2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Global Optimization of Three-input Systems using Multi-unit Extremum Seeking Control

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Abstract—**An efficient global optimization method based on multi-unit extremum seeking has been proposed recently for scalar and two-input systems. For scalar systems, the global optimum is obtained by controlling the finite-difference gradient and reducing the offset used for calculating this gradient. With two inputs, the uni-variate method is repeated on the circumference of a circle of reducing radius. In this paper, the concept is extended to three-input systems where the circle of varying radius sits on a shrinking sphere. The theoretical concepts are illustrated on the global optimization of several examples. The results show the capability of the proposed technique in deterministic convergence to the global optimum of the three-input systems.**

I. Introduction

Finding the global optimum of an industrial process has always been attractive in many engineering applications including chemical and petrochemical production plants. The global optimization has a rich volume of literature [6]. There is a large variety of different optimization algorithms which are applied to different problems. However, many of these optimization algorithms make assumptions about the properties of the objective functions which restrict their application. It is also important to note that the class of real-world optimization problems is often not easy to identify in order to apply the appropriate optimization algorithm. The model-based optimization methods are not always capable to find the best operating set points. Moreover, sometimes there is no appropriate model of the process and the only measurable data are input/output of the system.

Optimization problems which have few available knowledge about the properties of the problem admit an alternative design approach, the so-called "black-box optimization". These problems have an objective function that may not be easily differentiable. Black-box algorithms handle a more general class of problems and consider less assumption about the objective function [10]. The global optimization methods which take account of prior information about the characteristics and structure of the objective function cannot be considered as black-box optimization. Therefore a model-free global optimization strategy for black-box systems is needed in order to adjust the process on its best operating point. Different deterministic and stochastic methods have been developed to handle these kinds of optimization problems [8], [9]. On the other hand, extremum-seeking schemes are model-free methods which recast the optimization problem as a control problem and take advantage of sensitivity reduction and disturbance rejection. The construction of these techniques is suitable for optimizing the black-box objective functions [7]. However, the objective function value must be measured on line. The gradient estimation is the main difference between different alternatives of these techniques. In extremum seeking control based on perturbation, an external excitation signal is used in order to numerically compute the gradient [1], [11]. As an alternative, in multi-unit optimization framework the gradient is estimated by finite difference of the outputs of two identical units driven with the inputs that differ by an offset [14]. The gradient is pushed to zero by an integral controller. Although the latter method shows a faster convergence to the optimal point, however, the convergence of both of these techniques depends to their initial condition which makes the system to converge to the closest local optimum.

There has always been a debate on how efficient gradient estimation techniques of the continuous processes can be used for global optimization purposes without the intermediary of a model. Herein, some extremum seeking schemes based on perturbation have been used as a tool for global optimization [15]. However, the main drawback of these schemes is their restriction to a specific class of nonlinear maps. Towards this end, a deterministic global optimization technique using multi-unit adaptation framework for a general class of nonlinear scalar systems has been recently proposed by [4]. The idea of multi-unit extremum-seeking is to control the gradient evaluated using finite difference between two identical units operating with an offset. For scalar systems, it was shown that the global optimum could be obtained by reducing the offset to zero. For two-input systems, the uni-variate global optimization is performed on the circumference of a circle of reducing radius [5]. In this paper, the concept is extended to three-input systems where the circle of varying radius sits on a shrinking sphere. The idea is to perform uni-variate global optimization on the circle of varying radius that sweeps the surface of a shrinking sphere. The key contribution lies in formulating the rotation required to keep the best point

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found in the search domain. The dynamics of each unit is formulated in a way that the movement of each unit is towards a better local operating point on nonlinear map. The deterministic nature of this approach guarantees the convergence of the algorithm to the global optimum. Two key aspects are emphasized: (i) the rotation of the circle to keep the global optimum in the search domain and (ii) time-scale separation between the different layers of adaptation. Generalization of the proposed algorithm to higher dimensions is indeed a main challenge. In fact, extension from two dimensions to three requires "Rodrigues" rotation matrix. To reach higher than three dimensions the tools needed are even more involved. The paper is arranged as follows. Section II provides a brief overview on global optimization using multi-units for scalar and two-input systems. Section III presents the extension of the method to optimization of three-input systems. In section IV, the established algorithm is applied on several illustrative examples and its convergence is outlined. The conclusions are addressed in section V.

