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Improved conditions for reduced-order $\mathscr{H}_\infty$ filter design as a static output feedback problem

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***Abstract*— In this paper, the problem of reduced order $\mathscr{H}_\infty$ filter design for time-invariant discrete-time linear systems is investigated. The filtering problem is rewritten as a static output feedback control problem and the elimination lemma is applied to derive the design conditions for both precisely known and uncertain linear systems. An algorithm is proposed to solve the problem in two stages involving only linear matrix inequalities. A robust filter of arbitrary order is obtained by solving an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error dynamics. Numerical examples are presented to illustrate the advantages of the approach when compared to other techniques.**

I. INTRODUCTION

In the literature to date, the filtering problem for linear systems has been faced by many different approaches. Considering the Lyapunov theory, design conditions ranging from those obtained using quadratic Lyapunov functions, as for instance [1–3], to parameter-dependent ones [4, 5] can be observed. The strategies appeared so far can be used in different contexts as robust filtering [3, 6], gain scheduling filtering [7–10] and filtering of time-delayed systems [11], to mention some.

Several efforts were made to reduce the conservatism of filter design methods in cases where only sufficient conditions are available. It is significant the use of extra variables and polynomial relaxations in the search of better design conditions, which can be noted in many papers dealing with systems subject to parametric uncertainties, such as [6, 10, 12] and internal references. As discussed in [13], the design of robust filters for uncertain systems via parameter-dependent Lyapunov functions is an advanced topic, whose main objective is to reduce the conservatism of the quadratic approach.

What many of these results have in common is the way they treat the products involving the augmented system matrices, that is, the state space matrices obtained after coupling the filter to the plant. In general, the Lyapunov matrix is partitioned, auxiliary matrices are conveniently defined and congruence transformations applied to provide design conditions in terms of a optimization problem based on linear matrix inequalities (LMIs).

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In this paper, a different approach is proposed. The method consists of rewritten the augmented system as a closed-loop system by static output feedback where the filter matrices are embedded in a static gain K. This approach has been mainly explored in controller design problems, as for instance to design reduced order dynamic output compensators [14], with some results in the filtering context [15, 16]. The main contribution of this paper is to present a new filter design strategy based on static output feedback. An immediate advantage is the facility to deal with reduced order filter. The proposed approach does not require the definition of a partitioned Lyapunov matrix nor of the auxiliary matrices mentioned above. The robust filter is obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of LMI constraints. The optimization process is divided into two stages, similarly to the strategy proposed in [17–20]. First, a state feedback gain is obtained and, in the sequel, it is used to get the filter matrices at the second stage. Numerical simulations indicate that the realization and the quality ($\mathscr{H}_\infty$ performance) of the filter provided by the proposed method is related with the choice of the state feedback gain in the first stage. One can search for better $\mathscr{H}_\infty$ performance of the estimation error dynamics by simply exploring different strategies in the first stage. Numerical examples illustrate the advantages of the proposed approach when compared to other techniques from the literature.

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

Consider a stable time-invariant discrete-time linear system

$$\begin{aligned} x(k+1) &= A_s x(k) + B_w w(k) \\ z(k) &= C_z x(k) + D_{zw} w(k) \\ y(k) &= C_y x(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state space vector, $w(k) \in \mathbb{R}^m$ is the noise input belonging to $l_2[0,\infty)$, $z(k) \in \mathbb{R}^p$ is the signal to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output.

A robust proper filter of order r is investigated here, being given by

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f y(k) \\ z_f(k) &= C_f x_f(k) + D_f y(k) \end{aligned} \tag{2}$$

where $x_f(t) \in \mathbb{R}^r$ is the filter state space vector and $z_f(t) \in \mathbb{R}^p$ the estimated signal.

The estimation error dynamics is given by

$$\begin{aligned} \varsigma(k+1) &= \hat{A}\varsigma(k) + \hat{B}w(k) \\ e(k) &= \hat{C}\varsigma(k) + \hat{D}w(k) \end{aligned} \tag{3}$$

where $\varsigma(k) = [x(k)' \;\; x_f(k)']'$, $e(k) = z(k) - z_f(k)$ and

$$\hat{A} = \begin{bmatrix} A_s & \mathbf{0} \\ B_f C_y & A_f \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} B_w \\ \mathbf{0} \end{bmatrix}$$
$$\hat{C} = \begin{bmatrix} C_z - D_f C_y & -C_f \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D_{zw} \end{bmatrix} \tag{4}$$

The filtering problem considered is stated as follows.

