2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Tracking Control of a Nonholonomic Ground Vehicle

Veronica Medina-Garciadiego and Alexander Leonessa

Abstract— **The purpose of the presented work is to address the problem of controlling vehicles subject to nonholonomic constraints, specifically unicycle mobile robots, while tracking another dynamical model (reference system) that is less constrained. The closed loop stability of the tracking error dynamics is obtained using Lyapunov theory as well as properly designing the dynamics of the desired distance between the vehicle and the reference system. In particular, the vehicle maintains a time dependent distance to the reference system, which is tracking the desired trajectory. Results from the implementation of both the designed controller and a previously existing tracking controller are presented to verify and compare its performance and capabilities.**

Keywords: Control system design, tracking control, unicycle vehicles, nonholonomic.

I. Introduction

Unicycle vehicles are mechanical systems characterized by kinematic constraints that are not integrable and cannot be eliminated from the model equations. This class of mobile robots is subject to nonholonomic constraints, in fact they can have forward speed but are not capable of instantaneous lateral motion.

The problem of controlling unicycle mobile robots to track trajectories continues posing numerous challenges. Although many control algorithms have been proposed in the past decades ([2]–[5], [9], [11], [13]), the constrains on this vehicles induce different challenges according to the desired objectives, which span from having to stop in a precise position, in the case of parking ground vehicles, to having to maintain the velocity in a given range in the case of aerial vehicles.

Because of the nonholonomic constraints, the set of feasible trajectories for a unicycle vehicle is limited, and has to be considered when designing a tracking controller. Several methods have been developed to solve the problem of path following for underactuated vehicles ([2]–[5], [9], [11], [13]). In general, these consist of generating inputs to steer a unicycle vehicle through an admissible generated path as close as possible to a desired trajectory which is not subject to nonholonomic constraints.

Early research results on trajectory tracking control of unicycle vehicles include the work of Kanayama, who proposed using a sequence of straight lines to define the reference trajectory for the vehicle ([7]), and later used Lyapunov theory to design a local asymptotic tracking controller for a unicycle vehicle ([6]). Afterward, in [1], the authors proposed a method for adaptive tracking control of nonholonomic vehicles, while in [4] backstepping techniques are used to design an adaptive tracking controller for a nonholonomic kinematic model with unknown parameters.

In [3], a controller is designed using backstepping so that the tracking error converges to zero, and the velocity of the vehicle converges to the desired velocity. While this could be a desirable behavior, keeping a known distance to the virtual target or trajectory will allow the vehicle to have enough space to turn in case the target has lateral motion, which could help when we account for the fact that unicycle vehicles can not have instantaneous lateral motion. This is demonstrated in [11] and [12] where, using Lyapunov theory, a tracking controller is designed considering the vehicle dynamics and constraints, where the position error converges to the desired distance.

The vehicle should be close enough to the desired trajectory to be able to track it correctly, even if the target has instantaneous lateral motion but, at the same time, the distance should be large enough so that if the target undergoes sudden changes in speed, the vehicle will be able to mimic its behavior. Therefore, we propose a modification of the tracking controller introduced in [11] and [12], with the difference that we consider the distance between the vehicle and the virtual target as a time-varying parameter. The particular case where the distance directly depends on the velocity of the virtual target will be considered.

The paper is structured as follows. Section II describes the mathematical model for both a unicycle vehicle and the reference system. The tracking control is presented in Section III. The simulation results are discussed in Section IV, where the proposed controller is compared to the tracking controller from [11], to assess its efficacy. The paper is concluded in Section V.

II. System Description and Problem Statement

Consider the kinematic equations for a unicycle vehicle moving in the horizontal plane ([11])

$$\dot{p}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \nu(t)\cos(\theta(t)) \\ \nu(t)\sin(\theta(t)) \end{bmatrix}, \qquad (1)$$

$$\dot{\theta}(t) = \omega(t), \quad t \geq 0, \qquad (2)$$

where $p(t) \triangleq [x(t) \quad y(t)]^{\mathrm{T}} \in \mathbb{R}^2$, $t \geq 0$, is the vehicle position, $\nu(t) \in \mathbb{R}$, $t \geq 0$, is the longitudinal velocity, $\theta(t) \in \mathbb{R}$, $t \geq 0$, is the orientation, and $\omega(t) \in \mathbb{R}$, $t \geq 0$, is the angular velocity of the vehicle.