II. Multi-unit Extremum Seeking Control

The multi-unit optimization method is a real-time extremum seeking technique that estimates the gradient by the finite difference of the outputs of two identical units where the inputs differ by an offset Δ [14]. An integral controller then forces the gradient to zero. The basic schematic of this technique (where Δ is fixed) converges to a local optimum of the objective function. The extension of the local extremum seeking control using multi-units to global optimization of the static nonlinear systems was developed by Azar et al. [3, 4].

A. Global Optimization for Scalar Systems

The idea of global optimization using extremum-seeking is to monotonically decrease the offset to zero. The schematic is presented in figure 1.

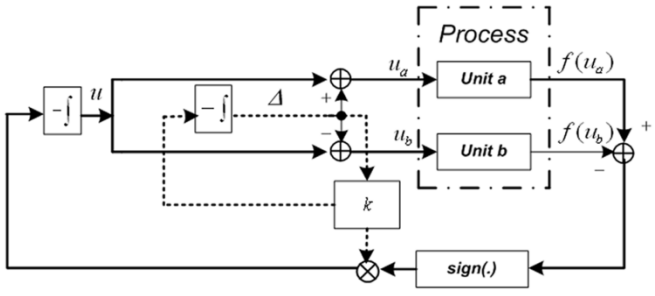

Fig. 1. Unconstrained global extremum-seeking with multi-units for scalar systems

The update equations and adaptation laws for global optimization (minimization) of a scalar and continuous function *f(u)* are given by,

$$u_a = u + \Delta \quad , \quad u_b = u - \Delta \tag{1}$$

$$\dot{u} = -k\Delta sign\left(f\left(u+\Delta\right) - f\left(u-\Delta\right)\right) \ , u(0) = u_0 \tag{2}$$

$$\dot{\Delta} = -k\Delta \quad , \quad \Delta(0) = \Delta_0 > 0 \ . \tag{3}$$

Depending on the rate of convergence *k*, the offset *Δ* decreases to zero exponentially. The proof of convergence for this algorithm has been provided using mathematical contradiction formalism. This algorithm was also extended to handle the constrained optimization problems [3].

B. Global Optimization for Two-input Systems

The scalar algorithm was extended to optimize the two-input black-box functions $f(u_1, u_2)$ [5], by repeating uni-variate global optimization on the circumference of a circle of reducing radius. Three iterative layers for the algorithm are considered: (*Layer 1)* Global optimization along the circumference of a circle (*Layer 2)* Recursive global optimization and (*Layer 3)* Reducing the radius of the circle.

In layer 1 the input values of the two units u_a and u_b evolve along the circumference of a circle,

$$\begin{aligned} u_{1b} &= u_1 + \Delta\cos(\theta_b) & u_{1a} &= u_1 + \Delta\cos(\theta_a) \\ u_{2b} &= u_2 + \Delta\sin(\theta_b) & u_{2a} &= u_2 + \Delta\sin(\theta_a) \end{aligned} \tag{4}$$

$$\theta_a = \theta + \Delta_\theta \quad , \quad \theta_b = \theta - \Delta_\theta \tag{5}$$

$$\dot{\theta} = -k_\theta \Delta_\theta sign\,(f_a - f_b) \tag{6}$$

$$\dot{\Delta}_\theta = -k_\theta \Delta_\theta \ . \tag{7}$$

This corresponds to uni-variate global optimization along the circle of radius Δ using the angle θ, and is depicted in figure 2.