Problem 1: Given an order r, find matrices $A_f \in \mathbb{R}^{r\times r}$, $B_f \in \mathbb{R}^{r\times q}$, $C_f \in \mathbb{R}^{p\times r}$ and $D_f \in \mathbb{R}^{p\times q}$, of the filter (2), such that the estimation error system (3) is asymptotically stable, and an upper bound γ to the $\mathscr{H}_\infty$ performance of the estimation error dynamics is minimized.

The following lemma, known as the discrete version of the bounded real lemma, provides a result that relates an upper bound γ to the $\mathscr{H}_\infty$ norm of a stable dynamic system with the existence of a Lyapunov function $v(x) = x'Px$ (see, for instance, [21, 22]). The lemma can be used in seeking a solution to Problem 1.

Lemma 1: If there exist filter matrices A_f, B_f, C_f, D_f, a scalar γ and a matrix $P = P' > 0$ such that[1]

$$\begin{bmatrix} -P & \hat{A}'P & \mathbf{0} & \hat{C}' \\ (\star) & -P & -P\hat{B} & \mathbf{0} \\ (\star) & (\star) & -\gamma^2\mathbf{I} & -\hat{D}' \\ (\star) & (\star) & (\star) & -\mathbf{I} \end{bmatrix} < 0 \tag{5}$$

then the estimation error system (3) is asymptotically stable with an $\mathscr{H}_\infty$ norm upper bounded by γ.

As presented Lemma 1 provides a nonlinear design condition due to the filter matrices appeared inside the augmented system matrices. In this case, some algebraic manipulations are required.

By rewriting the estimation error system (3) as

$$\begin{aligned} \varsigma(k+1) &= (A + B_2KC_2)\varsigma(k) + \tilde{B}w(k) \\ e(k) &= (C_1 + D_2KC_2)\varsigma(k) + \tilde{D}w(k) \end{aligned} \tag{6}$$

with matrices

$$A = \begin{bmatrix} A_{s(n\times n)} & \mathbf{0}_{(n\times r)} \\ \mathbf{0}_{(r\times n)} & \mathbf{0}_{(r\times r)} \end{bmatrix}, \quad B_2 = \begin{bmatrix} \mathbf{0}_{(n\times r)} & \mathbf{0}_{(n\times p)} \\ \mathbf{I}_{(r\times r)} & \mathbf{0}_{(r\times p)} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{z(p\times n)} & \mathbf{0}_{(p\times r)} \end{bmatrix}, \quad C_2 = \begin{bmatrix} \mathbf{0}_{(r\times n)} & \mathbf{I}_{(r\times r)} \\ C_{y((q\times n))} & \mathbf{0}_{(q\times r)} \end{bmatrix},$$

$$\tilde{B}' = \begin{bmatrix} (B_{w(n\times m)})' & (\mathbf{0}_{r\times m})' \end{bmatrix}, \quad D_2 = \begin{bmatrix} \mathbf{0}_{(q\times r)} & -\mathbf{I}_{(q\times q)} \end{bmatrix},$$

$$\tilde{D} = D_{zw(p\times m)}$$

and a static output feedback gain

$$K = \begin{bmatrix} A_{f(r\times r)} & B_{f(r\times q)} \\ C_{f(p\times r)} & D_{f(p\times q)} \end{bmatrix}, \tag{7}$$

a solution to Problem 1 is equivalently obtained by designing (7) that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance of system (6).

[1]The symbol $(\star)$ stands for symmetric blocks.

A. $\mathscr{H}_\infty$ filtering through a noisy-output measurement

As can be seen in (1), the filtering problem considered previously assumes only the presence of noise in the process, that is, in the equation that describes the dynamics, being the measured output free of disturbances. However, in most practical applications, the measurements made in physical systems are not free of errors caused by the presence of noise. In this case, the output $y(k)$ of (1) becomes

$$y(k) = C_y x(k) + D_{yw} w(k)$$

and the matrices $\hat{B}$ and $\hat{D}$ of the estimation error system (3)

$$\hat{B} = \begin{bmatrix} B_w \\ B_f D_{yw} \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D_{zw} - D_f D_{yw} \end{bmatrix}. \tag{8}$$