V. Medina-Garciadiego is with the Mechanical Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24060 , USA veromedin@gmail.com
A. Leonessa is with the Mechanical Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24060 , USA leonessa@vt.edu

The reference system is a virtual target with unitary mass that the vehicle will track, and is mathematically defined as

$$\ddot{p}_{\mathrm{r}}(t) = \begin{bmatrix} \ddot{x}_{\mathrm{r}}(t) \\ \ddot{y}_{\mathrm{r}}(t) \end{bmatrix} = \begin{bmatrix} f_x(t) \\ f_y(t) \end{bmatrix}, \quad t \geq 0, \tag{3}$$

where $p_{\mathrm{r}} \triangleq [x_{\mathrm{r}}(t) \quad y_{\mathrm{r}}(t)]^{\mathrm{T}} \in \mathbb{R}^2$, $t \geq 0$, is the position of the reference system, and $[f_x(t) \quad f_y(t)]^{\mathrm{T}} \in \mathbb{R}^2$, $t \geq 0$, are the components of a virtual force applied to the virtual target.

The linear velocity of the reference system is

$$v_{\mathrm{r}}(t) \triangleq \sqrt{\dot{x}_{\mathrm{r}}(t)^2 + \dot{y}_{\mathrm{r}}(t)^2}, \qquad t \geq 0, \tag{4}$$

where $v_{\mathrm{r}}(t) \geq 0, t \geq 0$.

The goal of the presented work is to derive a control law for a nonholonomic vehicle described by (1) and (2), so that it will track a virtual target (3) while guaranteing that the position error $\|p(t) - p_{\mathrm{r}}(t)\|$, $t \geq 0$, converges to a small neighborhood of a function $\delta(t) \triangleq [d(t) \quad 0]^{\mathrm{T}}$, $t \geq 0$, as $t \to \infty$, where $d(t)$, $t \geq 0$, is constructed so that it converges to a desired distance $d^\star(t), t \geq 0$.

To reach this objective, we assume in the following that $p_{\mathrm{r}}(t), t \geq 0$, is sufficiently smooth with bounded derivatives.

III. Tracking Control Design

Consider the reference system described by (3) where $f_x(t)$ and $f_y(t)$, $t \geq 0$, have been designed so that the virtual target follows a desired trajectory. By guarenteeing that the vehicle described by (1) and (2) approaches such a reference system, we can drive the vehicle along the desired trajectory as well.

Recalling that (1) and (2) describe the model of a unicycle vehicle, if the virtual target is placed on the side of the vehicle, considering that the vehicle is not capable of instantaneous lateral motion, it will need to overshoot, turn and move back to be able to follow the target. Instead, by designing a tracking controller, which allows the vehicle to maintain a distance $d(t)$, $t \geq 0$, from the reference point (which is supposed to be constant in [11] and [12]), the vehicle will have enough space to turn and direct itself to track the virtual target.

Assuming that the origin of the body-fixed coordinate frame $\mathcal{B}$ and the origin of the global inertial coordinate frame $\mathcal{U}$ coincide with the center of mass of the vehicle moving in the horizontal plane, we can define an orthonormal transformation matrix from $\mathcal{B}$ to $\mathcal{U}$

$$R(\theta) \triangleq \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{5}$$

where $\theta \in \mathbb{R}$. Using (1) and (3), we can define the position error as

$$e(t) \triangleq R^{\mathrm{T}}(\theta(t))(p_{\mathrm{r}}(t) - p(t)), \qquad t \geq 0, \tag{6}$$

where $e(t) \in \mathbb{R}^2, t \geq 0$. Following the approach introduced in [11] with the addition of a time dependent distance $d(t)$, $t \geq 0$, between the vehicle and the reference system, we introduce the following feedback law

$$\begin{bmatrix} \nu(t) \\ \omega(t) \end{bmatrix} = \Delta^{-1}(t)(-K \tanh(e(t) - \delta(t)) + R^{\mathrm{T}}(\theta(t))\dot{p}_{\mathrm{r}}(t) + \dot{\delta}(t)), \qquad t \geq 0, \tag{7}$$

with the distance parameters

$$\delta(t) \triangleq \begin{bmatrix} d(t) \\ 0 \end{bmatrix}, \quad \Delta(t) \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -d(t) \end{bmatrix}, \quad t \geq 0, \tag{8}$$

and K is a gain matrix

$$K \triangleq \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}, \tag{9}$$

where $k_x \in \mathbb{R}$ and $k_y \in \mathbb{R}$.

