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Flexible Aircraft Reduced-Order LPV Model Generation from a Set of Large-Scale LTI Models

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***Abstract*— In the civilian aeronautical industry, flexible aircraft models are often built and validated at frozen flight and mass configurations. Unfortunately, these medium(large)-scale models derived from high fidelity numerical tools are generally not well adapted for simulation, control and analysis. In this paper, a methodology to derive a reduced-order Linear Parameter Varying (LPV) model from a set of medium(large)-scale Linear Time Invariant (LTI) models describing a given system at frozen configurations is described. The proposed methodology is in three steps: (i) first, local model approximation is applied using recent advances in SVD-Krylov methods, (ii) then, an appropriate base change is applied to allow interpolation, (iii) and finally, an LPV model is derived and converted into a Linear Fractional Representation (LFR) of suitable size for analysis and control purposes. Results are thoroughly assessed on a set of industrial aeroelastic aircraft models.**

I. INTRODUCTION

A. Motivations

The increasing use of computer based modeling softwares often leads to an increasing number of variables and resources to manage, resulting in an expensive numerical cost. Moreover, from a control side, modern analysis and synthesis tools are drastically inefficient for such high dimensional dynamical systems. This is especially true in the flight dynamics domain, where several models are built for different flight and mass configurations (such as the Mach number, the airspeed and the various tanks filling levels…). Indeed, the entire models set describing the system over the complete parametric domain often becomes very big and leads to hard problems when stability, performance analyses and control design are performed (*e.g.* μ-analysis, $\mathcal{H}_{\infty,2}$ control…). These observations, supported by the recurrent industrial partners demand, are the underlying justification for this work.

Starting from a set of medium(large)-scale Linear Time Invariant (LTI) models describing a complex system at frozen configurations, the main contribution of this paper is to propose a methodology to obtain a reduced-order Linear Parameter Varying (LPV) model of suitable form, from which a Linear Fractional Representation (LFR) can be built to be used in place of the original LTI models. Another contribution is to apply the proposed approach to a set of real-world aeroelastical aircraft models.

B. Problem definition, structure & notations

Let us consider n_s stable SIMO LTI dynamical models $(\Sigma_i)_{i\in\{1,n_s\}}$ ($i \in \mathbb{N}$) of order n corresponding to given parametric configurations $(\delta^{(i)})_{i\in\{1,n_s\}}$:

$$\Sigma_i : \left\{ \begin{array}{rcl} \dot{x}_i(t) &=& A_i x_i(t) + b_i u(t) \\ y(t) &=& C_i x_i(t) + d_i u(t) \end{array} \right. , \; i = 1, \ldots, n_s \quad (1)$$

where $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m\times n}$ and $d_i \in \mathbb{R}^m$. The aim of this paper is to find a reduced-order parametrized model $\hat{\Sigma}(\delta)$ of order $r \ll n$, which approximates Σ_i when $\delta = \delta^{(i)}$:

$$\hat{\Sigma}(\delta) : \left\{ \begin{array}{rcl} \dot{\hat{x}}(t) &=& \hat{A}(\delta)\hat{x}(t) + \hat{b}(\delta)u(t) \\ \hat{y}(t) &=& \hat{C}(\delta)\hat{x}(t) + \hat{d}(\delta)u(t) \end{array} \right. \quad (2)$$

where $\hat{A}(\delta) \in \mathbb{R}^{r\times r}$, $\hat{b}(\delta) \in \mathbb{R}^r$, $\hat{C}(\delta) \in \mathbb{R}^{m\times r}$ and $\hat{d}(\delta) \in \mathbb{R}^m$. Another requirement is that the eigenvalues and the frozen frequency responses of $\hat{\Sigma}(\delta)$ evolve smoothly whatever the variation of δ inside the whole considered parametric domain.

The paper is organized as follows. In Section II, projection-based model approximation methods for SIMO LTI systems are briefly recalled. In Section III, an extension is proposed to properly interpolate the reduced-order models and construct a reduced-order parametrized model. In Section IV, the proposed methodology is validated on a set of industrial aircraft models, thus illustrating the consistency of the approach in a complex real-world application. Finally, Section V concludes and discusses the paper results.

In this paper, the state vectors of the original and the reduced-order systems are denoted $x \in \mathbb{R}^n$ and $\hat{x} \in \mathbb{R}^r$ respectively. W and V denote left and right projectors respectively. V_r denotes the first r columns of V. State-space (resp. transfer) form is denoted Σ (resp. $H(s)$) and $\delta \in \mathbb{R}^l$ gathers the varying parameters.