The main difference with respect to the case without noise occurs when one rewrite the estimation error system as a closed loop system by static output feedback. The noise matrices become similar to those of a static feedback of the noise, although there is not such physical interpretation. The system (6) becomes

$$\begin{aligned} \varsigma(k+1) &= (A + B_2KC_2)\varsigma(k) + (\tilde{B} + B_2K\tilde{D}_2)w(k) \\ e(k) &= (C_1 + D_2KC_2)\varsigma(k) + (\tilde{D} + D_2K\tilde{D}_2)w(k) \end{aligned} \tag{9}$$

with the new term

$$\tilde{D}_2' = \begin{bmatrix} (\mathbf{0}_{r\times m})' & (D_{yw(q\times m)})' \end{bmatrix}.$$

In order to design the output feedback gain K, an extension of the method proposed in [18, 19] is applied. It consists of adding extra variables to decouple the Lyapunov matrix and the static output gain, similarly to what has been done in the context of robust control in [23]. The output feedback gain is obtained as the solution of an LMI problem, after the choice of a state feedback controller.

For completeness, the elimination lemma, used in the proofs of the main results, is reproduced in the sequel [24]

Lemma 2: Given the matrices $\mathscr{U} \in \mathbb{C}^{n\times m}$, $\mathscr{V} \in \mathbb{C}^{k\times n}$ and $\Phi = \Phi^* \in \mathbb{C}^{n\times n}$, the following statements are equivalent:

i) there exists a matrix $X \in \mathbb{C}^{m\times k}$ satisfying

$$\mathscr{V}X\mathscr{U} + (\mathscr{V}X\mathscr{U})^* + \Phi < 0$$

ii) the following two conditions hold:

$$\begin{aligned} \mathscr{N}_v \Phi \mathscr{N}_v^* < 0 \quad &\text{or} \quad \mathscr{V}\mathscr{V}^* > 0 \\ \mathscr{N}_u^* \Phi \mathscr{N}_u < 0 \quad &\text{or} \quad \mathscr{U}^*\mathscr{U} > 0 \end{aligned}$$

where $\mathscr{N}_v$ and $\mathscr{N}_u^*$ are respectively orthogonal complement of $\mathscr{V}$ and $\mathscr{U}^*$, that is (considering appropriate matrix dimensions)

$$\mathscr{N}_v \mathscr{V} = 0, \quad \mathscr{N}_u^* \mathscr{U}^* = 0.$$

III. Main Results

Theorem 1: For a given positive scalar γ and state feedback gain $K_0 \in \mathbb{R}^{(r+p)\times(r+n)}$, if there exist symmetric matrices $P \in \mathbb{R}^{(n+r)\times(n+r)}$, $W \in \mathbb{R}^{m\times m}$ and matrices $G \in \mathbb{R}^{(r+p)\times(r+p)}$ and $L \in \mathbb{R}^{(r+p)\times(r+q)}$, such that

$$\begin{bmatrix} -P & A_0'P & C_2'L' - K_0'G' & \mathbf{0} & C_1' + K_0'D_2' \\ (\star) & -P & PB_2 & -P\tilde{B} & \mathbf{0} \\ (\star) & (\star) & -(G+G') & -L\tilde{D}_2 & D_2' \\ (\star) & (\star) & (\star) & -\gamma^2\mathbf{I} & -\tilde{D}' \\ (\star) & (\star) & (\star) & (\star) & -\mathbf{I}_{p\times p} \end{bmatrix} < 0 \quad (10)$$

where $A_0 = A + B_2K_0$, then there exists a robust filter in the form of (2) ensuring the asymptotic stability of the estimation error dynamics (3) and an upper bound γ to the $\mathscr{H}_\infty$ index of performance. The filter matrices are given by (7) with $K = G^{-1}L$.