The nonlinear term $\tanh(\cdot)$ in the controller allows the position error to increase the velocity up to a threshold. This implies that, for large position errors, the vehicle approaches the virtual target with a constant, maximum velocity until the position error is within a neighborhood of the origin.

In the proposed controller the distance $d(t), t \geq 0$, between the unicycle vehicle and the virtual target is a time-varying parameter which needs to be selected. We observe that, in order to be able to define the term $\Delta^{-1}(t), t \geq 0$, in (7), we need to guarantee that $d(t)>0$, $t \geq 0$. In particular, if this distance depends on the velocity of the reference system (4), the vehicle will be able to track the reference system even if the target has instantaneous changes of direction at larger velocities.

Theorem 3.1: Consider the system described by (1) and (2), the reference system described by (3), and the feedback controller (7). If the distance $d(t)$, $t \geq 0$, is updated according to the following

$$\dot{d}(t) = \dot{d}^\star(t) - \gamma(d(t) - d^\star(t)), \ t \geq 0, \ d(0) = d_0, \tag{10}$$

with $\gamma > 0$, then the distance $d(t), t \geq 0$, between the vehicle and the reference system converges to the desired distance $d^\star(t)$, $t \geq 0$, while the tracking error, $e(t)$, $t \geq 0$, given by (6), converges to the distance vector, $\delta(t) = [d(t) \quad 0]^{\mathrm{T}} \in \mathbb{R}^2$, $t \geq 0$.

Proof:

Consider the Lyapunov function candidate

$$V(e_1, d, t) \triangleq \frac{1}{2} e_1^{\mathrm{T}} e_1 + \frac{1}{2}(d(t) - d^\star(t))^2, \quad t \geq 0. \tag{11}$$

where

$$e_1(t) \triangleq e(t) - \delta(t), \qquad t \geq 0. \tag{12}$$

The time derivative of the tracking error $e(t)$, $t \geq 0$, given by (6), is

$$\dot{e}(t) = \dot{R}^{\mathrm{T}}(\theta(t), \omega(t))(p_{\mathrm{r}}(t) - p(t)) + R^{\mathrm{T}}(\theta(t))(\dot{p}(t) - \dot{p}_{\mathrm{r}}(t)), \quad t \geq 0, \tag{13}$$

while the derivative of the orthonormal transformation matrix $R(\theta)$ defined in (5) is

$$\dot{R}(\theta, \omega) = \begin{bmatrix} -\omega \sin(\theta) & -\omega \cos(\theta) \\ \omega \cos(\theta) & -\omega \sin(\theta) \end{bmatrix} = R(\theta) S(\omega), \tag{14}$$

where

$$S(\omega) \triangleq \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}. \quad (15)$$

Substituting (1), (12), and (14) into (13), we can express the error dynamics as

$$\begin{aligned}\dot{e}(t) &= S^{\mathrm{T}}(\omega(t))R^{\mathrm{T}}(\theta(t))(p(t)-p_{\mathrm{r}}(t)) - R^{\mathrm{T}}(\theta(t))\dot{p}_{\mathrm{r}}(t) \\ &\quad +R^{\mathrm{T}}(\theta(t))\begin{bmatrix} \nu(t)\cos\theta(t) \\ \nu(t)\sin\theta(t) \end{bmatrix} \\ &= -S(\omega(t))R^{\mathrm{T}}(\theta(t))(p(t)-p_{\mathrm{r}}(t)) \\ &\quad -R^{\mathrm{T}}(\theta(t))\dot{p}_{\mathrm{r}}(t) + \begin{bmatrix} \nu(t) \\ 0 \end{bmatrix} \\ &= -S(\omega(t))e(t) + \begin{bmatrix} \nu(t) \\ -d(t)\omega(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ d(t)\omega(t) \end{bmatrix} - R^{\mathrm{T}}(\theta(t))\dot{p}_{\mathrm{r}}(t) \\ &= -S(\omega(t))e(t) + \Delta(t)\begin{bmatrix} \nu(t) \\ \omega(t) \end{bmatrix} \\ &\quad +S(\omega(t))\delta(t) - R^{\mathrm{T}}(\theta(t))\dot{p}_{\mathrm{r}}(t) \\ &= -S(\omega(t))(e(t)-\delta(t)) - K\tanh(e(t)-\delta(t)) + \dot{\delta}(t) \\ &= -S(\omega(t))e_1(t) - K\tanh(e_1(t)) + \dot{\delta}(t), \quad t \geq 0. \quad (16)\end{aligned}$$

