2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

# Lyapunov-based economic model predictive control of nonlinear systems

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*Abstract*— **In this work, we focus on a class of general nonlinear systems and design a model predictive control (MPC) scheme which is capable of optimizing closed-loop performance with respect to general economic considerations taken into account in the construction of the cost function. Specifically, in the proposed design, the MPC optimizes a cost function which is related directly to certain economic considerations and is not necessarily dependent on a steady-state — unlike the conventional MPC designs. The proposed MPC is designed via Lyapunov-based techniques and has two different operation modes. The first operation mode corresponds to the periods in which the cost function should be optimized (e.g., normal production periods); and in this operation mode, the MPC maintains the closed-loop system state within a pre-defined stability region and optimizes the cost function to its maximum extent. The second operation mode corresponds to operation in which the system is driven by the MPC to an appropriate steady-state. In the MPC design, suitable constraints are incorporated to guarantee that the closed-loop system state is always bounded in the pre-defined stability region and is ultimately bounded in a small region containing the origin. The theoretical results are illustrated through a chemical process example.**

## I. INTRODUCTION

Maximizing profit has been and will always be the primary purpose of optimal process operations. Within process control, the economic optimization considerations of a plant are usually addressed via a real-time optimization (RTO) system (e.g., [1] and the references therein). In general, an RTO system includes two different layers: the upper layer that optimizes process operation set-points taking into account economic considerations using steady-state system models, and the lower layer (i.e., process control layer) whose primary objective is to design feedback control systems to force the process to track the set-points. Model predictive control (MPC) is widely adopted in industry in the process control layer because of its ability to deal with large multivariable constrained control problems and to account for optimization considerations [2], [3]. The key idea of a standard MPC is to choose control actions by repeatedly solving an on-line constrained optimization problem, which aims at minimizing a cost function that involves penalties on the state and control action over a finite prediction horizon. Typically, the cost function is in quadratic form including penalties on the deviations of the system state and control inputs from a desired steady-state. Because of the structure of the cost function, the control objective of a standard MPC is to drive the state of the closed-loop system to the desired steady-state. In MPC theory, the quadratic cost function is also widely used as a Lyapunov function to prove closed-loop stability (e.g., [4]). Even though in the standard MPC formulations, certain economic optimization considerations can be taken into account (e.g., optimal use of control action), general economic optimization considerations are usually not addressed. In order to account for general economic optimization considerations, the quadratic cost function used in standard MPC should be replaced by an economics-based cost function. Moreover, the standard MPC should be reformulated in an appropriate way to guarantee closed-loop stability.

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Within process control, there have been several calls for the integration of MPC and economic optimization of processes (e.g., [5]) as early as two decades ago; however, little attention has been given to the development of MPC accounting for general economic considerations in the cost function, except for a few recent important papers [6], [7]. In [6], general ideas of a combined steady-state optimization and linear MPC scheme as well as a case study were reported without rigorous stability analysis. In [7], MPC schemes using an economics-based cost function were proposed and the stability properties were established using a Lyapunov function. The MPC schemes in [7] adopt a terminal constraint which requires that the closed-loop system state settles to a steady-state at the end of each optimal input trajectory calculation (i.e., end of the prediction horizon). In addition, it is difficult to characterize, a priori, the set of initial conditions starting from where feasibility and closed-loop stability of the MPC schemes in [7] are guaranteed.

In this work, we focus on a class of general nonlinear systems and design a Lyapunov-based economic MPC (LEMPC) scheme which is able to take into account general economic considerations. The design of the LEMPC is based on uniting receding horizon control with explicit Lyapunov-based nonlinear controller design techniques and allows for an explicit characterization of the stability region of the closed-loop system (following [8]). In the proposed design, the LEMPC optimizes a cost function which is related directly to certain economic considerations and is not necessarily dependent on a steady-state — unlike the standard MPC designs. Specifically, the proposed LEMPC framework has two different operation modes. The first operation mode corresponds to operation periods in which the cost function should be optimized (e.g., normal production periods); and in this operation mode, the LEMPC maintains the closed-

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loop system state within a pre-defined stability region and optimizes the cost function. The second operation mode corresponds to operation periods in which the system settles down to an appropriate steady-state; and in this operation mode, the LEMPC drives the state of the system to the steady-state. In the LEMPC design, suitable constraints are incorporated to guarantee that the closed-loop system state is always bounded in a pre-defined stability region and is ultimately bounded in a small region containing the origin. The theoretical results are illustrated through a chemical process example. In [9], the results presented in this paper have been extended to account for bounded disturbances and asynchronous, delayed measurements.