Proof: First, note that LMI (10) can be rewritten as statement *i*) of Lemma 2 with matrices $X = G$,

$$\mathscr{U} = \begin{bmatrix} S & \mathbf{0} & -\mathbf{I} & -\tilde{S} & \mathbf{0} \end{bmatrix}, \quad \mathscr{V}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $S = KC_2 - K_0$, $\tilde{S} = K\tilde{D}_2$, and

$$\Phi = \begin{bmatrix} -P & A_0'P & \mathbf{0} & \mathbf{0} & C_1' + K_0'D_2' \\ (\star) & -P & PB_2 & -P\tilde{B} & \mathbf{0} \\ (\star) & (\star) & \mathbf{0} & \mathbf{0} & D_2' \\ (\star) & (\star) & (\star) & -\gamma^2\mathbf{I} & -\tilde{D}' \\ (\star) & (\star) & (\star) & (\star) & -\mathbf{I} \end{bmatrix}$$

together with the change of variables $GK = L$. Second, defining $\mathscr{N}_u$ and $\mathscr{N}_v$ as follows

$$\mathscr{N}_v = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathscr{N}_u = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ S & \mathbf{0} & -\tilde{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

the inequalities $\mathscr{N}_u^*\Phi\mathscr{N}_u < 0$ and $\mathscr{N}_v\Phi\mathscr{N}_v^* < 0$ yield, respectively

$$\begin{bmatrix} -P & \mathscr{F}_{12} & \mathbf{0} & (C_1 + D_2KC_2)' \\ (\star) & -P & -P(\tilde{B} + B_2K\tilde{D}_2) & \mathbf{0} \\ (\star) & (\star) & -\gamma^2\mathbf{I} & -(\tilde{D} + D_2K\tilde{D}_2)' \\ (\star) & (\star) & (\star) & -\mathbf{I} \end{bmatrix} < 0$$
$$\mathscr{F}_{12} = (A + B_2KC_2)'P$$

and

$$\begin{bmatrix} -P & (A + B_2K_0)'P \\ P(A + B_2K_0) & -P \end{bmatrix} < 0$$

where the first one is equivalent to (5) when Lemma 1 is applied to system (6) and the second one certifies the stability of $A + B_2K_0$. Then, in accordance with Lemma 2, one can conclude that the estimation error dynamics (3) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ index of performance, what ends the proof. ■

Theorem 1 was obtained using the elimination lemma (specially interesting in the context of uncertain systems, as discussed in the sequel) extending the method proposed in [19] by considering the $\mathscr{H}_\infty$ index of performance, similar to [17, 18] in the context of $\mathscr{H}_2$ controller design. The solution proposed here is given in two steps as shown bellow.

Algorithm 1:

1) Find a state feedback control law $u = K_0\varsigma(k)$ that stabilizes the system

$$\begin{aligned} \varsigma(k+1) &= A\varsigma(k) + B_2u(k) + (\tilde{B} + B_2K\tilde{D}_2)w(k) \\ e(k) &= C_1\varsigma(k) + D_2u(k) + (\tilde{D} + D_2K\tilde{D}_2)w(k) \end{aligned} \quad (11)$$

and minimizes the $\mathscr{H}_\infty$ index of performance with respect to the noise input $w(k)$;

2) Fix K_0, γ and r and solve Theorem 1.

Theorem 1 provides a way to solve the $\mathscr{H}_\infty$ filtering problem as an equivalent static output feedback problem by appropriately exploring Algorithm 1. Some advantages of this approach, known from the results concerned with controller design problems, include the facility in dealing with reduced, or augmented, order filter design by manipulating the parameter r. Moreover, the methods appeared so far in the literature to design static output feedback controllers can be explored in the search of a solution to Problem 1.

It is worth mentioning the importance of the first step in Algorithm 1 in the solution of Theorem 1, and its relation with the quality of the filter to be obtained. In the case of discrete systems, the hypothesis of stability of system (1) implies the stability of the estimation error dynamics (6) (note that the eigenvalues of the matrix A are equal to the eigenvalues of the matrix A_s plus 'r 'extra eigenvalues in the origin). Consequently, $K_0 = \mathbf{0}$ is a feasible choice for the second step of the proposed algorithm and obviously a feasible K_0 will always exists. However, this choice may eventually lead to unfeasible filters (unfeasible gains, realizations). Note by the augmented system matrices in (4) for example, that a static filter ($A_f = \mathbf{0}$) provides an estimation error dynamics asymptotically stable under the assumption of A_s be stable. In this case, despite the asymptotic stability of the estimation error dynamics, the $\mathscr{H}_\infty$ performance of the filter may be affected. One possible way to overcome this situation is by using a performance criterion in both stages of Algorithm 1, especially in the first step. Other heuristics could be used to find the best filter provided by the conditions of Theorem 1, by letting γ as a variable in both steps of Algorithm 1. In this paper, K_0 has been selected as the stabilizing state feedback gain that minimizes the $\mathscr{H}_\infty$ norm of the transfer function from the noise input $w(k)$ to the error $e(k)$. By minimizing an upper bound to the $\mathscr{H}_\infty$ performance in the first step, it is possible to avoid trivial solutions such as $A_f = \mathbf{0}$, $B_f = \mathbf{0}$, $C_f = \mathbf{0}$, $D_f = \mathbf{0}$, arising from choices as $K_0 = \mathbf{0}$. Exceptions occur when $K_0 = \mathbf{0}$ coincides with the lowest value of $\mathscr{H}_\infty$ norm, or when Theorem 1 has no feasible solution.