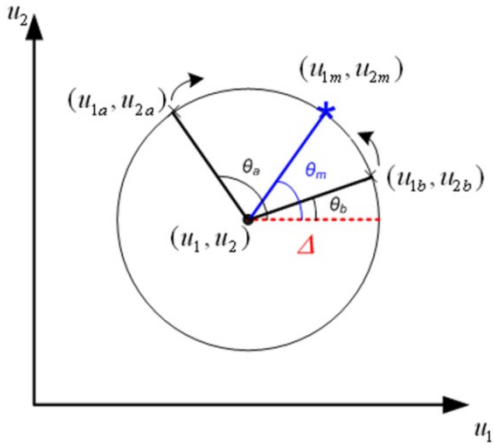

Fig. 2. Global optimization along the circumference of a circle

The adaptation laws (for minimization) along the circumference of the circular path stay similar to the uni-variate case. Here, θ_a and θ_b are the angles of the two units and Δ_θ is the offset between these angles. f_a and f_b are objective function values provided by two units "a" and "b" and $k_\theta > 0$ is a parameter that determines the rate at which Δ_θ is reduced. In layer 2 the initial conditions of the (6) and (7) are reinitialized periodically as follows,

$$\theta(iT_{\theta+}) = \pi + \theta_{mi} \tag{8}$$

$$\Delta_\theta(iT_{\theta+}) = \pi \ . \tag{9}$$

The optimization along the circumference of the circle is repeated every T_θ time units. "i" denotes the number of iteration (i = 0,1,2,…) and at the beginning of each iteration, Δ_θ is initialized to π in order to cover the entire circle. θ_{mi} corresponds to the converged value and it represents the global optimum along the circumference of the circle of iteration "i-1". At the beginning of first iteration (i.e. i=0), the initial value of θ_{m0} is arbitrarily set at zero. In the next iterations, θ_{mi} is computed from the values of θ_a and θ_b at the end of the previous iteration as,

$$\theta_{mi} = \begin{cases} \theta_a(iT_{\theta-}) & if \quad f_a < f_b \\ \theta_b(iT_{\theta-}) & if \quad f_a \geq f_b \end{cases} \tag{10}$$

In layer 3 the radius of the circle is monotonically reduced to zero i.e.,

$$\dot{\Delta} = -k_c\Delta \quad , \Delta(0) = \Delta_0 > 0 \, , \tag{11}$$

where $k_c > 0$ is a parameter that determines the rate at which Δ is reduced. The coordinates which correspond to the global optimum of each iteration are as follows (Figure 2),

$$\begin{aligned} u_{1m} &= u_1 + \Delta\cos(\theta_{mi}) \\ u_{2m} &= u_2 + \Delta\sin(\theta_{mi}) \end{aligned} \tag{12}$$

The adaptation laws of the center of the circle are so chosen to keep the global optimum found. In other words, the circle with the radius Δ and center (u_1,u_2) is contracted in such a manner as to keep (u_{1m},u_{2m}) at the same point i.e. $(\dot{u}_{1m},\dot{u}_{2m}) = (0,0)$. So, the adaptation laws are given by,

$$\begin{aligned} \dot{u}_1 &= -\dot{\Delta}\cos(\theta_{mi}) \qquad u_1(0) = u_{10} \\ \dot{u}_2 &= -\dot{\Delta}\sin(\theta_{mi}) \qquad u_2(0) = u_{20} \end{aligned} \tag{13}$$

One of the key aspects of this algorithm is the time-scale separation. The radius of the circle needs to be reduced much slower than the rate at which the circumference of the circle is swept. This means that $k_c << k_\theta$.

III. Construction of Algorithm for Three Input Systems

Consider the problem of minimizing, $y = f(u_1, u_2, u_3)$, where f: $R^3 \rightarrow R$, is a non-convex continuous nonlinear function. The problem may have multiple local optima, $(u_{1k}^*, u_{2k}^*, u_{3k}^*)$ k = 1, 2… m, but a unique global minimum, $(u_1^{**}, u_2^{**}, u_3^{**})$. In the rest of the paper, it is assumed that the global minimum is unique. The proposed algorithm uses the spirit of the global optimization by multi-units for two-input systems. Similarly, we need two units referred to as "a" and "b". The idea for generalizing the algorithm to three dimensions is to perform the global optimization on the circumference of a circle of varying radius that sweeps the surface of a shrinking sphere. In order to mathematically formulate the above mentioned methodology, five iterative layers for the proposed optimization algorithm are considered.