## II. Preliminaries

### A. Notation and class of nonlinear systems

The operator $|\cdot|$ is used to denote Euclidean norm of a vector, and a continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The symbol $\Omega_r$ is used to denote the set $\Omega_r := \{x \in R^{n_x} : V(x) \leq r\}$ where $V$ is a scalar function, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^{n_x} : x \in A, x \notin B\}$. The symbol $diag(v)$ denotes a matrix whose diagonal elements are the elements of vector $v$ and all the other elements are zeros. We consider a class of nonlinear systems which can be described by the following state-space model:

$$\dot{x}(t) = f(x, u_1, \ldots, u_m) \tag{1}$$

where $x(t) \in R^{n_x}$ denotes the vector of state variables of the system and $u_i(t) \in R$, $i = 1, \ldots, m$, denote $m$ control (manipulated) inputs. The $m$ control inputs are restricted to be in $m$ nonempty convex sets $U_i \subseteq R$, $i = 1, \ldots, m$, which are defined as $U_i := \{u_i \in R : |u_i| \leq u_i^{\max}\}$ where $u_i^{\max}$, $i = 1, \ldots, m$, are the magnitudes of the input constraints. We assume that $f$ is a locally Lipschitz vector function and that the origin is an equilibrium point of the unforced nominal system (i.e., system of Eq. 1 with $u_i(t) = 0$, $i = 1, \ldots, m$) which implies that $f(0, 0, \ldots, 0) = 0$. We further assume that the system state measurements are available and sampled at synchronous time instants $t_k = t_0 + k\Delta$ where $t_0$ is the initial time and $\Delta$ is the sampling time.

### B. Lyapunov-based controller

We assume that there exists a Lyapunov-based controller $h(x) = [h_1(x) \cdots h_m(x)]^T$ with $u_i = h_i(x)$, $i = 1, \ldots, m$, which renders the origin of the nominal closed-loop system asymptotically stable while satisfying the input constraints for all the states $x$ inside a given stability region. We note that this assumption is essentially equivalent to the assumption that the system is stabilizable or that the pair $(A, B)$ in the case of linear systems is stabilizable. Using converse Lyapunov theorems [10], [11], this assumption implies that there exist class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ and a continuously differentiable Lyapunov function $V(x)$ for the nominal closed-loop system which is continuous and bounded in $R^{n_x}$, that satisfy the following inequalities:

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \\ \frac{\partial V(x)}{\partial x} f(x, h_1(x), \ldots, h_m(x)) \leq -\alpha_3(|x|) \\ \left|\frac{\partial V(x)}{\partial x}\right| \leq \alpha_4(|x|),\ h_i(x) \in U_i,\ i = 1, \ldots, m \end{aligned} \tag{2}$$

for all $x \in D \subseteq R^{n_x}$ where $D$ is an open neighborhood of the origin. We denote the region $\Omega_\rho \subseteq D$ as the stability region of the closed-loop system under the Lyapunov-based controller $h(x)$. Note that explicit stabilizing control laws that provide explicitly defined regions of attraction for the closed-loop system have been developed using Lyapunov techniques for specific classes of nonlinear systems, particularly input-affine nonlinear systems; the reader may refer to [12], [11], [13], [14] for results in this area including results on the design of bounded Lyapunov-based controllers by taking explicitly into account constraints for broad classes of nonlinear systems.

By continuity, the local Lipschitz property assumed for the vector field $f$ and taking into account that the manipulated inputs $u_i$, $i = 1, \ldots, m$ are bounded, there exists a positive constant $M$ such that

$$|f(x, u_1, \ldots, u_m)| \leq M \tag{3}$$

for all $x \in \Omega_\rho$ and $u_i \in U_i$, $i = 1, \ldots, m$. In addition, by the continuous differentiable property of the Lyapunov function $V(x)$ and the Lipschitz property assumed for the vector field $f$, there exists a positive constant $L_x$ such that

$$\left|\frac{\partial V}{\partial x} f(x, u_1, \ldots, u_m) - \frac{\partial V}{\partial x} f(x', u_1, \ldots, u_m)\right| \leq L_x |x - x'| \tag{4}$$

for all $x, x' \in \Omega_\rho$ and $u_i \in U_i$, $i = 1, \ldots, m$.