The time derivative of the Lyapunov function (11) is

$$\dot{V}(t) = e_1^{\mathrm{T}}(t)\dot{e}_1(t) + (d(t)-d^\star(t))(\dot{d}(t)-\dot{d}^\star(t)), \quad t \geq 0, \quad (17)$$

and, by substituting (16) and (10) into (17), we obtain

$$\begin{aligned}\dot{V}(t) &= e_1(t)^{\mathrm{T}}(-S(\omega(t))e_1(t) - K\tanh(e_1(t))) + (d(t) \\ &\quad -d^\star(t))(\dot{d}(t)-\dot{d}^\star(t)) \\ &= -e_1^{\mathrm{T}}(t)S(\omega(t))e_1(t) - e_1^{\mathrm{T}}(t)K\tanh(e_1(t)) \\ &\quad -\gamma(d(t)-d^\star(t))^2 \\ &= -e_1^{\mathrm{T}}(t)K\tanh(e_1(t)) - \gamma(d(t)-d^\star(t))^2 \\ &= -(e(t)-\delta(t))^{\mathrm{T}}K\tanh(e(t)-\delta(t)) \\ &\quad -\gamma(d(t)-d^\star(t))^2, \qquad t \geq 0. \quad (18)\end{aligned}$$

Therefore, $\dot{V}(t) < 0$, $t \geq 0$, which proves uniform convergence of the position error $e(t)$, $t \geq 0$, to the distance vector $\delta(t)$, $t \geq 0$, between the vehicle and the reference system, while $d(t)$ converges uniformly to the desired position $d^\star(t)$, $t \geq 0$. ∎

IV. Simulation Results

To illustrate the performance of the tracking control given by (7), we compare its behavior with the one obtained in [11], with both vehicles following the same virtual target, and the same initial conditions, vehicle position and orientation. The tracking controller presented in [11], is given by

$$u_{\mathrm{d}}(t) = \bar{\Delta}^{-1}(-K\tanh(e(t)-\bar{\delta}) + R^{\mathrm{T}}(\theta)\dot{p}_{\mathrm{r}}(t)), \quad (19)$$

where the distance matrix $\bar{\Delta} \in \mathbb{R}$, and the distance vector $\bar{\delta} \in \mathbb{R}$ are defined as

$$\bar{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & -\bar{d} \end{bmatrix}, \qquad \bar{\delta} = \begin{bmatrix} \bar{d} \\ 0 \end{bmatrix}, \quad (20)$$

and the distance $\bar{d} \in \mathbb{R}$ from the vehicle to the virtual target is constant. The desired reference trajectory is a sine-shaped curve traced with constant velocity in the x-direction, which is obtained from (3) with

$$f_x(t) \triangleq 0, \quad (21)$$

$$f_y(t) \triangleq -20\sin(x_{\mathrm{r}}(t)), \qquad t \geq 0. \quad (22)$$

The corresponding reference and vehicle trajectories with initial conditions $x(0) = x_{\mathrm{r}}(0) = 0$, $\dot{x}(0) = \dot{x}_{\mathrm{r}}(0) = 1$, $y(0) = y_{\mathrm{r}}(0) = 0$, $\dot{y}(0) = \dot{y}_{\mathrm{r}}(0) = 20$, are shown in Figure 1. The same gain matrix $K = I_2$ was used for both vehicles, while the distance for the tracking controller from [11] was $\bar{d} = 2$. The desired distance needed in (7) for the proposed approach was defined as

$$d^\star(t) \triangleq \alpha\, v_{\mathrm{r}}(t) + \beta, \qquad t \geq 0, \quad (23)$$

with $\alpha = 0.1$, $\beta = 0.1$, and $\gamma = 1$.

