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Salient Point Quadrature Nonlinear Filtering

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Abstract— **In this paper, a new nonlinear filter named Salient Point Quadrature Filter (SPQF) using a sparse grid method is proposed. The filter is derived using the so-called salient points to approximate the integrals in the Bayesian estimation algorithm. The univariate salient points are determined by the moment match method and then the sparse-grid theory is used to extend the univariate salient point sets to multi-dimensional cases. Compared with the other point-based methods, the estimation accuracy level of the new filter can be flexibly controlled and the filter algorithm is computationally more efficient since the number of salient points for SPQF increases polynomially with the dimension, which alleviates the curse of the dimensionality for high dimensional problems. Another contribution of this paper is to show that the Unscented Kalman Filter (UKF) is a subset of the SPQF with the accuracy level 2. The performance of this new filter was demonstrated by the orbit determination problem. The simulation results show that the new filter has better performance than the Extended Kalman Filter (EKF) and UKF.**

I. INTRODUCTION

Nonlinear filtering has been intensively studied and widely applied in many science and engineering disciplines. The general filtering problem can be addressed from the Bayesian estimation theory [1]. In general, exact finite-dimensional solutions to the integral equations in the Bayesian estimation do not exist. Numerous approximate nonlinear filters have been proposed. A large class of them is based on the assumption that the state probability density function (pdf) is Gaussian. These filters include the Gauss-Hermite Quadrature Filter (GHQF) [2, 3], the Unscented Kalman Filter (UKF) [4, 5], the Cubature Kalman Filter (CKF) [6], the Central Difference Filter (CDF) [2], and the Divided Difference Filter (DDF) [7]. All the Gaussian approximation filters involve using numerical quadratures to approximate the expectations of nonlinear functions with respect to Gaussian pdfs. UKF uses the so-called sigma points with the number of points increasing linearly with the dimension. CDF [2] and DDF [7] are based on interpolation formula using the similar deterministic sampling approach. The common limitation of those methods including UKF, CDF, DDF and CKF is that they are hard to be extended to achieve any higher order estimation accuracy.

In this paper, we proposed a new nonlinear filtering algorithm named Salient Point Quadrature Filter (SPQF) using a general moment-match method for one-dimensional integral approximation and a sparse grid method for multi-dimensional extension. The sparse grid method was originally used to alleviate the curse of dimensionality problem in the numerical integration [8,9]. It is utilized in the approximate nonlinear filter design in this paper using a special linear combination of lower-dimensional tensor products such that the number of necessary points is dramatically less than that of using the direct tensor product rule. As a result, the computational cost does not increase exponentially using the sparse grid method.

The new SPQF proposed in this paper is more flexible to use than UKF, CKF, CDF, and DDF, and capable of achieving higher accuracy levels with moderately increased number of points. Another contribution of this paper is to prove that UKF is a subset of SPQF with the accuracy level 2.

The rest of this paper is organized as follows: the Gaussian approximation filters are briefly reviewed in Section II. Section III introduces the Salient Point Quadrature rule and filter. Section IV presents the simulation results for the orbit determination and makes comparisons with EKF and UKF. Some conclusion remarks are given in Section V.

II. GAUSS-APPROXIMATION FILTERS

2.1 *Gaussian approximation filters*

Consider a nonlinear discrete-time system with additive process noise and measurement noise:

$$\mathbf{x}_k = f\left(\mathbf{x}_{k-1}\right) + \mathbf{v}_{k-1} \tag{1}$$

$$\mathbf{y}_k = h\left(\mathbf{x}_k\right) + \mathbf{n}_k \tag{2}$$

where $\mathbf{v}_{k-1}$ and $\mathbf{n}_k$ are white Gaussian process noise and measurement noise with the covariances $\mathbf{Q}_{k-1}$ and $\mathbf{R}_k$ respectively.

Gaussian approximation filters such as GHQF and UKF assume that the pdf of the states is Gaussian and calculate the state mean and covariance as follows [2]:

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \int_{R^n} f\left(\mathbf{x}_{k-1}\right) N\left(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1|k-1}\right) d\mathbf{x}_{k-1} \tag{3}$$

$$\mathbf{P}_{k|k-1} = \int_{R^n} f\left(\mathbf{x}_{k-1}\right) f\left(\mathbf{x}_{k-1}\right)^T N\left(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1|k-1}\right) d\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k|k-1}\hat{\mathbf{x}}_{k|k-1}^T + \mathbf{Q}_{k-1} \tag{4}$$

where $N\left(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1|k-1}\right)$ denotes the multivariate normal

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distribution with the mean $\hat{\mathbf{x}}_{k-1}$ and the covariance $\mathbf{P}_{k-1|k-1}$.