## III. Lyapunov-based economic MPC

In the proposed design, the MPC maximizes a cost function which takes into account specific economic considerations and it has two operation modes. In the first operation mode, the MPC maintains the system state within a predefined stability region and optimizes the cost function; in the second operation mode, the MPC tries to drive the state of the system to the desired steady-state.

We propose to design the MPC via Lyapunov-based MPC techniques [8] to take advantage of the stability properties of the Lyapunov-based controller $h(x)$. Specifically, from the initial time $t_0$ up to a specific time $t'$, the LEMPC operates at the first operation mode and the state of the system is enforced to be in the stability region $\Omega_\rho$ of the Lyapunov-based controller (i.e., $x \in \Omega_\rho$) while maximizing the cost function; after the time $t'$, the LEMPC operates at the second operation mode and calculates the inputs in a way such that the Lyapunov function of the system continuously decreases to steer the state of the system to a neighborhood of the desired steady-state. This proposed LEMPC provides more degrees of freedom to the state of the system to obtain its optimal trajectory in the invariant set $\Omega_\rho$ and eventually regulates the system state at the desired steady-state. For simplicity and without loss of generality in the rest of this paper, we assume that the specific time $t'$ is an integer multiple of the sampling time of the MPC, $\Delta$. In the sequel, we describe these steps in details.

### *A. Lyapunov-based economic MPC formulation*

In this subsection, we describe the design of the proposed LEMPC. At time $t_k$, $k = 0, 1, 2, \ldots$, the MPC is evaluated to obtain the future input trajectories. Specifically, the optimization problem of LEMPC at sampling time $t_k$ is as follows:

$$\max_{u_1,\ldots,u_m \in S(\Delta)} \int_{t_k}^{t_{k+N}} L(x(\tau), u_1(\tau), \ldots, u_m(\tau)) d\tau \quad (5a)$$

$$\text{s.t. } \dot{\tilde{x}}(t) = f(\tilde{x}(t), u_1(t), \ldots, u_m(t)) \quad (5b)$$

$$u_i(t) \in U_i, \ i = 1, \ldots, m \quad (5c)$$

$$\tilde{x}(t_k) = x(t_k) \quad (5d)$$

$$V(\tilde{x}(t)) \leq \rho, \ \forall t \in [t_k, t_{k+N}) \quad (5e)$$

$$\begin{aligned} &\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u_1(t_k), \ldots, u_m(t_k)) \\ &\quad \leq \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), h_1(x(t_k)), \ldots, h_m(x(t_k))), \\ &\quad \text{if } t_k \geq t' \end{aligned} \quad (5f)$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period $\Delta$, $N$ is the prediction horizon, $L(x(\tau), u_1(\tau), \ldots, u_m(\tau))$ is the economic measure which defines the cost function, the state $\tilde{x}$ is the predicted trajectory of the system with $u_1, \ldots, u_m$ computed by the LEMPC and $x(t_k)$ is the state measurement obtained at time $t_k$. The optimal solution to this optimization problem is denoted by $u_i^*(t|t_k)$, $i = 1, \ldots, m$, which is defined for $t \in [t_k, t_{k+N})$.

In the optimization problem of Eq. 5, the constraints of Eqs. 5b-5e are active in both operation modes; the constraint of Eq. 5f is only active in the second operation mode. Specifically, the constraint of Eq. 5b is the system model used in the LEMPC; the constraint of Eq. 5c defines the input constraints on all the inputs; the constraint of Eq. 5d defines the initial condition; the constraint of Eq. 5e ensures that the state of the system is maintained in the stability region $\Omega_\rho$. This constraint allows the LEMPC to obtain the optimal trajectory from an economic standpoint while guaranteeing that the state of the system is always within the stability region. The constraint of Eq. 5f enforces that, after time $t'$ (i.e., in the second operation mode), the Lyapunov function of the system decreases at least at the rate given by the Lyapunov-based controller $h(x)$ implemented in a sample-and-hold fashion.

The manipulated inputs of the proposed control design from time $t_k$ to $t_{k+1}$ ($k = 0, 1, 2, \ldots$) are defined as follows:

$$u_i(t) = u_i^*(t|t_k), \ i = 1, \ldots, m, \ \forall t \in [t_k, t_{k+1}). \quad (6)$$

### *B. Stability analysis*

As it will be proved in Theorem 1 below, the proposed LEMPC takes advantage of the constraints of Eqs. 5e and 5f to compute the optimal trajectories $u_1, \ldots, u_m$ such that the state of the system is always restricted in the stability region $\Omega_\rho$ and eventually the Lyapunov function value $V(x(t_k))$ is a decreasing sequence with a lower bound and achieves the closed-loop stability of the system.