Figure 4 shows that the distance $d(t)$, $t \geq 0$, converging to $d^\star(t)$, $t \geq 0$. Note that the value of the distance $\bar{d}$ used in (19) is in the same range of values of $d^\star(t)$, $t \geq 0$.

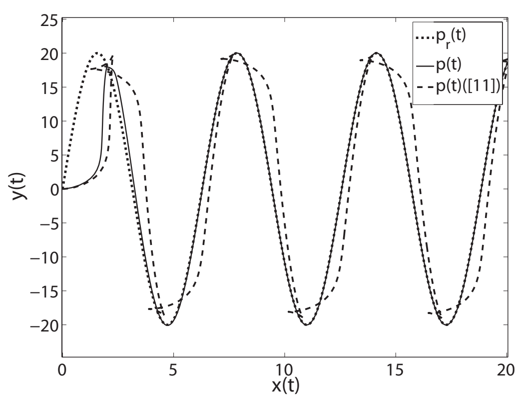

Fig. 1. Trajectories of the reference system and the vehicles

As the virtual target moves along the desired trajectory, the vehicles change their orientation $\theta(t)$, $t \geq 0$, as shown in Figure 2, while in Figure 3 the linear and angular velocities are represented.

Figure 4 shows how the proposed controller allows the vehicle to get closer to the target point when the velocity of the reference decreases, where the minimum distance between the vehicle and the reference system is defined by the parameter β, which needs to be a positive number to guarantee $d(t) > 0$, $t \geq 0$.

In order to prevent the vehicle to get too far from the virtual target, α can be defined as

$$\alpha \triangleq \frac{d^\star_{max} - \beta}{v_{\mathrm{r,max}}}, \qquad t \geq 0, \quad (24)$$

where $d^\star_{max} \in \mathbb{R}, t \geq 0$, is the maximum desired distance, $v_{\mathrm{r,max}} \in \mathbb{R}$, $t \geq 0$, is the maximum velocity that the virtual target will have, and, as stated before, β is the minimum distance.