Update:
$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{z}_k\right) \tag{5}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{L}_k\mathbf{P}_{xz}^T \tag{6}$$
where $$\mathbf{L}_k = \mathbf{P}_{xz}\left(\mathbf{R}_k + \mathbf{P}_{zz}\right)^{-1} \tag{7}$$
$$\mathbf{z}_k = \int_{\boldsymbol{R}^n} h\left(\mathbf{x}_k\right) N\left(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}\right) d\mathbf{x}_k \tag{8}$$
$$\mathbf{P}_{xz} = \int_{\boldsymbol{R}^n} \left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right)\left(h\left(\mathbf{x}_k\right) - \mathbf{z}_k\right)^T N\left(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}\right) d\mathbf{x}_k \tag{9}$$
$$\mathbf{P}_{zz} = \int_{\boldsymbol{R}^n} \left(h\left(\mathbf{x}_k\right) - \mathbf{z}_k\right)\left(h\left(\mathbf{x}_k\right) - \mathbf{z}_k\right)^T N\left(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}\right) d\mathbf{x}_k \tag{10}$$

The integrals in (3)-(4) and (8)-(10) can be approximated by the Gauss-Hermite Quadrature (GHQ), Unscented Transformation (UT) or Cubature Rule [6]. The corresponding filtering algorithm is given below.

Prediction:
$$\hat{\mathbf{x}}_{k|k-1} = \sum_{i=1}^{N_p} W_i f\left(\boldsymbol{\xi}_i\right) \tag{11}$$
$$\mathbf{P}_{k|k-1} = \sum_{i=1}^{N_p} W_i \left(f\left(\boldsymbol{\xi}_i\right) - \hat{\mathbf{x}}_{k|k-1}\right)\left(f\left(\boldsymbol{\xi}_i\right) - \hat{\mathbf{x}}_{k|k-1}\right)^T + \mathbf{Q}_{k-1} \tag{12}$$
$$\mathbf{P}_{k-1|k-1} = \mathbf{S}\mathbf{S}^T \;; \quad \boldsymbol{\xi}_i = \mathbf{S}\boldsymbol{\gamma}_i + \hat{\mathbf{x}}_{k-1|k-1} \tag{13}$$

where N_p is the total number of points; $\boldsymbol{\gamma}_i$ and W_i are the points and weights used to approximate the integral $\int_{\boldsymbol{R}^n} f\left(\mathbf{x}_{k-1}\right) N\left(\mathbf{x}; \mathbf{0}, \mathbf{I}_n\right) d\mathbf{x}_{k-1}$; $\boldsymbol{\xi}_i$ is the transformed point obtained from the covariance decomposition.

Update:
$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{L}_k\left(\mathbf{y}_k - \mathbf{z}_k\right) \tag{14}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{L}_k\mathbf{P}_{xz}^T \tag{15}$$
where $$\mathbf{z}_k = \sum_{i=1}^{N_p} W_i h\left(\tilde{\boldsymbol{\xi}}_i\right) \tag{16}$$
$$\mathbf{P}_{xz} = \sum_{i=1}^{N_p} W_i \left(\tilde{\boldsymbol{\xi}}_i - \hat{\mathbf{x}}_{k|k-1}\right)\left(h\left(\tilde{\boldsymbol{\xi}}_i\right) - \mathbf{z}_k\right)^T \tag{17}$$
$$\mathbf{P}_{zz} = \sum_{i=1}^{N_p} W_i \left(h\left(\tilde{\boldsymbol{\xi}}_i\right) - \mathbf{z}_k\right)\left(h\left(\tilde{\boldsymbol{\xi}}_i\right) - \mathbf{z}_k\right)^T \tag{18}$$

$\tilde{\boldsymbol{\xi}}_i$ is the transformed point obtained from the predicted covariance decomposition, i.e.
$$\mathbf{P}_{k|k-1} = \tilde{\mathbf{S}}\tilde{\mathbf{S}}^T \;; \quad \tilde{\boldsymbol{\xi}}_i = \tilde{\mathbf{S}}\boldsymbol{\gamma}_i + \hat{\mathbf{x}}_{k|k-1} \tag{19}$$

There are many rules to choose the quadrature point $\boldsymbol{\gamma}_i$ and the weight W_i such as GHQ rule [2, 3] and UT [4]. The multivariate GHQ extends the univariate GHQ by the tensor product rule [2, 3]. The univariate GHQ with *m* points is exact up to the ($2m-1$)$^{\text{th}}$ order of polynomials [3]. The multivariate GHQ rule is exact for all polynomials of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ with $1 \le i_j \le 2m-1$ [2]. However, the total number of points $N_p = m^n$ increases exponentially with the dimension n.

For the UT with $2n+1$ points [4], $\boldsymbol{\gamma}_i$ and W_i are given by

$$\begin{cases} \boldsymbol{\gamma}_1 = \left[0, 0, \cdots 0, 0\right]^T; & W_1 = \dfrac{\kappa}{n+\kappa} \\ \boldsymbol{\gamma}_i = \sqrt{n+\kappa}\,\mathbf{e}_{i-1}; & W_i = \dfrac{1}{2\left(n+\kappa\right)} \;;\; 2 \le i \le n+1 \\ \boldsymbol{\gamma}_i = -\sqrt{n+\kappa}\,\mathbf{e}_{i-n-1}; & W_i = \dfrac{1}{2\left(n+\kappa\right)} \;;\; n+2 \le i \le 2n+1 \end{cases} \tag{20}$$

where $\mathbf{e}_{i-1}$ is the unit vector in $\mathbf{R}^n$ with the $(i\text{-}1)^{\text{th}}$ element being 1 and κ is a tuning parameter with the suggested optimal value of $\kappa = 3 - n$ for Gaussian distributions [4].