II. APPROXIMATION BY PROJECTION METHODS FOR SIMO LTI SYSTEMS

Model reduction is an active research field covering both numerical and control communities and where many approaches have been developed (see *e.g.* [1]). Without loss of generality, the projection framework, which consists of projecting the original system, lying on an initial space, onto a reduced one, is clearly the most appropriate for (very) large-scale systems reduction [2], [1]. For SIMO LTI systems, this problem can be formulated as follows.

Definition 1 (Projections-based approximation problem): Given the following SIMO LTI system (with $x \in \mathbb{R}^n$):

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + bu(t) \\ y(t) &=& Cx(t) + du(t) \end{array} \right. \quad (3)$$

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the projection-based reduction problem consists of finding $V, W \in \mathbb{R}^{n\times r}$ ($W^TV = I_r$, $r \ll n$) such that the reduced-order system $\hat{\Sigma}$ (with $\hat{x} \in \mathbb{R}^r$), defined as:

$$\hat{\Sigma} : \left\{ \begin{array}{rcl} \dot{\hat{x}}(t) & = & \hat{A}\hat{x}(t) + \hat{b}u(t) \\ \hat{y}(t) & = & \hat{C}\hat{x}(t) + \hat{d}u(t) \end{array} \right. \tag{4}$$

accurately approximates Σ. In this case, we have $\hat{A} = W^TAV$, $\hat{b} = W^Tb$, $\hat{C} = CV$ and $\hat{d} = d$.

Projection-based model reduction techniques can be classified in two broad categories: (i) *SVD* (and Sylvester like) (ii) *Krylov*. The former, widely used in the control community, derives from the fact that the related reduction is based on SVD (Singular Value Decomposition), while the second one, well known from the numerical community, is based on the construction of Krylov subspaces. As each approach presents specific and complementary advantages and bottlenecks, deep attention has recently been given to a third category, the *SVD-Krylov* one, proposed by *Gugercin* [3]. In the following, these three approaches are briefly recalled, together with their main properties.

A. SVD-based approaches

SVD-based methods are grounded on two main points: the solution of two *Lyapunov equations* and the *SVD* computation. Practically, the balanced realization is the key point.

1) Balanced realization: A SIMO LTI system of the form (3), it is said to be balanced iff $\mathcal{P} = \mathcal{Q} = \textbf{diag}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n$ denote the Hankel singular values (sorted in decreasing order), and where $\mathcal{P}$ and $\mathcal{Q}$ denote the controllability and the observability Gramians, solutions of the Lyapunov equations:

$$\begin{array}{rcl} A\mathcal{P} + \mathcal{P}A^T + bb^T & = & 0 \\ A^T\mathcal{Q} + \mathcal{Q}A + C^TC & = & 0 \end{array} \tag{5}$$

If so, the balanced realization Σ_b is given as:

$$\Sigma_b : \left\{ \begin{array}{rcl} \dot{x}_b(t) & = & W^TAVx_b(t) + W^Tbu(t) \\ y(t) & = & CVx_b(t) + du(t) \end{array} \right. \tag{6}$$

where $x_b \in \mathbb{R}^n$, $V = UZ\Sigma^{-1/2}$ and $W = LY\Sigma^{-1/2}$ (with $\mathcal{P} = UU^T$, $\mathcal{Q} = LL^T$ and $U^TL = Z\Sigma Y^T$).

2) Balanced Truncation (BT): Assume that Σ_b is given in a balanced form, and let V_r and W_r denote the first r columns of V and W. As described in Definition 1, the reduced-order system $\hat{\Sigma}$, obtained by balanced truncation, is achieved by applying V_r and W_r projectors on the initial model Σ.

3) Properties and remarks: This reduction method consists of removing the states which are simultaneously difficult to control and to observe. When applied to a stable system, such approximation approach preserves stability and guarantees an upper bound on the approximation error (in the $\mathcal{H}_\infty$ norm sense). Despite these very nice properties, the bottleneck of SVD-based approaches is that they are numerically costly [4]. Moreover, as illustrated in Section IV, they are not optimal in term of $\mathcal{H}_2$ error.

B. Krylov-based approaches

While the previous approaches consist of projecting the initial system onto the dominant subspace spanned by the eigenvectors of the controllability and observability Gramians product, the Krylov-based approaches lead to a projection onto the reachability and/or detectability subspaces. The underlying idea of Krylov-based methods is the moment matching problem [2], together with the *Arnoldi algorithm*. The moment matching is guaranteed by the choice of the projection matrices. *Krylov subspaces* is a numerical tool to allow moment matching without computing them explicitly.