A. Layer 1: Global optimization along the circumference of a circle on a three-dimensional sphere

Consider the surface of a sphere centred at the input values $u = (u_1, u_2, u_3)$ and a radius of Δ. Let (φ, θ) represent the elevation and azimuth angles in the three-dimensional polar system. The circle on which the uni-variate optimization will be performed is traced by having the same elevation but varying azimuth. So, the angle φ is common between the coordinates of vectors "a" and "b", while θ_a and θ_b vary. The input values of the two units are given by,

$$\mathbf{u}_a = \mathbf{u} + \mathbf{M}\mathbf{u}_{ra} \quad , \quad \mathbf{u}_b = \mathbf{u} + \mathbf{M}\mathbf{u}_{rb} \, , \tag{14}$$

where,

$$\mathbf{u}_{ra} = \begin{bmatrix} \Delta\cos(\varphi) \\ \Delta\sin(\varphi)\cos(\theta_a) \\ \Delta\sin(\varphi)\sin(\theta_a) \end{bmatrix}, \mathbf{u}_{rb} = \begin{bmatrix} \Delta\cos(\varphi) \\ \Delta\sin(\varphi)\cos(\theta_b) \\ \Delta\sin(\varphi)\sin(\theta_b) \end{bmatrix} \tag{15}$$

For the moment, consider $\mathbf{M}=\mathbf{I}_{3\times3}$ (the identical matrix). This rotation matrix will be used later. The adaptation laws (for minimization) along the circumference of the circle on the sphere stay the same as (5), (6), (7). Here, these equations correspond to the uni-variate global optimization along the circle of radius $\Delta\sin(\varphi)$ on a three-dimensional sphere.

B. Layer 2: Recursive global optimization along the circle

The initialization of the initial conditions also remains the same as (8), (9). Here, θ_{mi} represents the azimuth angle in three-dimensional space that is associated to the best optimum at the end of each iteration. θ_{mi} is updated according to (10).

C. Layer 3: Variation of the radius of the circle

The key differences between the two-input and three-input cases lie in this layer. In the two-input case, the radius of the circle was reduced to zero. However, in the three-input case, it needs to be so arranged to sweep the surface of the sphere. This can be done by exponentially decreasing the angle φ as shown below,

$$\dot{\varphi} = -k_\varphi\varphi \, . \tag{16}$$

Note that though φ decreases monotonically from 2π to zero, the radius of the circle $\Delta\sin(\varphi)$ would initially increase and then decrease to zero when φ goes from 2π to π. Then again, the radius will increase and decrease as φ goes from π to 0. To make further development easier, the elevation angle will be kept constant during an iteration and will be changed from one iteration to another using,

$$\varphi_{i+1} = \varphi_i(1 - k_\varphi T_\theta) \, . \tag{17}$$

In the two-input case, the center of the circle was moved to compensate for the shrinking of the circle so as to keep the best point found in the search domain. In the three-input case, the circle has to be rotated on the surface of the sphere to keep the best point found. The idea is to compensate for the variation of φ from one iteration to another by an appropriate rotation. The coordinates of the best point found

at the end of iteration (i) is,

$$\mathbf{u}_{mi} = \mathbf{u} + \mathbf{M}_i \mathbf{u}_{rmi} \tag{18}$$

$$\mathbf{u}_m = \begin{bmatrix} u_{1m} \\ u_{2m} \\ u_{3m} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{u}_{rmi} = \begin{bmatrix} \Delta\cos(\varphi_i) \\ \Delta\sin(\varphi_i)\cos(\theta_{mi}) \\ \Delta\sin(\varphi_i)\sin(\theta_{mi}) \end{bmatrix} \tag{19}$$