In order to find the state feedback gain K_0 that minimizes the $\mathscr{H}_\infty$ norm of system (11), the method proposed in [22, Theorem 10] was applied. The main issue that appears at this point is concerned with the use of the matrices that multiply the noise vector $w(k)$, that is, $(\tilde{B} + B_2K\tilde{D}_2)$ and $(\tilde{D} + D_2K\tilde{D}_2)$. For obvious reasons, the variable K can not be used as an input parameter in the first step of Algorithm 1 because its value will be determined only at the second step. The proposed alternative is to use random values different from zero. The goal is to achieve an effect similar to the matrices $(\tilde{B}+B_2K\tilde{D}_2)$ and $(\tilde{D}+D_2K\tilde{D}_2)$, meanwhile reducing the chances of having only trivial solutions ($K_0=\mathbf{0}$). In the second stage of the algorithm the original noise matrices (8), along with the designed gain K_0, are used.

The difficulty appeared in the design of the state feedback gain is related to the use of the elimination lemma to introduce extra variables and decouple the Lyapunov matrix from the static feedback gain. If the value of the $\mathscr{H}_\infty$ norm found in the second stage is not satisfactory, the method can be applied again with different values for the state feedback gain. New values of K_0 can be tunned using the information contained in the matrices $\tilde{B}$ and $\tilde{D}$ with random choices for the terms $(B_2K\tilde{D}_2)$ and $(KD_2\tilde{D}_2)$, or with different performance criteria in the first stage (as for instance, $\mathscr{H}_2$ norm instead of $\mathscr{H}_\infty$, pole placement, besides others).

A. Robust filter

Assuming system (1) uncertain, with matrices belonging to the polytope

$$\tilde{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A_s(\alpha) & B_w(\alpha) \\ \hline C_z(\alpha) & D_{zw}(\alpha) \\ \hline C_y(\alpha) & D_{yw}(\alpha) \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_{si} & B_{wi} \\ \hline C_{zi} & D_{zwi} \\ \hline C_{yi} & D_{ywi} \end{array}\right] \right\} \tag{12}$$

where $\alpha \in \mathscr{U}_N$ models the uncertainties, with

$$\mathscr{U}_N = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0\ ,\ i=1,\ldots,N \right\},$$

Theorem 1 can be extended as follows.

Theorem 2: For a given positive scalar γ and state feedback gain $K_0 \in \mathbb{R}^{(r+p)\times(r+n)}$, if there exist symmetric matrices $P_i \in \mathbb{R}^{(n+r)\times(n+r)}$, $W_i \in \mathbb{R}^{m\times m}$, $i=1,\ldots,N$, and matrices $G \in \mathbb{R}^{(r+p)\times(r+p)}$ and $L \in \mathbb{R}^{(r+p)\times(r+q)}$, such that

$$\Psi_i = \begin{bmatrix} -P_i & \mathscr{G}_{12} & C_{2i}'L' - K_0'G' & \mathbf{0} & C_{1i}' + K_0'D_{2i}' \\ (\star) & -P_i & P_iB_{2i} & -P_i\tilde{B}_i & \mathbf{0} \\ (\star) & (\star) & -(G+G') & -L\tilde{D}_{2i} & D_{2i}' \\ (\star) & (\star) & (\star) & -\gamma^2\mathbf{I} & -\tilde{D}_i' \\ (\star) & (\star) & (\star) & (\star) & -\mathbf{I}_{p\times p} \end{bmatrix} < 0 \tag{13}$$