1) Moment matching problem: Given an initial SIMO LTI system (3), its associated transfer matrix $H(s) = C(sI - A)^{-1}b + d \in \mathbb{C}^m$ can be decomposed through a Taylor series around a given shift point $\sigma \in \mathbb{C}$, as follows:

$$H(s)|_\sigma = d + \sum_{i=0}^{\infty} \eta_i(s-\sigma)^i \ , \ i \in \mathbb{N} \tag{7}$$

where $\eta_i = -C(A - \sigma I)^{-(i+1)}b \in \mathbb{C}^m$ is the ith moment at σ. The approximation problem consists of seeking:

$$\hat{H}(s)|_\sigma = d + \sum_{i=0}^{\infty} \hat{\eta}_i(s-\sigma)^i \ , \ i \in \mathbb{N} \tag{8}$$

such that $\eta_i = \hat{\eta}_i$ at σ for $i = 0, \ldots, q(r)$, where $q(r) \in \mathbb{N}$ denotes the number of moments matched by the reduced-order model. Since moments are ill-conditioned, [2] proposes a numerically efficient way to guarantee moment matching without computing them explicitly, through the use of projection matrices constructed to span the *Krylov subspaces*.

Definition 2 (Krylov subspaces): Given $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $C \in \mathbb{R}^{m\times n}$, the r order Krylov $\mathcal{K}_r$ subspace is the linear subspace spanned by the images of b (resp. C^T) under the first r powers of A (resp. A^T), that is:

$$\begin{array}{rcl} \mathcal{K}_r(A, b) & := & \textbf{span}\{b, Ab, \ldots, A^{r-1}b\} \\ \mathcal{K}_r(A^T, C^T) & := & \textbf{span}\{C^T, A^TC^T, \ldots, A^{T(r-1)}C^T\} \end{array} \tag{9}$$

Based on Definition 2, Theorem 1 holds.

Theorem 1 (Krylov spaces and moment matching [2]): Given $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{m\times n}$, $r_b \in \mathbb{N}^*$, $r_c \in \mathbb{N}^*$ and $\sigma \in \mathbb{C}$, if:

$$\begin{array}{c} \mathcal{K}_{r_b}\big((A - \sigma I)^{-1}, b\big) \subseteq \mathcal{V} = \textbf{span}(V) \\ \mathcal{K}_{r_c}\big((A^T - \overline{\sigma} I)^{-1}, C^T\big) \subseteq \mathcal{W} = \textbf{span}(W) \end{array} \tag{10}$$

where $W^TV = I_r$, and σ is chosen such that $A - \sigma I$ is invertible, then, the moments of Σ and $\hat{\Sigma}$ satisfy:

$$\eta_i|_\sigma = \hat{\eta}_i|_\sigma \ , \text{ for } i = 0, 1, \ldots, q(r) \tag{11}$$

where $q(r) = r_b + \lfloor r_c/m \rfloor - 1$.

Practically, due to the increasing power of the $A - \sigma I$ matrix, these subspaces tend to rapidly converge toward their eigenvectors, leading to poorly ranked matrices. To avoid this, a numerically efficient solution consists of generating an orthogonal basis of this subspace, using the Arnoldi algorithm (see [5]). Applied to our reduction problem, through Theorem 1, the reduced-order model $\hat{\Sigma}$ can be obtained using the projection given in Definition 1. This kind of approach well approximates the original system around σ.

2) (Iterative) Rational Krylov Algorithm: The Rational Krylov Algorithm is an extension of the previous single moment matching problem, allowing for multiple point interpolation [2], [6]. It is formalized in Theorem 2.

Theorem 2 (Rational Krylov subspace [2]): Given $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{m\times n}$, $r_{b_k} \in \mathbb{N}^{K*}$, $r_{c_k} \in \mathbb{N}^{K*}$ and $\sigma_k \in \mathbb{C}^K$ such that $A - \sigma_k I$ are invertible, if:

$$\begin{aligned}\bigcup_{k=1}^{K} \mathcal{K}_{r_{b_k}}\left((A-\sigma_k I)^{-1}, (A-\sigma_k I)^{-1} b\right) = \mathbf{span}(V) \\ \bigcup_{k=1}^{K} \mathcal{K}_{r_{c_k}}\left((A^T-\overline{\sigma}_k I)^{-1}, (A^T-\overline{\sigma}_k I)^{-T} C^T\right) = \mathbf{span}(W)\end{aligned} \tag{12}$$

(denoted $\mathcal{K}_{r_{b_k}}(A, b, \sigma_k)$ and $\mathcal{K}_{r_{c_k}}(A^T, C^T, \overline{\sigma}_k)$ respectively, with $W^T V = I_r$), then the moments of Σ and $\hat{\Sigma}$ satisfy $\hat{\eta}_{\sigma_k}^{(j_k)} = \eta_{\sigma_k}^{(j_k)}$ for $j_k = 0, \ldots, q(r)$, where $k = 1, \ldots, K$ and $q(r) = r_{b_k} + \lfloor r_{c_k}/m \rfloor - 1$.