III. Salient Point Quadrature Filter

In this section, the Salient Point Quadrature (SPQ) is introduced to calculate the integrals in (3)-(4) and (8)-(10) using a linear combination of lower-dimensional tensor products based on the Smolyak's rule, given by [9]

$$\begin{aligned} &\int_{\boldsymbol{R}^n} f\left(\mathbf{x}\right) N\left(\mathbf{x}; \mathbf{0}, \mathbf{I}_n\right) d\mathbf{x} \approx I_{n,L}\left(f\right) \\ &= \sum_{q=L-n}^{L-1} \left(-1\right)^{L-1-q} \mathbf{C}_{n-1}^{L-1-q} \sum_{\boldsymbol{\Xi} \in \mathbf{N}_q^n} \left(I_{i_1} \otimes \cdots I_{i_n}\right)\left(f\right) \end{aligned} \tag{21}$$

where $\mathbf{x} = \left[x_1, x_2, \cdots, x_p, \cdots, x_n\right]^T$. $I_{n,L}\left(f\right)$ is an approximation to the *n*-dimensional integral of the function f with respect to $N\left(\mathbf{x}; \mathbf{0}, \mathbf{I}_n\right)$ with the accuracy level $L \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. By accuracy level L it means that $I_{n,L}\left(f\right)$ is exact for all polynomials of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ with $\sum_{j=1}^{n} i_j \le 2L-1$ [9]. $\mathbf{C}_{n-1}^{L-1-q}$ is the binomial coefficient; $\otimes$ denotes the tensor product. I_{i_j} is the univariate SPQ rule with the accuracy level of at least $i_j \in \Xi$, where $\Xi \triangleq \left(i_1, \cdots, i_n\right)$ is an accuracy level sequence of *n* natural numbers. By accuracy level i_j, it means that I_{i_j} is exact to at least the ($2i_j - 1$)$^{\text{th}}$ order of all univariate polynomials. $\mathbf{N}_q^n$ is a set of accuracy level sequences defined by

$$\begin{cases} \mathbf{N}_q^n = \left\{\Xi : \sum_{d=1}^{n} i_d = n+q\right\} & \text{for } q \ge 0 \\ \mathbf{N}_q^n = \varnothing & \text{for } q < 0 \end{cases} \tag{22}$$

The more explicit form of Eq. (21) can be written as

$$I_{n,L}\left(f\right) = \sum_{q=L-n}^{L-1} \sum_{\boldsymbol{\Xi} \in \mathbf{N}_q^n} \sum_{x_1 \in X_{i_1}} \cdots \sum_{x_n \in X_{i_n}} f\left(x_1, \ldots, x_n\right) \left\{ \left(-1\right)^{L-1-q} \mathbf{C}_{n-1}^{L-1-q} \prod_{p=1}^{n} w_{i_p} \right\} \tag{23}$$

where X_{i_j} is the univariate salient point set with the accuracy level i_j, which contains i_j or more points. The choice of X_{i_j} is not unique, which will be discussed afterwards. w_{i_p} is the weight in I_{i_p} associated with the state variable x_p and the term enclosed by the braces is the weight associated with a multivariate salient point (*n*-dimensional) determined by the element in N_q^n. For a multivariate salient point that appears

multiple times, the weight of this point is the sum of the weights on the point over all combinations of $X_{i_1} \otimes X_{i_2} \cdots \otimes X_{i_n}$ containing the point, which will be shown in Algorithm I.

The set of n-dimensional salient points, $X_{n,L}$, with the accuracy level L is given by

$$X_{n,L} = \bigcup_{q=L-n}^{L-1} \bigcup_{\Xi \in \mathbf{N}_q^n} \left(X_{i_1} \otimes \cdots \otimes X_{i_n} \right) \tag{24}$$

where $\bigcup$ denotes the union operation of the point sets. Each element in $\mathbf{N}_q^n$, i.e. $\Xi = (i_1, \cdots, i_n)$, determines a tensor product sequence of X_{i_j} where $i_j \in \Xi \in \mathbf{N}_q^n$.

The SPQ method includes two steps:

Step 1: choose salient points and weights for univariate integral approximation using the moment match approach.

The first step is to ensure that the univariate SPQ with the level i_j is exact for univariate polynomials of the order $2i_j$-1 in order for the multivariate SPQ with the level L to be exact for all multivariate polynomials with the total order of $2L$-1. This is shown in the Theorem 3.1:

Theorem 3.1[9]: Assume that the sequence of univariate quadrature rules $I = \left\{ I_{i_j} : i_j \in N \right\}$ is defined such that I_{i_j} is exact for all univariate polynomials of the order up to $2i_j - 1$. Then the Smolyak rule $I_{n,L}$ (21) using I as the univariate basis sequence is exact for n-variate polynomials of the total order up to $2L-1$.