$$i = 1,\ldots,N$$
$$\mathscr{G}_{12} = A_i'P_i + K_0'B_{2i}'P_i$$

$$\Psi_{ij} = \begin{bmatrix} -P_i-P_j & \mathscr{T}_{12} & \mathscr{T}_{13} & \mathbf{0} & \mathscr{T}_{15} \\ (\star) & -P_i-P_j & \mathscr{T}_{23} & \mathscr{T}_{24} & \mathbf{0} \\ (\star) & (\star) & \mathscr{T}_{33} & \mathscr{T}_{34} & \mathscr{T}_{35} \\ (\star) & (\star) & (\star) & -2\gamma^2\mathbf{I} & \mathscr{T}_{45} \\ (\star) & (\star) & (\star) & (\star) & -2\mathbf{I}_{p\times p} \end{bmatrix} < 0 \tag{14}$$

$$i=1,\ldots,N-1,\ j=i+1,\ldots,N$$
$$\mathscr{T}_{12} = A_i'P_j + A_j'P_i + K_0'B_{2j}'P_i + K_0'B_{2i}'P_j$$
$$\mathscr{T}_{13} = C_{2i}'L' + C_{2j}'L' - 2K_0'G'$$
$$\mathscr{T}_{15} = C_{1i}' + C_{1j}' + K_0'D_{2i}' + K_0'D_{2j}'$$
$$\mathscr{T}_{23} = P_iB_{2j} + P_jB_{2i}$$
$$\mathscr{T}_{24} = -P_i\tilde{B}_j - P_j\tilde{B}_i$$
$$\mathscr{T}_{33} = -2(G+G')$$
$$\mathscr{T}_{34} = -L\tilde{D}_{2i} - L\tilde{D}_{2j}$$
$$\mathscr{T}_{35} = D_{2i}' + D_{2j}'$$
$$\mathscr{T}_{45} = -\tilde{D}_i' - \tilde{D}_j'$$

then there exists a robust filter in the form of (2) ensuring the asymptotic stability of the estimation error dynamics (3) and an upper bound γ to the $\mathscr{H}_\infty$ index of performance. The filter matrices are given by (7) with $K = G^{-1}L$.

Proof: Applying the following operation [25]

$$\Psi(\alpha) = \sum_{i=1}^{N} \alpha_i^2 \Psi_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i\alpha_j\Psi_{ij} \tag{15}$$

to the LMIs (13) and (14) one gets

$$\Psi(\alpha) = \begin{bmatrix} -P(\alpha) & \mathscr{X}_{12}(\alpha) & \mathscr{X}_{13}(\alpha) & \mathbf{0} & \mathscr{X}_{15}(\alpha) \\ (\star) & -P(\alpha) & \mathscr{X}_{23}(\alpha) & \mathscr{X}_{24}(\alpha) & \mathbf{0} \\ (\star) & (\star) & \mathscr{X}_{33}(\alpha) & \mathscr{X}_{34}(\alpha) & D_2(\alpha)' \\ (\star) & (\star) & (\star) & -\gamma^2\mathbf{I} & -\tilde{D}(\alpha)' \\ (\star) & (\star) & (\star) & (\star) & -\mathbf{I}_{p\times p} \end{bmatrix}$$

$$\mathscr{X}_{12}(\alpha) = A(\alpha)'P(\alpha) + K_0'B_2(\alpha)'P(\alpha)$$
$$\mathscr{X}_{13}(\alpha) = C_2(\alpha)'L' - K_0'G'$$
$$\mathscr{X}_{15}(\alpha) = C_1(\alpha)' + K_0'D_2(\alpha)'$$
$$\mathscr{X}_{23}(\alpha) = P(\alpha)B_2(\alpha)$$
$$\mathscr{X}_{24}(\alpha) = -P(\alpha)\tilde{B}(\alpha)$$
$$\mathscr{X}_{33}(\alpha) = -(G+G')$$
$$\mathscr{X}_{34}(\alpha) = -L\tilde{D}_2(\alpha)$$

with $\Psi(\alpha) < 0$.

From this step on the proof follows similar to the proof of Theorem 1 but with respect to a parameter-dependent version of Lemma 1. ■

The use of Lemma 2 makes possible to develop an interesting design procedure to face Problem 1 by an equivalent static output feedback problem. By appropriately exploring the first stage of Algorithm 1 one may find a sequence of decreasing $\mathscr{H}_\infty$ upper bound γ. Several strategies can be used to select the matrices of the noise in the first stage, such as random inputs, proposed earlier, or more elaborate techniques such as evolutionary algorithms (genetic algorithms, ant colonies, among others). Moreover, in the case of uncertain systems, Lemma 2 can be applied in an iterative process to provide more extra variables to the problem, increasing the degree of freedom during the solution of Theorem 2. As can be seen, the $\mathscr{H}_\infty$ robust filters design methods presented in theorems 1 and 2 may be extended in many different ways. These topics are under investigation by the authors.