$\mathbf{M}_i$ is the rotation matrix that is updated at each iteration. At the beginning of iteration ($i+1$), φ_i changes to φ_{i+1}, and to compensate for this change $\mathbf{M}_i$ needs to be changed to $\mathbf{M}_{i+1}$. In order not to change the best point found,

$$\mathbf{M}_{i+1}\mathbf{v}_{s(i+1)} = \mathbf{v}_{mi}, \tag{20}$$

where,

(21)

$$\mathbf{v}_{s(i+1)} = \begin{bmatrix} \Delta\cos(\varphi_{i+1}) \\ \Delta\sin(\varphi_{i+1})\cos(\theta_{mi}) \\ \Delta\sin(\varphi_{i+1})\sin(\theta_{mi}) \end{bmatrix}, \quad \mathbf{v}_{mi} = \mathbf{M}_i \begin{bmatrix} \Delta\cos(\varphi_i) \\ \Delta\sin(\varphi_i)\cos(\theta_{mi}) \\ \Delta\sin(\varphi_i)\sin(\theta_{mi}) \end{bmatrix}$$

The rotation matrix can be obtained by the "Rodrigues" rotation matrix formula [12], (22)

$$\mathbf{M} = \begin{bmatrix} \cos\gamma + \Gamma_1^2(1-\cos\gamma) & \Gamma_1\Gamma_2(1-\cos\gamma) - \Gamma_3\sin\gamma & \Gamma_2\sin\gamma + \Gamma_1\Gamma_3(1-\cos\gamma) \\ \Gamma_3\sin\gamma + \Gamma_1\Gamma_2(1-\cos\gamma) & \cos\gamma + \Gamma_2^2(1-\cos\gamma) & -\Gamma_1\sin\gamma + \Gamma_2\Gamma_3(1-\cos\gamma) \\ -\Gamma_2\sin\gamma + \Gamma_1\Gamma_3(1-\cos\gamma) & \Gamma_1\sin\gamma + \Gamma_2\Gamma_3(1-\cos\gamma) & \cos\gamma + \Gamma_3^2(1-\cos\gamma) \end{bmatrix}$$

where the axis of rotation $\mathbf{\Gamma}$ and the angle of rotation γ can be computed as,

$$\gamma_{mi} = \arccos\left(\frac{\mathbf{v}_{si}^T \mathbf{v}_{m(i-1)}}{\left|\mathbf{v}_{si}\right|\left|\mathbf{v}_{m(i-1)}\right|} \right) \tag{23}$$

$$\mathbf{\Gamma}_{mi} = \mathbf{v}_{si} \times \mathbf{v}_{m(i-1)}. \tag{24}$$

D. Layer 4: Recursive global optimization along the sphere

In order to repeatedly sweep the surface of the sphere, the initial condition for the adaptation law (16) is set to,

$$\varphi(jT_{\varphi+}) = 2\pi \tag{25}$$

where "j" denotes the number of sweeps on the sphere (j = 0,1,2,…) The period of having expansion-contraction of the circle twice on the sphere is indicated by T_φ. Note that since from the perspective of the angle, 2π and 0 are the same, this re-initialization does not affect the rotation matrices. The information on the best point found is present in the rotation matrix.