According to *Theorem 3.1*, we will use the moment match method to ensure that the univariate SPQ with the level i_j is exact for univariate polynomials of the order $2i_j$-1.

In this paper, we use $2i_j$-1 symmetric salient points for the univariate point set with the level i_j. Other point selection methods are also possible. Take L=3 as an example. According to the Smolyak's rule, or Eq. (22), we need level 1, 2, and 3 univariate SPQ. For the level 1, the point set is chosen to be $\{0\}$ with the corresponding weight of 1. For the level 2, we choose the symmetric point set as $\{-\hat{p}_1, 0, \hat{p}_1\}$ with the corresponding weight sequence $(\hat{w}_2, \hat{w}_1, \hat{w}_2)$. For the level 3, we choose the symmetric point set as $\{-\hat{p}_3, -\hat{p}_2, 0, \hat{p}_2, \hat{p}_3\}$ with the corresponding weight sequence $(\hat{w}_5, \hat{w}_4, \hat{w}_3, \hat{w}_4, \hat{w}_5)$. According to Theorem 3.1, the point sets for the level 2 and level 3 should match univariate polynomials of the 3rd order and the 5th order respectively. The general moment match formula (one-dimensional Gaussian type integral) is

$$M_j = \int_{-\infty}^{\infty} x^j N(x;0,1)\, dx = \sum_{i=1}^{N_p} \hat{w}_i \left(\hat{p}_i \right)^j \tag{25}$$

where M_j is the j^{th} order moment; N_p is the number of points; $\hat{w}_i$ and $\hat{p}_i$ are the weight and point respectively.

From the Eq. (25), for the level 2, the following equations should be satisfied:

$$\begin{cases} \hat{w}_1 + 2\hat{w}_2 = M_0 = 1 \\ 2\hat{w}_2 p_1^2 = M_2 = 1 \end{cases} \tag{26}$$

which leads to $\hat{w}_1 = 1 - 1/\hat{p}_1^2$; $\hat{w}_2 = 1/(2\hat{p}_1^2)$ (27)

Similarly, for the level 3, the Eq. (28) should be satisfied

$$\begin{cases} \hat{w}_3 + 2\hat{w}_4 + 2\hat{w}_5 = M_0 = 1 \\ 2\hat{w}_4 \hat{p}_2^2 + 2\hat{w}_5 \hat{p}_3^2 = M_2 = 1 \\ 2\hat{w}_4 \hat{p}_2^4 + 2\hat{w}_5 \hat{p}_3^4 = M_4 = 3 \end{cases} \tag{28}$$

Solving this equation set (if $\hat{p}_3 \neq \hat{p}_2$) yields

$$\begin{aligned} &\hat{w}_3 = 1 - 2\hat{w}_4 - 2\hat{w}_5; \qquad \hat{w}_4 = (0.5 - \hat{w}_5 \hat{p}_3^2)/\hat{p}_2^2 \\ &\hat{w}_5 = (3 - \hat{p}_2^2)/[2\hat{p}_3^2(\hat{p}_3^2 - \hat{p}_2^2)] \end{aligned} \tag{29}$$

In the equation sets (26) and (28), only the even moments are considered because the symmetry of the points implies the odd moments are matched automatically.

When the position of the points $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ are given, the weights can be determined from Eqs. (27) and (29).

Step 2: extend the univariate SPQ to the multi-dimensional SPQ using the sparse grid method, i.e. Eq. (23).

To better illustrate the sparse grid method, we use the Smolyak's rule to construct $X_{2,3} (n = 2,\ L = 3)$ from the univariate salient point sets X_1, X_2 and X_3 containing 1, 3, and 5 points, respectively, as shown in Fig. 1 [10].

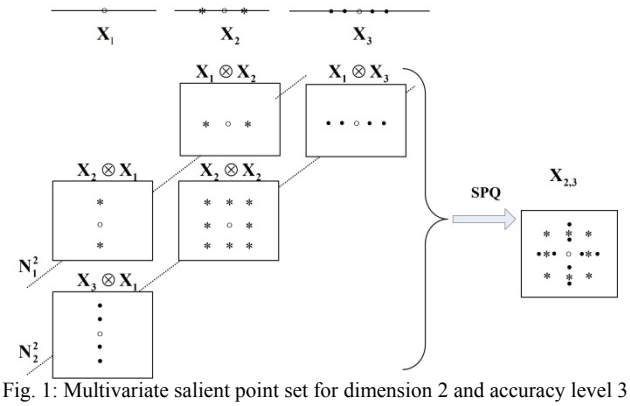

Fig. 1: Multivariate salient point set for dimension 2 and accuracy level 3

From Eq. (23), q can be 1 or 2 and then we have $\mathbf{N}_1^2 = \{(1,2),(2,1)\}$ and $\mathbf{N}_2^2 = \{(1,3),(2,2),(3,1)\}$. $\mathbf{N}_1^2$ and $\mathbf{N}_2^2$ contain the accuracy level sequences that determine the tensor products of X_1, X_2 and X_3 as shown in Fig.1. The final sparse-grid point set $X_{2,3}$ yields 17 points as shown on the bottom right of Fig. 1.