Finally, it is important to stress that reduced-order filters are very important from a practical point of view, especially for implementation purposes. As an example, one can cite the multirate filter bank design problem, as can be seen in [15, 16], where the authors also rewrite the filtering problem as a static output feedback problem, but apply a different algorithm to solve it (a comparison with [16] is presented in the next section). The use of the proposed method in the design of filter banks is also under investigation by the authors.

IV. Numerical Experiments

All the experiments have been performed in a PC equipped with: Linux Ubuntu 9.04, Athlon 64 X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using the SDP solver SeDuMi [26] interfaced by the parser YALMIP [27], MATLAB 7.0.1.

Example I (Precisely known case)

Consider system (1) with the following matrices

$$A_s = \begin{bmatrix} 0.6 & 0.1 & 0.2 & -0.3 & -0.2 & 0 \\ 0 & 0.4 & -0.3 & 0.2 & 0.1 & 0.1 \\ 0.3 & -0.2 & 0.1 & -0.1 & 0 & -0.2 \\ -0.1 & 0.3 & 0.1 & -0.3 & 0.1 & 0.05 \\ 0.1 & 0.2 & -0.1 & 0.1 & 0.3 & 0.1 \\ 0.3 & 0.1 & -0.2 & 0.3 & 0.2 & -0.3 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C'_y = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 1 \end{bmatrix}, \quad C'_z = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and null D_{zw}. This system was also considered in [16]. Theorem 1 was applied using random values for matrices $(B_2 K \tilde{D}_2)$ and $(D_2 K \tilde{D}_2)$ in the first stage of Algorithm 1, given a reduced order filter with better performance than the one proposed in [16]. The results are summarized in Table I.

TABLE I

RESULTS FOR EXAMPLE I.

Method	Filter order	γ
[16]	3	0.56
Theorem 1	3	0.47

The reduced order ($n = 3$) filter matrices synthesized by the proposed conditions are

$$A_f = \begin{bmatrix} 0.73 & 1.21 & 0.01 \\ -0.91 & -1.42 & 0.02 \\ 1.32 & 1.20 & -0.67 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0.0057 \\ -0.0001 \\ -0.0284 \end{bmatrix}, \quad C'_f = \begin{bmatrix} 37.47 \\ 60.82 \\ -1.21 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0.41 \end{bmatrix}$$

Example II (Uncertain case)

Consider system (1) with the following matrices

$$A_s = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix}, B_w = \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C'_y = \begin{bmatrix} -100 \\ 10 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C'_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and null D_{zw} with $|\delta| \leq 0.45$. This system was also considered in [6] and can be represented by a polytope with two vertices. Theorem 2 was applied using random values for matrices $(B_2 K \tilde{D}_2)$ and $(D_2 K \tilde{D}_2)$ in the first stage of Algorithm 1. The results are summarized in Table II.

As can be noticed, the reduced order ($n = 1$) filter designed by the proposed method outperforms the full order filters designed by the methods appeared in [4] and [28] and has a performance close to the full order filter proposed in [6] (15% greater).

TABLE II

RESULTS FOR EXAMPLE II.

Method	Filter order	γ
[6]	2	1.82
[28]	2	2.15
[4]	2	3.20
Theorem 2	1	2.11

V. CONCLUSION

This paper presented an $\mathscr{H}_\infty$ robust filter design procedure for both uncertain and precisely known systems. The proposed approach rewrites the problem of filtering as a static output feedback problem. The main feature of the proposed conditions is in the separation of the design matrix variables from the Lyapunov matrix and in the use of slack variables. The filter is obtained by solving an optimization problem with LMIs constraints. The reduced-order filter design can also be faced by the proposed conditions by simply setting a specific parameter. The proposed algorithm for solving the problem can be exploited in different ways in the search of better $\mathscr{H}_\infty$ performance, as some suggestions presented, providing more flexibility for the designer. The numerical experiment section reinforces the discussion presented throughout the text.

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