By assuming $r_{b_k} = r_{c_k} = r_k$, *Gugercin et al.* [6] show how to select the σ_k interpolation points in order to reach the first order $\mathcal{H}_2$ optimality conditions (see Algorithm 1).

Algorithm 1 Iterative Rational Krylov (IRKA) [6]

Require: A, b, C, $\sigma_k^{(0)}$, r_k
1: Construct $\mathbf{span}(V) = \mathcal{K}_{r_k}(A, b, \sigma_k^{(0)})$
2: Construct $\mathbf{span}(W) = \mathcal{K}_{r_k}(A^T, C^T, \overline{\sigma}_k^{(0)})$
3: Set $W = W(W^T V)^{-T}$
4: **while** $|\sigma_k^{(i)} - \sigma_k^{(i-1)}| > \epsilon$ **do**
5: $\quad i \leftarrow i + 1$
6: $\quad \hat{A} = W^T A V$, $\sigma_k^{(i)} = -\lambda_k(\hat{A})$
7: $\quad$ Construct $\mathbf{span}(V) = \mathcal{K}_{r_k}(A, b, \sigma_k^{(i)})$
8: $\quad$ Construct $\mathbf{span}(W) = \mathcal{K}_{r_k}(A^T, C^T, \overline{\sigma}_k^{(i)})$
9: $\quad$ Set $W = W(W^T V)^{-T}$
10: **end while**
Ensure: $V, W \in \mathbb{R}^{n\times r}$ and $W^T V = I_r$

From an initial shift selection $\sigma_k^{(0)}$, Algorithm 1 constructs V, W (step 1-2) and step 3 ensures that $W^T V = I_r$. Note that stopping at step 3 leads to the classical Rational Krylov Algorithm. The iterative version is developed from step 4-10 and consists of adjusting the shift selection by reusing the reduced model poles images as new shifts (see *e.g.* [6]).

3) Properties and remarks: The Krylov-based methods achieve moments matching of varying order at multiple points through a numerically efficient procedure. As a consequence, it well approximates the frozen frequency responses of the initial system over different points. On the other hand, the main drawback of these approaches is that stability is not a priori guaranteed, global error bound does not exist and convergence is not guaranteed. However, our experience shows that this algorithm often provides stable reduced models but requires a deflation mechanism in the SIMO case.

C. SVD-Krylov based approaches

Based on the SVD and Krylov advantages and bottlenecks recalled in II-A and II-B, a third methodology has emerged: the *SDV-Krylov-based approach* which gathers the advantages of each approach.

1) (Iterative) SVD-Rational Krylov Algorithm: The SVD-Rational Krylov approach computes a single Gramian (here, the detectability one), and then computes the Krylov subspace associated with the input matrix. The Iterative SVD-Krylov Algorithm, is given in Algorithm 2 (see [3]).

Algorithm 2 Iterative SVD-Rational Krylov (ISRKA) [3]

Require: A, b, C, $\sigma_k^{(0)}$, r_k
1: Construct $\mathbf{span}(V) = \mathcal{K}_{r_k}(A, b, \sigma_k^{(0)})$
2: Compute the observability Gramian $\mathcal{Q}$
3: Compute $W = \mathcal{Q}V(V^T \mathcal{Q} V)^{-1}$
4: **while** $|\sigma_k^{(i)} - \sigma_k^{(i-1)}| > \epsilon$ **do**
5: $\quad i \leftarrow i + 1$
6: $\quad \hat{A} = W^T A V$, $\sigma_k^{(i)} = -\lambda_k(\hat{A})$
7: $\quad$ Construct $\mathbf{span}(V) = \mathcal{K}_{r_k}(A, b, \sigma_k^{(i)})$
8: $\quad$ Compute $W = \mathcal{Q}V(V^T \mathcal{Q} V)^{-1}$
9: **end while**
Ensure: $V, W \in \mathbb{R}^{n\times r}$ and $W^T V = I_r$

Algorithm 2 constructs V using the Rational Krylov approach, and W using the Gramian approach (step 1-3). The iteration is performed as in Algorithm 1. As illustrated on our flexible aircraft application (see Section IV, Fig. 2), such an adaptation mechanism allows to rapidly converge toward the $\mathcal{H}_2$ Wilson first-order condition [7].