The salient points and weights are generated in Algorithm I.

Algorithm I: Generate Salient Points and Weights

$[\chi, W] = \text{SPQ}[n, L]$

(χ : salient point set with the element of χ_i ; W: weight sequence with the element of W_i)

FOR $q = L-n : L-1$

Determine $\mathbf{N}_q^n$

FOR each element $\Xi=(i_1,\cdots,i_n)$ in $\mathbf{N}_q^n$, form $X_{i_1}\otimes X_{i_2}\cdots\otimes X_{i_n}$

For each point χ_i in $X_{i_1}\otimes X_{i_2}\cdots\otimes X_{i_n}$

IF the point is new, add it to χ, assign a new index i to the point and calculate its weight as

$$W_i=(-1)^{L-1-q}\mathbf{C}_{n-1}^{L-1-q}\prod_{p=1}^{n}w_{i_p} \tag{30}$$

w_{i_p} is the weight of the point in X_{i_p} and chosen from $\hat{w}_j$, $j=1,\cdots,(N_p+1)/2$

ELSE (the point is already existing) update the old weight by

$$W_i=W_i+(-1)^{L-1-q}\mathbf{C}_{n-1}^{L-1-q}\prod_{p=1}^{n}w_{i_p} \tag{31}$$

END IF

END FOR

END FOR

END FOR

Remark 3.1: The sparse Gauss-Hermite Quadrature [10] can be viewed as a special case of SPQ.

The following theorem shows that UKF is a subset of SPQF.

Theorem 3.2: The points and weights generated by UT [4] are identical to the points and weights generated by the SPQ method with the accuracy level 2 (L=2) if three symmetric points are used for the level 2 univariate salient point set.

Proof: For level 1 univariate salient point quadrature rule, the point set is $\{0\}$ with the corresponding weight 1. For level 2 univariate SPQ, the point set is $\{-\hat{p}_1,0,\hat{p}_1\}$ with the corresponding weight sequence $(\hat{w}_2,\hat{w}_1,\hat{w}_2)$.

Since the accuracy level is L=2, the value of q can be 0 or 1.

When $q=0$, $\mathbf{N}_0^n=\left\{\underbrace{(1,1,\cdots,1,1)}_{n\text{ elements}}\right\}$, the salient point corresponding to $\mathbf{N}_0^n$ will be $[0,0,\cdots 0,0]^T$ with the weight $(-1)^{2-1-0}\times\mathbf{C}_{n-1}^{2-1-0}\times\left(\overbrace{1\times1\times\cdots\times1\times1}^{n-1\text{ elements}}\right)=-(n-1)$ (Eq. (30)).

When $q=1$,

$$\mathbf{N}_1^n=\left\{\underbrace{\overbrace{(2,1,\cdots1,1)}^{n},\overbrace{(1,2,\cdots1,1)}^{n},\cdots\overbrace{(1,\cdots1,2,1,\cdots1)}^{n},\cdots\overbrace{(1,1,\cdots1,2)}^{n}}_{n\text{ elements}}\right\}.$$

Corresponding to the combination $(2,1,\cdots1,1)$, there are two points $[-\hat{p}_1,0,\cdots0,0]^T$ and $[\hat{p}_1,0,\cdots0,0]^T$ with the same weight $W_2=(-1)^{2-1-1}\times\mathbf{C}_{n-1}^{2-1-1}\times\left(\overbrace{1\times1\times\cdots\times1}^{n-1\text{ elements}}\times\hat{w}_2\right)=\hat{w}_2$ (Eq. (30)). Similarly, corresponding to the combination $(1,\cdots1,2,1,\cdots1)$, there are two points $[0,\cdots0,-\hat{p}_1,0,\cdots0]^T$ and $[0,\cdots0,\hat{p}_1,0,\cdots0]^T$ with the same weight $W_2=\hat{w}_2$. The weight for the point $[0,0,\cdots0,0]^T$ using (31) is

$$W_1=(-1)^{2-1-0}\times\mathbf{C}_{n-1}^{2-1-0}\times\left(\overbrace{1\times1\times\cdots\times1\times1}^{n\text{ elements}}\right)+(-1)^{2-1-1}\times\mathbf{C}_{n-1}^{2-1-1}$$
$$\times\left\{\overbrace{\left(\hat{w}_1\times\overbrace{1\times1\times\cdots\times1}^{n-1\text{ elements}}\right)+\left(1\times\hat{w}_1\times\overbrace{1\times1\times\cdots\times1}^{n-2\text{ elements}}\right)+\cdots+\left(\overbrace{1\times1\times\cdots\times1}^{n-1\text{ elements}}\times\hat{w}_1\right)}^{n\text{ elements}}\right\}$$
$$=-(n-1)+n\cdot\hat{w}_1$$