E. Layer 5: Reducing the radius of the sphere

It is assumed that the feasible global minimum lies within the initial sphere (centered at the initial inputs ($u_1(0)$,$u_2(0)$,$u_3(0)$) with the radius of $\Delta(0)$). This radius is monotonically reduced to zero i.e.,

$$\dot{\Delta} = -k_s\Delta \qquad \Delta(0) = \Delta_0 > 0. \tag{26}$$

$k_s > 0$ is a parameter that determines the rate at which Δ is reduced to ε. On the other hand, the adaptation laws of the center of the sphere are so chosen to keep $\mathbf{u}_{mi}$ at the same point. To do this, the derivative of the coordinates of the global optimum found at each iteration (equation 18) must be set at zero i.e.,

$$\dot{\mathbf{u}}_{mi} = \dot{\mathbf{u}} + \overline{\dot{\mathbf{M}}_i \mathbf{u}_{rmi}} = 0. \tag{27}$$

In the time scale of the adaptation of the centre, $\dot{\theta}_{mi}$, $\dot{\varphi}_{mi}$ and $\dot{\mathbf{M}}_{mi} = 0$. So, the rate of change in the coordinates of the center of the sphere is calculated as follows,

$$\dot{\mathbf{u}} = -\overline{\dot{\mathbf{M}}_i \mathbf{u}_{rmi}} = -\dot{\Delta}\,\mathbf{M}_i \begin{bmatrix} \cos(\varphi_i) \\ \sin(\varphi_i)\cos(\theta_{mi}) \\ \sin(\varphi_i)\sin(\theta_{mi}) \end{bmatrix}. \tag{28}$$

It is important to note that there are three different time scales in this problem, i.e. (i) the time-scale of azimuth evolution, determined by k_θ, (ii) the time-scale of the evolution of elevation angle, determined by k_φ and (iii) the time-scale for the shrinking of the sphere, determined by k_s. It can be easily computed that the period corresponding to each of these time scales are given by,

$$T_\theta = \int_\varepsilon^{2\pi} \frac{d\theta}{k_\theta\theta} = \frac{1}{k_\theta}\ln\left(\frac{2\pi}{\varepsilon}\right) \tag{29}$$

$$T_\varphi = \int_\varepsilon^{2\pi} \frac{d\varphi}{k_\varphi\varphi} = \frac{1}{k_\varphi}\ln\left(\frac{2\pi}{\varepsilon}\right) \tag{30}$$

$$T_s = \int_\varepsilon^{\Delta_0} \frac{d\Delta}{k_s\Delta} = \frac{1}{k_s}\ln\left(\frac{\Delta_0}{\varepsilon}\right) \tag{31}$$

For the algorithm to work well, pseudo-steady state assumptions have to be done at each stage, which in turn means that it is important that these three time-scales are well separated, i.e., $T_\theta << T_\varphi << T_s$. If an enough time scale separation between the dynamics of the angles θ and φ is considered, the set of expanding and contracting rotating circles would cover the entire surface of the sphere.

IV. Illustrative Examples

A. Test Problems

Three test problems were selected for evaluation of the developed algorithm in three variables case. These test problems have been widely used in global optimization literature. The analytical results are presented in details.

Example 1. Rosenbrock function [13],

$$\begin{aligned} &Min \quad f(u) = \sum_{i=1}^{n-1}\left[100\left(u_i^2 - u_{i+1}\right)^2 + \left(u_i - 1\right)^2\right] \\ &s.t. \quad -5 \le u_i \le 10 \\ &\qquad u \in R^n \quad , \quad f_{global} = f(1,1,\ldots,1) = 0 \end{aligned} \tag{32}$$

This function has several local minima.

Example 2. Sphere function,

$$Min \quad f(u) = \sum_{i=1}^{n} u_i^2$$
$$s.t. \quad -5.12 \le u_i \le 5.12 \qquad (33)$$
$$u \in R^n, \quad f_{global} = f(0,0,...,0) = 0$$

Example 3. Hartman function H3 [2],

$$Min \quad f(u) = -\sum_{k=1}^{4} C_k \exp\left(-\sum_{i=1}^{3} A_{ki}(u_i - P_{ki})^2\right)$$
$$s.t. \quad 0 \le u_i \le 1$$
$$C_k > 0, \quad u, A_{ki}, P_{ki} \in R^n$$
$$f_{global} = f(0.114614,\ 0.555649,\ 0.852547) = -3.86278 \qquad (34)$$