2) Properties and remarks: Thanks to the Gramian computation, asymptotic stability is preserved, while moment matching is guaranteed by the Krylov subspace construction. Therefore, this approach provides both good modal and frequential matching, while keeping the reduced system eigenvalues in the left half plane, and ensures minimal $\mathcal{H}_2$ error (see [3]).

III. Extension to parametrized models, toward LPV and LFR formulations

In the above section, projection-based approximation methods were briefly recalled in the LTI framework. Here instead, we consider a set of LTI systems (possibly with inconsistent state vectors) as described in (1), with the objective to obtain a parametrized reduced-order model of the form (2). The idea is to (i) apply local reduction to each model Σ_i, (ii) find a transformation that allows interpolation, *i.e.* that forces the states to belong to the same basis [8], [9], [10], and finally (iii) find appropriate polynomials which interpolate the reduced models. The procedure is described in the following subsections.

A. Reduction by local projectors

The first step consists of applying n_s local projectors $(V_i)_{i=\{1,n_s\}}, (W_i)_{i=\{1,n_s\}} \in \mathbb{R}^{n\times r}$ to each LTI model through the use of Algorithm 2, guaranteeing then that moments are matched and stability is preserved. Then, n_s local reduced-order models are obtained:

$$\hat{\Sigma}_i : \left\{ \begin{array}{ccc} \dot{\hat{x}}_i & = & \hat{A}_i \hat{x}_i + \hat{b}_i u \\ \hat{y} & = & \hat{C}_i \hat{x}_i + \hat{d}_i u \end{array} \right. , \quad i = 1, \ldots, n_s \tag{13}$$

where $\hat{A}_i = W_i^T A_i V_i^T$, $\hat{b}_i = W_i^T b_i$, $\hat{C}_i = C_i V_i^T$ and $\hat{d}_i = d_i$. At this point, since the reduced-order models have been obtained with different changes of state coordinates, they cannot be rigorously interpolated. A solution to this point has been proposed in [8], which consists of applying an additional state transformation.

B. Projection on the same basis

Since $x_i = V_i \hat{x}_i$, by following [8], let a linear transformation $R \in \mathbb{R}^{n\times r}$ be defined such that $\hat{x}_i^* = R^T x_i$. In an interpolation perspective, the objective is to force all state vectors $\hat{x}_i^*$ to be equal. In order to have this, note that it is always possible to find a linear transformation R such that:

$$R^T V_1 \hat{x}_1 = R^T V_2 \hat{x}_2 = \cdots = R^T V_{n_s} \hat{x}_{n_s} = \hat{x}^* \tag{14}$$

By noting $\hat{x}_i = T_i^{-1}\hat{x}^*$ (where $T_i = R^T V_i$), each new reduced-order system $\hat{\Sigma}_i^*$ will thus be given as:

$$\hat{\Sigma}_i^* : (A_i^*, b_i^*, C_i^*, d_i^*) \tag{15}$$

where $A_i^* = T_i W_i^T A_i V_i T_i^{-1}$, $b_i^* = T_i W_i^T b_i$, $C_i^* = C_i V_i T_i^{-1}$ and $d_i^* = d_i$. All local models have now the same state vector, making interpolation possible (see Section III-C).

The problem is to define the linear transformation R. It should span all the dynamics of the local models. Hence, an intuitive choice can be to select the most significant transformations as follows:

$$USZ^T = \mathbf{SVD}([V_1, \ldots, V_{n_s}]) \tag{16}$$

where V_i denotes the local projectors obtained for each local model. Hence, to keep the most significant singular values, one can choose $R = U_r$, *i.e.* the r first columns of the unitary matrix U. Notice that other approaches to generate R exist [10], but still this one provides nice results (see Section IV).

C. Reduced-order models interpolation

The element-wise interpolation of the state-space matrices of $(\hat{\Sigma}_i^*)_{i\in\{1,\,n_s\}}$ can now be achieved. Let $(z_i)_{i\in\{1,\,n_s\}}$ be the values taken by any of these matrix elements for all the parametric configurations $(\delta^{(i)})_{i\in\{1,n_s\}}$. In the perspective of building an LFR, either a polynomial or a rational interpolation must be performed. The former prevents the appearance of discontinuities and is preferred here. The following expression is thus assumed (see also [11]):

$$z(\delta) = \sum_{k=1}^{n_p} \gamma_k p_k(\delta) \tag{17}$$

where $(p_k)_{k\in\{1,n_p\}}$ is a set of multivariate polynomials and $(\gamma_k)_{k\in\{1,n_p\}}$ are parameters to be determined. Let:

$$P = \begin{pmatrix} p_1(\delta^{(1)}) & \ldots & p_{n_p}(\delta^{(1)}) \\ \vdots & \ddots & \vdots \\ p_1(\delta^{(n_s)}) & \ldots & p_{n_p}(\delta^{(n_s)}) \end{pmatrix} = \begin{pmatrix} P_1 & \ldots & P_{n_p} \end{pmatrix} \tag{18}$$

$$\Gamma^T = \begin{pmatrix} \gamma_1 & \ldots & \gamma_{n_p} \end{pmatrix}, \; Z^T = \begin{pmatrix} z_1 & \ldots & z_{n_s} \end{pmatrix} \tag{19}$$

where $\delta^{(i)}$ is the value of δ for the i^{th} parametric configuration. The objective is to minimize the quadratic error between $z(\delta)$ and $(z_i)_{i\in\{1,n_s\}}$, *i.e.* to compute:

$$\Gamma_{opt} = \arg\min_{\Gamma\in\mathbb{R}^{n_p}} \Big((Z - P\Gamma)^T (Z - P\Gamma)\Big) \tag{20}$$

An intuitive choice for $(p_k)_{k\in\{1,n_p\}}$ is:

$$\left\{ \begin{array}{ccc} \mathbb{R}^l & \rightarrow & \mathbb{R} \\ (\delta_1, ..., \delta_l) & \rightarrow & \delta_1^{i_1} ... \delta_l^{i_l} \end{array}, i_1 \leq d_1, ..., i_l \leq d_l \right\} \tag{21}$$

where δ_j is the j^{th} element of δ, l is the length of δ and $d_1, \ldots, d_l$ are user-defined integers. In this context, the solution of (20) is:

$$\Gamma_{opt} = (P^T P)^{-1} P^T Z \tag{22}$$

Once polynomial approximations $\hat{A}(\delta), \hat{b}(\delta), \hat{C}(\delta), \hat{d}(\delta)$ of $(\hat{A}_i^*, \hat{b}_i^*, \hat{C}_i^*, \hat{d}_i^*)_{i\in\{1,\,n_s\}}$ are available, the structured tree decomposition algorithm of [12] is applied to get an LFR. This algorithm is implemented in the function `symtreed` of the LFR Toolbox for Matlab [13].

IV. Application to an industrial aircraft system

The methodology proposed in Sections II and III is now applied to a set of aeroelastic aircraft models.

A. Modeling objectives and challenging issues

Three open-loop longitudinal models $(\Sigma_i)_{i\in\{1,3\}}$ are considered here, which describe both the rigid and the flexible dynamics of a civilian passenger aircraft for different configurations of the center tank (empty, half-filled and filled). The first objective is to obtain reduced-order models with consistent state space matrices and accurate frequency responses. With reference to Fig. 1, the issue is then to convert these reduced-order models successively into an LPV model and an LFR, where y_r denotes the vertical wind velocity w_z, while y consists of the vertical load factor at the rear of the aircraft fuselage N_{z_r} and the pitch rate q.

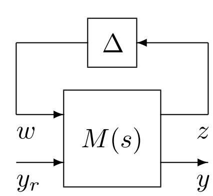

Fig. 1. Structure of the open-loop LFR.

The LFR should be highly representative of the initial full-order models, in the sense that its frozen frequency responses should almost exactly match those of the initial models for the three considered parametric configurations. A special attention should also be paid to its eigenvalues and frequency responses to ensure that their variations are as smooth as possible on the whole parametric domain. Moreover, its complexity should remain compatible with the use of robustness analysis tools. It is worth being emphasized that generating such an LFR is a challenging task:

- The reference models $(\Sigma_i)_{i\in\{1,3\}}$ have about 300 states due to several flexible modes and aerodynamic delays.
- The grid is very coarse. Achieving a good fit over the whole parametric domain is thus a demanding task.
- The parametric structure of the models is unknown.

It thus prevents the direct use of standard reduction methods to obtain a suitable low-order LFR. Some methodologies have already been proposed in [14], [15], but a tedious preprocessing step is usually required to reduce the reference models and ensure modal consistency for the whole set of reduced-order models.