since this point is a repeated point. The salient points and weights for other combinations in $\mathbf{N}_1^n$ can be similarly derived. To summarize it, the salient points and weights with the accuracy level 2 are

$$\begin{cases}\boldsymbol{\gamma}_1=[0,0,\cdots0,0]^T\\ \boldsymbol{\gamma}_i=\hat{p}_1\mathbf{e}_{i-1},\quad 2\le i\le n+1\\ \boldsymbol{\gamma}_i=-\hat{p}_1\mathbf{e}_{i-n-1},\quad n+2\le i\le 2n+1\end{cases} \tag{32}$$

and

$$\begin{cases}W_1=-(n-1)+n\cdot\hat{w}_1\\ W_i=\hat{w}_2,\quad i=2\cdots2n+1\end{cases} \tag{33}$$

respectively. If we choose $\hat{p}_1=\sqrt{n+\kappa}$, by Eq. (27), $\hat{w}_1=1-\frac{1}{(n+\kappa)}$ and $\hat{w}_2=\frac{1}{2(n+\kappa)}$. Hence

$$\begin{cases}W_1=-(n-1)+n\cdot\left(1-\frac{1}{(n+\kappa)}\right)=\frac{\kappa}{n+\kappa}\\ W_i=\frac{1}{2(n+\kappa)},\quad i=2\cdots2n+1\end{cases} \tag{34}$$

Comparing the points and weights of UT in Eq. (20) and those in SPQ, Eqs. (32) and (34), they are identical. ■

The number of salient points can be calculated by analyzing the number of points corresponding to all different accurate level sequences in $\mathbf{N}_q^n$. However, we can use the highest accurate level set to calculate the number of points when the condition in the following proposition is satisfied.

Proposition 3.1: If $X_1\subseteq X_2$, $1<q<n$, the point sets corresponding to the accuracy level set $\mathbf{N}_{q-1}^n$ will be contained in the point sets corresponding to $\mathbf{N}_q^n$.

Proof: By the definition, $\mathbf{N}_{q-1}^n=\left\{\Xi:\sum_{d=1}^{n}i_d=n+q-1\right\}$ and $\mathbf{N}_q^n=\left\{\Xi:\sum_{d=1}^{n}i_d=n+q-1+1\right\}$. If the combination $(i_1,i_2,\cdots,i_j,\cdots i_n)$ belongs to $\mathbf{N}_{q-1}^n$, then the combination $(i_1,i_2,\cdots,i_j+1,\cdots i_n)$ belongs to $\mathbf{N}_q^n$. For $\mathbf{N}_q^n$, each element sequence in it contains at most q elements that are greater than 1. Because $1<q<n$ and there are at most q elements

greater than 1, we can always find an $i_j = 1,\ 1 < j < n$. In this case, the points corresponding to $\left(i_1, i_2, \cdots, i_j, \cdots i_n\right)$ will be generated by $X_{i_1} \otimes X_{i_2} \otimes \cdots X_1 \cdots \otimes X_{i_n}$. The points corresponding to $\left(i_1, i_2, \cdots, i_j + 1, \cdots i_n\right)$ will be generated by $X_{i_1} \otimes X_{i_2} \otimes \cdots X_2 \cdots \otimes X_{i_n}$. Because $X_1 \subseteq X_2$, any point generated by $\mathbf{N}_{q-1}^n$ will be contained in the point sets generated by $\mathbf{N}_q^n$. ■

For the level 3 SPQ, we can give more detailed analysis of the SPQ using the same procedure in the proof of Theorem 3.1. For convenience, the SPQ points and weights with the accuracy level 3 for three different cases are given below.

A. Case 1: $\hat{p}_1 = \hat{p}_2 = \hat{p}_3$

In this case, there are three variables, $\hat{p}_1$, $\hat{w}_1$ and $\hat{w}_2$. By the Eq. (26) and Eq. (28), we can get $\hat{p}_1 = \sqrt{3}$, $\hat{w}_1 = \frac{2}{3}$, $\hat{w}_2 = \frac{1}{6}$. Because the univariate salient point set with different accuracy levels satisfies the condition of *Proposition 3.1*, we only need to use $\mathbf{N}_{L-1}^n$ to calculate the multivariate salient points. Note that L=3 and thus the largest q in $\mathbf{N}_q^n$ is L-1.

$$\mathbf{N}_{L-1}^n = \mathbf{N}_2^n = \left\{ \underbrace{(3,1,\cdots,1),\cdots,(1,1,\cdots,3)}_{n \text{ elements}}, \underbrace{(2,2,\cdots,1),\cdots,(1,\cdots,2,2)}_{C_n^2 \text{ elements}} \right\}$$

The combination $(3,1,\cdots,1)$ generates 2 points $\left[\pm\hat{p}_1, 0, \cdots, 0\right]^T$ and the combination $(2,2,\cdots,1)$ generates 4 new points ($\left[\pm\hat{p}_1, \hat{p}_1, \cdots, 0\right]^T$, $\left[\pm\hat{p}_1, -\hat{p}_1, \cdots, 0\right]^T$) and one old point $\left[0, \cdots, 0, \cdots, 0, \cdots 0\right]^T$. Similar results can be derived for other combinations.

