 1.0kg, \quad l_1 = 1m, \quad d_1 = 0.5m, \quad I_1 = 0.083N.m^2 \\ &m_2 = 2.0kg, \quad l_2 = 2m, \quad d_2 = 1.0m, \quad I_2 = 0.667N.m^2 \end{aligned} \quad (53)$$

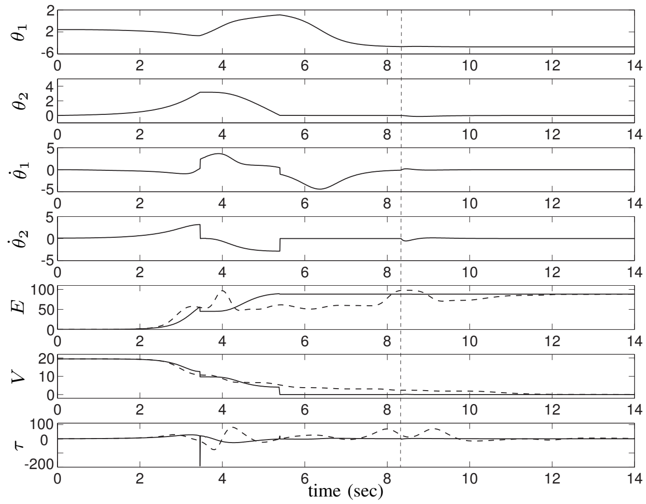

Fig. 3. Plot of joint angles (rad), joint angle velocities (rad/s), total energy (J), Lyapunov-like function and control input (N.m) for the first simulation. In the last three plots, solid and dashed lines show the results for our algorithm and the algorithm presented in [7].

Consistent with the choice in [7], the control gains and initial conditions were chosen as:

$$\begin{aligned} &k_p = 1, \quad k_d = 1, \quad k_e = 0.005, \quad k_c = 1.37 \\ &(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (-\frac{\pi}{2}, 0, 0, 0.1) \end{aligned} \quad (54)$$

For the parameters given in (53), E_{des} was computed to be $88.29\ J$. The simulation results are shown in Fig.3 and they plot the joint angles and angular velocities, the total energy of the system, the positive definite function V, and the control input. The total energy, the function V and the control input are shown for our controller in solid line and for the algorithm in [7] in dashed line. Our controller used impulsive braking twice, at times $t = 3.45\ s$ and $t = 5.38\ s$ approximately. At these times, there is a sudden jump in the values of $\dot{\theta}_1$, $\dot{\theta}_2$, V and E. Using our controller, swing-up is achieved in $8.33\ s$ (shown by the vertical dotted line in Fig.3), after which the linear controller is invoked, as compared to the 18 seconds required by the controller in [7]. The maximum continuous torque for our controller is approximately 28 N.m, which is considerably less than the maximum continuous torque of $80\ Nm$ required by the controller in [7]. The impulsive torque required to brake the second link has a large magnitude and we allay concerns regarding its magnitude in our remark after the next simulation.

For the second simulation, we used the kinematic and dynamic parameters from Xin and Kaneda [6]:

$$\begin{aligned} &m_1 = 1.0kg, \quad l_1 = 1m, \quad d_1 = 0.5m, \quad I_1 = 0.083N.m^2 \\ &m_2 = 1.0kg, \quad l_2 = 2m, \quad d_2 = 1.0m, \quad I_2 = 0.33N.m^2 \end{aligned} \quad (55)$$

The control gains and initial conditions were chosen as:

$$\begin{aligned} &k_p = 61.2, \quad k_d = 61.2, \quad k_e = 1, \quad k_c = 75.6 \\ &(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (-1.4, 0, 0, 0) \end{aligned} \quad (56)$$

For the parameters in (55), E_{des} was computed to be $49.05\ J$. The simulation results are shown in Fig.4. As in the last

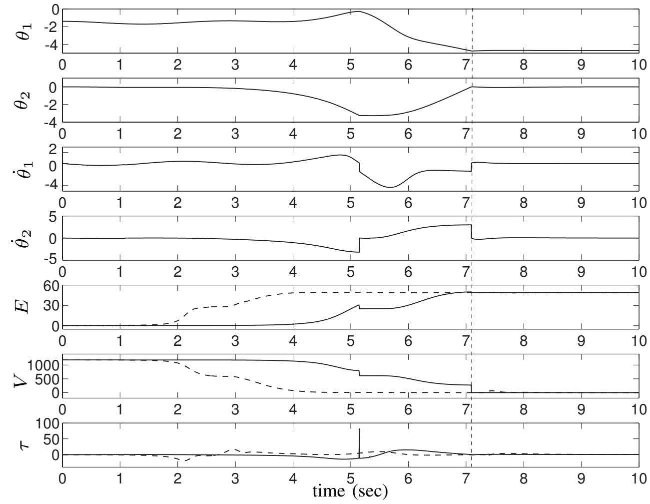

Fig. 4. Plot of joint angles (rad), joint angle velocities (rad/s), total energy (J), Lyapunov-like function and control input (N.m) for the second simulation. In the last three plots, solid and dashed lines show the results for our algorithm and the algorithm presented in [6].

simulation, the solid and dashed lines in the plots for energy, function V and control input correspond to our controller and the controller in [6], respectively. It can be seen from Fig.4 that two impulses were applied at $t = 5.14\ s$ and $t = 7.09\ s$, which cause jumps in the velocities and sudden changes in the function V and energy. With our controller, swing-up was achieved and the linear controller was invoked at $t = 7.09\ s$ which is shown by the vertical dotted line in Fig.4. This is marginally better than the time required by the controller in [6], which is $7.33\ s$. The maximum continuous torque required by our controller is also lower, approximately 15 N.m as compared to 20 N.m required by the controller in [6].

Remark 1: Although the impulsive torques were assumed to provide instantaneous braking of the second link, the numerical simulations above indicate that the impulsive torques need not to be Dirac delta functions and the swing-up algorithm is effective even when the magnitude of the impulse is bounded and its time support is not infinitesimal. The magnitude of the impulsive inputs are however much larger than the continuous torque required by our controller, or the controllers proposed in [6] and [7]. This should not be seen as a problem since actuators such as motors can apply substantially larger torques[2] than the maximum continuous torque over small time intervals.

VI. Conclusion

We proposed an efficient algorithm for swing-up control of the acrobot. The new algorithm uses a continuous torque derived from a positive definite Lyapunov-like function and applies impulsive braking torques at specific instants of time. The impulsive braking torque causes jumps in certain states of the system and thereby results in sudden decrease in the Lyapunov-like function. The asymptotic convergence of the acrobot configuration to the upright equilibrium configuration is proved using stability theory for impulsive dynamical systems. For the sake of comparison, the proposed algorithm is implemented using two sets of acrobot parameters that have appeared in the literature [6], [7]. Both simulation results show faster swing-up and lower maximum continuous torques as compared to the results in [6] and [7]. Our future work will focus on experimental verification of the proposed algorithm for further development of control methods that include impulsive inputs.

[2]This is referred to as the peak torque [19] and it can be twice to ten times larger than the maximum continuous torque for different motors.

VII. Acknowledgments

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