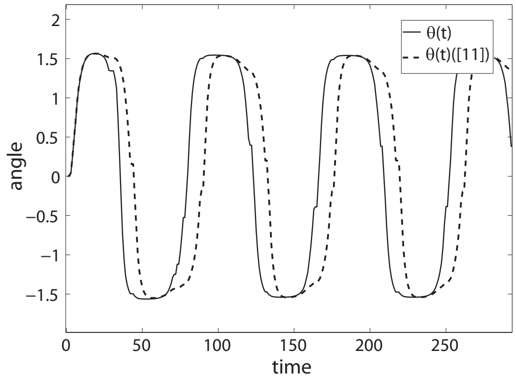

Fig. 2. Orientation of the vehicles

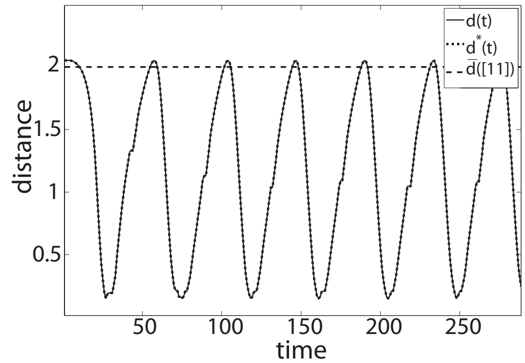

Fig. 4. Distances of the vehicles

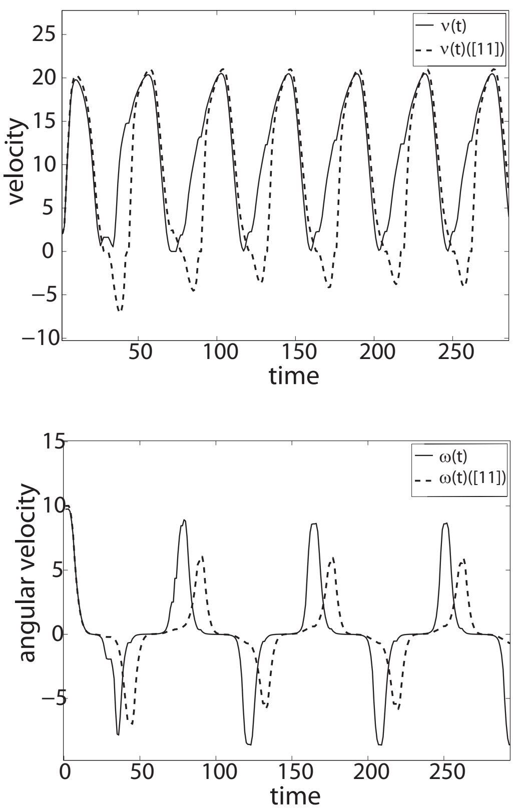

Fig. 3. Linear and angular velocities of the vehicles

V. Conclusion

This paper presents a tracking controller for unicycle vehicles based on the strategy proposed in [11] and [12]. With the proposed controller, a vehicle subject to nonholonomic constrains can track a reference system represented by a virtual target capable of instantaneous lateral motion, by maintaining a time varying distance $d(t),\ t \geq 0$, to the reference system, which in our example, depends on the linear velocity of said system.

We used a Lyapunov function approach to design the tracking controller, and proved the convergence of the position error $e(t),\ t \geq 0$, to a neighborhood of the distance vector $\delta(t) = [d(t) \quad 0]^{\mathrm{T}},\ t \geq 0$, from the unicycle vehicle to the reference system, while this distance $d(t)$ converges to the desired distance $d^{\star}(t),\ t \geq 0$.

In future work, we will test this controller on real unicycle four-wheeled vehicle. Additionally, the designed tracking controller can be easily extended to achieve obstacle avoidance and collaborative tracking control when considering swarms of unicycle vehicles.

References

[1] Y. Chang and B. Chen, “Adaptive tracking control design of nonholonomic mechanical systems,” in *Proc. 35rd IEEE Conf. Dec. Contr.*, 1996.

[2] D. DeVon and T. Bretl, “Kinematic and dynamic control of a wheeled mobile robot,” in *Proc. IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, 2007.

[3] R. Fierro and F. L. Lewis, “Control of a nonholonomic mobile robot: Backstepping kinematics into dynamics,” *Journal of Robotic systems*, vol. 14, pp. 149–163, 1997.

[4] T. Fukao, H. Nakawa, and N. Adachi, “Adaptive tracking control of a nonholonomic mobile robot,” *IEEE Transactions on Robotics and Automation*, vol. 16, pp. 609–615, 2008.

[5] R. Ghabcheloo, A. P. Aguiar, A. Pascoal, C. Silvestre, and I. K. J. Hespanha, “Coordinated path-following control of multiple underactuated autonomous vehicles in the presence of communication failures,” in *Proc. 45th IEEE Conf. Dec. Contr.*, December 2006.

[6] Y. Kanayama, Y. Kimura, F. Miyazaki, and T. Noguchi, “A stable tracking control method for an autonomous mobile robot,” in *proc. IEEE Int. Conf. Robotics and Automation*, 1990, pp. 384–389.

[7] Y. Kanayama and Shin’ichiYuta, “Vehicle path specification by a sequence of straight lines,” *IEEE Journal of Robotics and Automation*, vol. 4, pp. 265–276, 1988.

[8] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.

[9] J. Ren, H. Zhang, K. A. McIsaac, and R. V. Patel, “Stable motion planning for dynamic nonholonomic mobile robot,” in *Proc. IEEE Int. Conf. Automation Science and Eng.* IEEE, 2006.

[10] I. Rush D. Robinett and D. G. Wilson, “Collective plune tracing: A minimal information approach to collective control,” in *Proc. Amer. Contr. Conf.*, 2007, pp. 4884–4891.

[11] F. Vanni, A. P. Aguiar, and J. Gaspar, “Control of unicycle type robots,” in *Proc. IV Jornadas de Engenharia Electronica e Telecomunicacoes e de Computadores*, 2008, pp. 180–185.

[12] F. Vanni, A. P. Aguiar, and A. M. Pascoal, “Cooperative path-following of underactuated autonomous marine vehicle with logic-based communication,” in *Proc. IFAC Workshop on Navigation, Guidance and Control of Underwater Vehicles*, April 2008, pp. 1–6.

[13] D. Wang and G. Xu, “Full-state tracking and internal dynamics of nonholonomic wheeled mobile robots,” *IEEE/ASME Transactions on Mechatronics*, vol. 8, pp. 203–214, 2003.