Therefore, the total number of points is

$$4\mathbf{C}_n^2 + 2\mathbf{C}_n^1 + 1 = 2n^2 + 1.$$

B. Case 2: $\hat{p}_1 = \hat{p}_2 \neq \hat{p}_3$

In this case, we have seven variables, $\hat{p}_1$, $\hat{p}_3$, $\hat{w}_1$, $\hat{w}_2$, $\hat{w}_3$, $\hat{w}_4$ and $\hat{w}_5$, where $\hat{p}_1$, $\hat{p}_3$ are two tunable variables.

Following the similar deduction as Case 1, the total number of points can be derived to be

$$4\mathbf{C}_n^2 + 4\mathbf{C}_n^1 + 1 = 2n^2 + 2n + 1.$$

C. Case 3: $\hat{p}_1 \neq \hat{p}_2 \neq \hat{p}_3$

In this case, there are eight variables, $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$, $\hat{w}_1$, $\hat{w}_2$, $\hat{w}_3$, $\hat{w}_4$ and $\hat{w}_5$, where $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ are three tunable variables.

Following the similar deduction as Case 1, the total number of points can be derived to be

$$4\mathbf{C}_n^1 + 4\mathbf{C}_n^2 + 2\mathbf{C}_n^1 + 1 = 2n^2 + 4n + 1.$$

There are some important merits of the SPQ compared with other point-based methods.

1) The accuracy can be flexibly controlled by the accuracy levels of SPQ. Theoretically, if the nonlinear function f can be approximated sufficiently well by polynomials, then the integral $\int_{-\infty}^{\infty} f(\mathbf{x})\mathbf{N}(\mathbf{x};\mathbf{0},\mathbf{I})d\mathbf{x}$ can always be calculated with sufficient accuracy by increasing the level of SPQ. Other point based methods are difficult to extend to higher accuracy levels. For example, the integral $\int_{-\infty}^{\infty}\left(1+\sum x_i^2\right)^n \mathbf{N}(\mathbf{x};\mathbf{0},\mathbf{I})d\mathbf{x}$ (x_i is the i^{th} element of $\mathbf{x}$) can be calculated by SPQ with level 4 and level 5 exactly when n equals 3 or 4 whereas UKF is difficult to achieve that.

2) The number of SPQ points for a fixed accuracy level increases polynomially with the dimension due to the sparse grid method. Therefore, the filter algorithm does not demand high computational load and is very efficient for high dimensional estimation problems.

3) The tunable parameters (location of the salient points) make this filter more flexible. For different nonlinear functions, different point sets with the same accuracy level may have very different performance.

IV. Numerical Results and Analysis

In this section, SPQF is applied to the orbit determination problem and compared with EKF, UKF and GHQF.

The Near-Earth satellite dynamics can be described by [11]

$$\ddot{\mathbf{r}} = -\mu\mathbf{r} / \|\mathbf{r}\|^3 + \mathbf{a}_G + \mathbf{a}_D + \boldsymbol{\nu} \tag{35}$$

where $\mathbf{r} = \left[x, y, z\right]^T$ is the position of satellite in the inertial coordinate frame; $\boldsymbol{\nu}$ is the white Gaussian process noise. $\mathbf{a}_G$ and $\mathbf{a}_D$ are the instantaneous acceleration due to the J_2 perturbation and atmospheric drag respectively.

Measurement model is given by

$$\begin{cases} az = \tan^{-1}\left(\dfrac{\rho_e}{\rho_n}\right) + n_{az} \\ el = \tan^{-1}\left(\dfrac{\rho_u}{\sqrt{\rho_e^2 + \rho_n^2}}\right) + n_{el} \\ \|\boldsymbol{\rho}\| = \sqrt{\rho_u^2 + \rho_e^2 + \rho_n^2} + n_\rho \end{cases} \tag{36}$$

where the azimuth (*az*), the elevation (*el*), and the range $\boldsymbol{\rho} = \left[\rho_u \quad \rho_e \quad \rho_n\right]^T$ can be measured by the radar site on the ground with respect to the local observer coordinate system. $\boldsymbol{\rho}$ can be related to the range vector in the inertial frame by the coordinate transformation [11] given by

$$\begin{bmatrix} \rho_u \\ \rho_e \\ \rho_n \end{bmatrix} = \begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \|\mathbb{R}\|\cos\lambda\cos\theta \\ y - \|\mathbb{R}\|\cos\lambda\sin\theta \\ z - \|\mathbb{R}\|\sin\lambda \end{bmatrix} \tag{37}$$

$\|\mathbb{R}\| = 6378.1363km$ is the earth radius; λ and θ are the latitude and local sidereal time of the observer respectively; n_{az}, n_{el}, n_ρ are the white Gaussian measurement noises.

Since it is hard to track satellite all the time, we assume the track time is 5 minutes and the measurement period is 5

seconds. In this paper, we use the accuracy level 2 and level 3 for the SPQF algorithm, which is sufficient to give satisfactory results.