This function has four local minima. The coefficients c_k, a_{ki}, p_{ki} are as follows,

$$P_{ki} = \begin{bmatrix} 0.3689 & 0.1170 & 0.2673 \\ 0.4699 & 0.4387 & 0.7470 \\ 0.1091 & 0.8732 & 0.5547 \\ 0.03815 & 0.5743 & 0.8828 \end{bmatrix}, \quad A_{ki} = \begin{bmatrix} 3 & 10 & 30 \\ 0.1 & 10 & 35 \\ 3 & 10 & 30 \\ 0.1 & 10 & 35 \end{bmatrix}$$
$$C_k = [1 \quad 1.2 \quad 1 \quad 3.2]^T \qquad (35)$$

B. Application of global multi-unit optimization method

For example 1 (Rosenbrock function), the initial settings (u_{10}, u_{20}, u_{30}) = (3,-0.3,-3), and $\Delta_0 = 5$ were considered such that the global minimum among the other local ones lies in the sphere composed by the centre of (u_{10}, u_{20}, u_{30}) and the radius of Δ_0. The key condition to satisfy is the inequality $(u_1^{**} - u_{10})^2 + (u_2^{**} - u_{20})^2 + (u_3^{**} - u_{30})^2 \le \Delta_0^2$ which is in fact verified by choosing Δ_0 big enough. The other parameters used for all examples were $k_s = 0.001$, $k_\varphi = 0.01$, $k_\theta = 0.1$, $\varepsilon = 0.01$. Applying the global optimization algorithm using multi-units makes the system inputs to converge to the global minimum. The time evolution of the inputs and Δ are shown in figure 3.

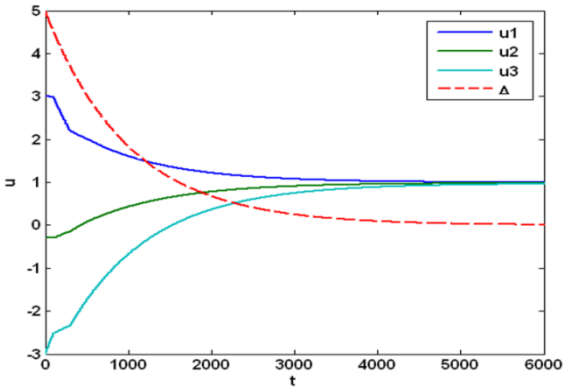

Fig. 3. Evolution of the inputs and Δ for example 1

For example 2, the initial inputs were (u_{10}, u_{20}, u_{30}) = (0.5, 0.5, 0.5) and $\Delta_0 = 1$.The contraction of the sphere is depicted in figure 4. Since the space of rotations is continuous, the centre of the sphere is expected to converge to the global optimum of the nonlinear map when Δ reaches to zero. It is clear from this figure that the continuously shrinking spheres never come out of the prior one.

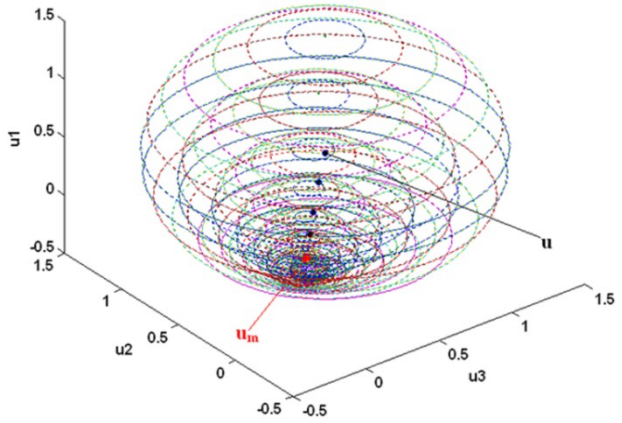

Fig. 4. Contraction of the sphere to the global optimum for example 2

In example 2, figure 5 illustrates a part of repetitive expansion of the rotating circles latched on the global optimum found on their circumference iteratively ($\mathbf{u}_{mi}$).