B. Local models order reduction

As described in Sections III-A and III-B, the strategy consists of reducing each LTI model and applying a state transformation in order to obtain reduced-order models with the same state vector. First, to emphasize the interest of the SVD-Krylov approach (ISRKA, Algorithm 2) with respect to the Balanced Truncation (BT) one (obtained with the `balred` Matlab function), local model reduction is applied to the three systems. Then, the following error criteria:

$$\epsilon_{\mathcal{H}_2} = \frac{100}{n_s}\sum_{i=1}^{n_s}\frac{||\Sigma_i - \hat{\Sigma}_i||_2}{||\Sigma_i||_2}\;,\; \epsilon_{\mathcal{H}_\infty} = \frac{100}{n_s}\sum_{i=1}^{n_s}\frac{||\Sigma_i - \hat{\Sigma}_i||_\infty}{||\Sigma_i||_\infty} \quad (23)$$

are evaluated and compared for various reduction orders. Fig. 2 gathers the results obtained for $r = 14, \ldots, 34$ (for all simulations, Algorithm 2 is empirically initialized with $\sigma_k = [z\ \overline{z}]^T \in \mathbb{C}^r$, $z = [1\ \ldots\ 1000]^T j \in \mathbb{C}^{r/2}$ ($j = \sqrt{-1}$), $r_k = [1\ \ldots\ 1]^T \in \mathbb{R}^r$ and $\epsilon = 10^{-2}$).

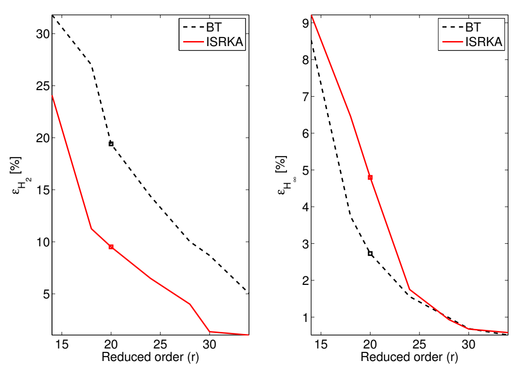

Fig. 2. Approximation error comparison between the BT and ISRKA methods for local model reduction. Left frame, $\epsilon_{\mathcal{H}_2}$; Right frame, $\epsilon_{\mathcal{H}_\infty}$.

From Fig. 2, the following comments can be done:

- By increasing the reduced order r, both errors decrease.
- Compared to the BT, the IRSKA approach provides significantly better results in terms of $\mathcal{H}_2$ relative error while keeping the $\mathcal{H}_\infty$ one almost similar.
- The state vectors obtained with the ISRKA procedure are in the same basis, unlike the BT ones.
- For $r = 20$, we observe quite nice results for both relative errors; indeed, the BT approach achieves $\epsilon_{\mathcal{H}_2} = 19.40\%$ and $\epsilon_{\mathcal{H}_\infty} = 2.73\%$ while the ISRKA achieves $\epsilon_{\mathcal{H}_2} = 9.85\%$ and $\epsilon_{\mathcal{H}_\infty} = 4.80\%$.

From now on, we will consider $r = 20$ and focus on the SVD-Krylov approach. Fig. 3 compares the frequency responses of the reduced-order and the initial high-order models corresponding to a half-filled tank - *i.e.* $\hat{\Sigma}_2^*$ *vs.* Σ_2.

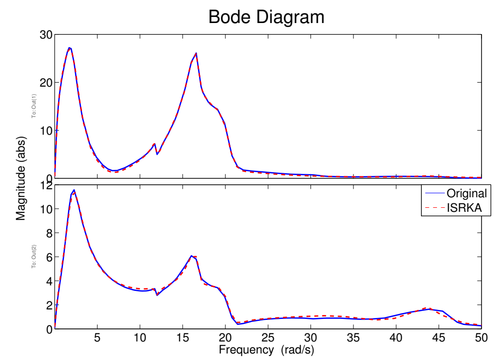

Fig. 3. Comparison between the frequency responses (top: from w_z to N_{z_r} - bottom: from w_z to q) of the initial Σ_2 (solid lines) and the reduced-order $\hat{\Sigma}_2^*$ ($r = 20$, dashed lines) models.

The frequency responses of $\hat{\Sigma}_2^*$ are very satisfactory and reproduce accurately the main frequency peaks. Analogous results are obtained for the two other configurations, but they are not presented in details due to space limitations. Now that the models have the same state vector, the interpolation can be achieved as described in Section III-C.

C. LPV model and LFR generation

In order to have as few occurrences of δ as possible in the LPV formulation (2), and therefore to obtain an LFR whose Δ block in as small as possible, an additional change of state coordinates is performed and the reduced-order models $(\hat{\Sigma}_i^*)_{i=\{1,3\}}$ are all written in the companion form. Indeed, our experience shows that this form, as well as the modal form, usually produce good results in terms of both eigenvalues and frequency responses [14], [15].