We use three different parameters for SPQF in Table 1:

Table 1: Parameter values for different SPQFs

	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$
1st SPQF (Case 1)	$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$
2nd SPQF (Case 2)	1.71	1.71	2.5
3rd SPQF (Case 3)	1.76	1	2.5

The 2nd SPQF has two tunable variables $\hat{p}_1 (= \hat{p}_2)$ and $\hat{p}_3$, and the 3rd SPQF has three tunable variables $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$.

The initial true value is assumed to be $x_0 = \begin{bmatrix} x_0^p & x_0^v \end{bmatrix}$,

where $x_0^p = [6949.599783, 1045.733299, 64.918535]$ km

and $x_0^v = [-0.902571, 5.697655, 4.841182]$ km/s

The initial estimate is $\hat{x}_0 = \begin{bmatrix} \hat{x}_0^p & \hat{x}_0^v \end{bmatrix}$ where

$\hat{x}_0^p = [7252.009273, 1358.407862, 383.904071]$ km

and $\hat{x}_0^v = [-0.613101, 5.991868, 5.138553]$ km/s.

The initial variance is $P_0 = \text{diag}\left(\left[10^4, 10^4, 10^4, 10^{-2}, 10^{-2}, 10^{-2}\right]\right)$

The latitude and longitude of the radar site is -10.749° and -70.5983°, respectively.

The process and measurement noise covariance are

$$Q = \text{diag}\left(\left[0, 0, 0, 10^{-16}, 10^{-16}, 10^{-16}\right]\right)$$

and $$R = \text{diag}\left(\left[\left(0.015^\circ\right)^2, \left(0.015^\circ\right)^2, 0.025^2 \text{km}\right]\right).$$

The performance of SPQF is compared with EKF, UKF, and GHQF using the root-mean square error (RMSE) for 50 simulation runs. The simulation results of the estimation error for the position and the velocity are shown in Fig. 2 and Fig. 3 respectively. All SPQFs exhibit much better performance than EKF. The 1st, 2nd and 3rd SPQFs (level 2) have very close performance and all the level 3 SPQFs have no distinguishable difference. In addition, the level 3 SPQFs has better performance than the level 2 SPQFs. This is because the level 3 SPQ has higher accuracy than the level 2 SPQ. Note that the level 2 SPQF is identical with the UKF with $\hat{p}_1 = \sqrt{n+\kappa}$. The 1st SPQF (level 2) is the same as the UKF with the optimal parameter $\kappa = 3 - n$. Although SPQFs (level 3) has the similar performance with GHQF, SPQF (level 3) uses 73, 85, and 97 points for the 1st, 2nd, 3rd cases, respectively whereas GHQF uses 3^6 (=729) points. Thus, the SPQF is computationally more efficient than the GHQF.

V. Conclusion

In this paper, a new nonlinear filter, salient point quadrature filter (SPQF) was developed using the moment match and the sparse grid method. It is proven that UKF is a subset of SPQF with the accuracy level 2. In addition, the new filter is flexible to use since higher accuracy level can be easily achieved by moderately increasing the number of SPQ points polynomially with the dimension. The simulation results demonstrated that the new filtering algorithm exhibits much better performance than EKF and UKF.

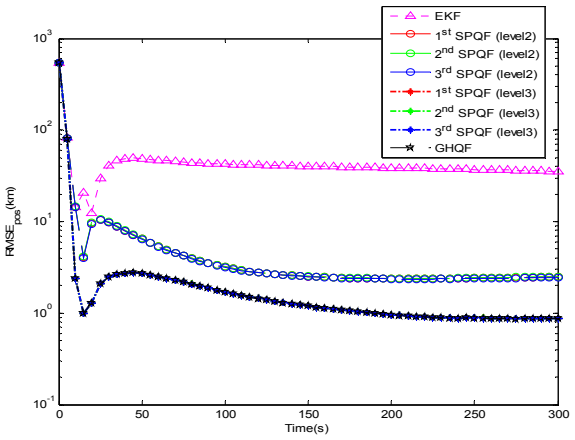

Fig. 2: RMSE for position error of EKF, SPQFs and GHQF

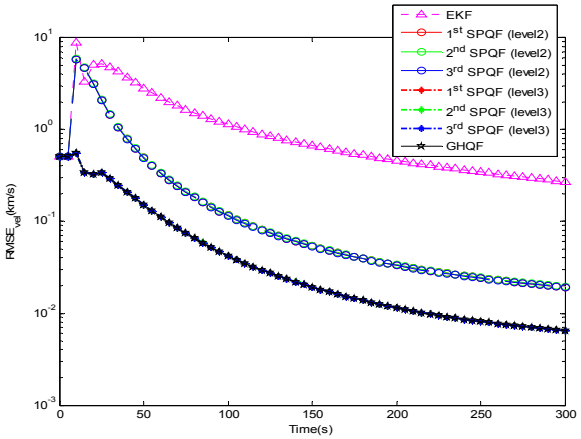

Fig. 3: RMSE for velocity error of EKF, SPQFs and GHQF

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