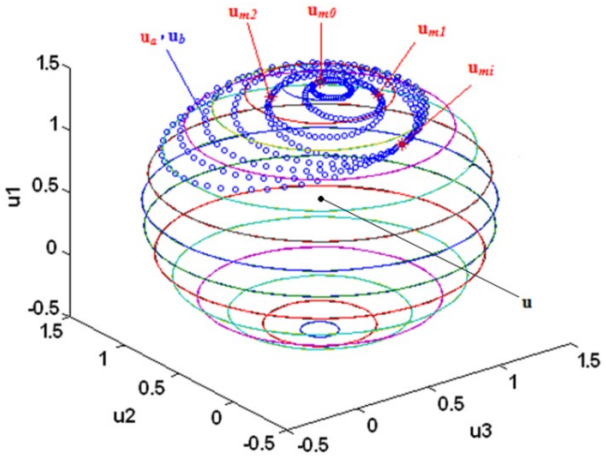

Fig. 5. Illustration of repetitive expansion of the rotating circles on the sphere toward the global optimum ($\mathbf{u}_{mi}$)

In example 3 (Hartman function), the initial conditions are chosen to be (u_{10}, u_{20}, u_{30}) = (0.9, 0.1, 0.3) and $\Delta_0 = 1$. Other tuning parameters remain the same. Applying the global optimization algorithm using multi-units makes the system input to converge to the global minimum at (u_1^{**},u_2^{**}, u_3^{**})= (0.2143, 0.5533 , 0.8519), depicted in figure 6 (red line).

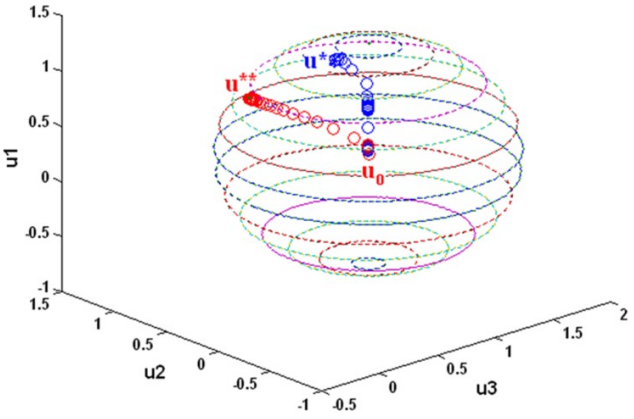

Fig. 6. Evolution of the centre of sphere with T_s/T_θ=66 (red line) and T_s/T_θ=10 (blue line) for example 3

If the condition $k_s \ll k_\varphi \ll k_\theta$ is not satisfied i.e. if the fraction k_θ / k_φ or k_φ / k_s are not large enough, the shrinking of the circle (or sphere) become too fast and the global optimum could be missed. In this example, the period of each iteration on the circumference of the rotating circle, the period of a complete extraction-contraction of this circle on the sphere and finally the total time of integration were T_θ=60.443 (s), T_φ=600.443 (s) and T_s=4000.6052 (s) respectively. It is clear that after T_φ/T_θ =10 iterations the rotating "expanding-contracting" circles complete a whole coverage of the surface of the shrinking sphere from zero to 2π. The number of total iterations (rotations on the sphere) was T_s/ T_θ =66. However, for instance if this ratio is reduced to 10 (with k_s = 0.01, k_φ = 0.05, k_θ = 0.1), the multi-unit optimization would converge to the local optimum at (0.8775, 0.4061, 1.0285) rather than the global one. This is depicted in figure 6 using the blue line.

V. Conclusions

A deterministic unconstrained global optimization method using multi-unit extremum seeking control for three-input systems is proposed. This technique converges to the global optimum of any nonlinear, continuous and static objective function with three variables, provided the global optimum is present in the initial spherical search space. Development of the proposed algorithm to multi-input systems and to constrained optimization problems are the next steps considered in this research framework. A more complex rotation scheme needs to be devised for higher dimensions.

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