Since three center tank configurations are considered, choosing $n_p = 3$ and applying the interpolation method proposed in Section III-C allows to perform an exact interpolation. Once the LPV model $\hat{\Sigma}(\delta)$ is obtained, where δ denotes the amount of fuel in the center tank, it can easily be transformed into an LFR as described in Section III-C. With reference to Fig. 1, $M(s)$ is an LTI system of order $r = 20$, while $\Delta = \delta I$ is a 6×6 real diagonal matrix, which is, to our knowledge, a very nice result. Although no comparison can be made outside the grid points, a strong requirement is that the LFR behavior remains realistic, *i.e.* that its eigenvalues and frequency responses vary as smoothly as possible. This is actually the case, as illustrated in Figs. 4 and 5.

The resulting LFR can then be used both for analysis and design purposes. This is out of the scope of the paper, but a few results are presented here to show that its size is compatible with existing tools. The μ-analysis based method of [16] is first applied and allows to validate *a posteriori* the stability of the LFR on the whole parametric domain. The highest $\mathcal{H}_\infty$ norm of the transfer function from w_z to N_{z_r} is then evaluated when Δ takes all possible values in the considered domain. More precisely, the methods of [16] and [17] are successively applied to compute upper and lower bounds. A guaranteed upper bound on the whole

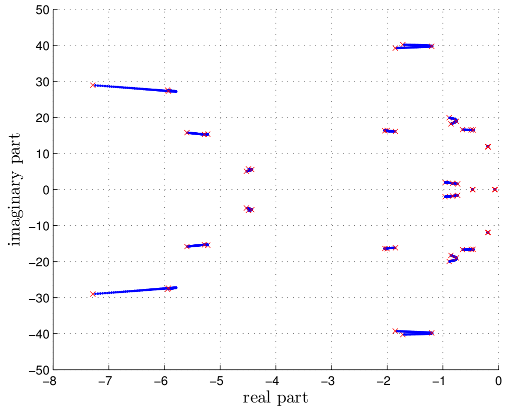

Fig. 4. Modal trajectories of $\hat{\Sigma}(\delta)$ with respect to the center tank filling level (crosses: reduced-order LTI models $(\hat{\Sigma}_i^*)_{i=\{1,3\}}$).

parametric domain is $\gamma_{UB} = 29.62$, while a worst-case configuration $\gamma_{LB} = 29.33$ is detected for a filled tank at the frequency $\omega = 1.57$ rad/s. The bounds are very tight and the computational time is only 6s. Moreover, this result is consistent with what can be observed on Fig. 5 (top).

V. Conclusions

In this paper, the problem of generating a reduced-order parametrized model from a set of large-scale LTI systems, describing a complex plant modeled at frozen configurations, is treated. The contribution gathers different results from control and numerical communities to derive a methodology that solves this problem in an almost direct manner. The proposed approach is successfully assessed on a set of aeroelastic aircraft models, leading to a low-order LFR whose size makes it compatible with several analysis and synthesis tools.

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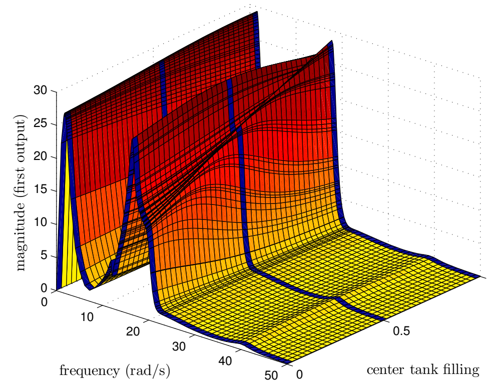

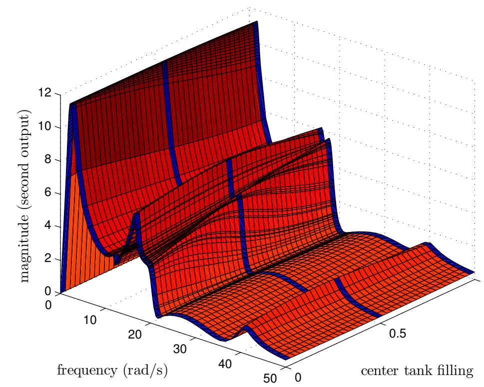

Fig. 5. Frozen frequency responses of $\hat{\Sigma}(\delta)$ (top: from w_z to N_{z_r} - bottom: from w_z to q) with respect to the center tank filling level (solid lines: reduced-order LTI models $(\hat{\Sigma}_i^*)_{i=\{1,3\}}$).

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