*Theorem 1: Consider the system of Eq. 1 in closed-loop under the LEMPC design of Eq. 5 based on a controller $h(x)$ that satisfies the conditions of Eq. 2. Let $\epsilon_w > 0$, $\Delta > 0$ and $\rho > \rho_s > 0$ satisfy the following constraint:*

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L_x M \Delta \leq -\epsilon_w / \Delta. \quad (7)$$

*If $x(t_0) \in \Omega_\rho$ and if $\rho^* \leq \rho$ where $\rho^* = \max\{V(x(t+\Delta)) : V(x(t)) \leq \rho_s\}$, then the state $x(t)$ of the closed-loop system is always bounded in $\Omega_\rho$ and is ultimately bounded in $\Omega_{\rho^*}$.*

*Proof:* The proof consists of two parts. We first prove that the optimization problem of Eq. 5 is feasible for all states $x \in \Omega_\rho$. Subsequently, we prove that, under the LEMPC design of Eq. 5, the state of the system of Eq. 1 is always bounded in $\Omega_\rho$ and is ultimately bounded in a region that contains the origin.

*Part 1:* When $x(t)$ is kept in the region $\Omega_\rho$ (which will be proved in the Part 2), the feasibility of the LEMPC of Eq. 5 follows because all input trajectories $u_i(t)$ such that $u_i(t) = h_i(x(t_{k+j}))$, $\forall t \in [t_{k+j}, t_{k+j+1})$ with $j = 0, \ldots, N-1$ are feasible solutions to the optimization problem of the LEMPC since all such trajectories satisfy the input constraint of Eq. 5c and the Lyapunov-based constraints of Eqs. 5e and 5f. This is guaranteed by the closed-loop stability property of the Lyapunov-based controller in $\Omega_\rho$.

*Part 2:* We first note that according to Proposition 1 in [15], the predicted state trajectory in both operation modes lies in the stability region of the Lyapunov-based controller $h(\cdot)$ when it is applied in a sample-and-hold fashion. So, $V(\tilde{x}(\tau)) \leq \rho$ for both operation modes. We consider the two different operation modes in this proof and prove that in the first operation mode, the state of the closed-loop system is always bounded in the region $\Omega_\rho$, and that in the second operation mode, the Lyapunov function values $V(x(t_k))$ of the closed-loop system is a decreasing sequence with a lower bound.

First, we assume that the LEMPC is in the first operation mode (i.e., $t_k < t'$). Because of the constraint of Eq. 5e, it ensures that the state of the system of Eq. 1 is always bounded in $\Omega_\rho$.

Second, we assume that the LEMPC is in the second operation mode (i.e., $t_k \geq t'$). When $x(t_k) \in \Omega_\rho$, from the inequalities of Eq. 2 and the inequality of Eq. 5f, the following inequality can be written:

$$\begin{aligned} &\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u_1^*(t_k|t_k), \ldots, u_m^*(t_k|t_k)) \\ &\leq \frac{\partial V(x(t_k))}{\partial x} (f(x(t_k), h_1(x(t_k)), \ldots, h_m(x(t_k))) \\ &\leq -\alpha_3(|x(t_k)|). \end{aligned} \quad (8)$$

The time derivative of the Lyapunov function along the actual state trajectory $x(t)$ of the system of Eq. 1 in $t \in [t_k, t_{k+1})$

is given by:

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x}(f(x(t), u_1^*(t_k|t_k), \ldots, u_m^*(t_k|t_k)). \tag{9}$$

Adding and subtracting $\frac{\partial V(x(t_k))}{\partial x}(f(x(t_k), u_1^*(t_k|t_k), \ldots,$ $u_m^*(t_k|t_k))$ into the right-and-side of Eq. 9 and taking Eq. 8 into account, we obtain the following inequality:

$$\begin{aligned}\dot{V}(x(t)) \leq & -\alpha_3(|x(t_k)|) \\ & + \frac{\partial V(x(t))}{\partial x}(f(x(t), u_1^*(t_k|t_k), \ldots, u_m^*(t_k|t_k)) \\ & - \frac{\partial V(x(t_k))}{\partial x}(f(x(t_k), u_1^*(t_k|t_k), \ldots, u_m^*(t_k|t_k)).\end{aligned} \tag{10}$$

From Eq. 2, Eq. 4 and the above inequality, the following inequality is obtained for all $x(t_k) \in \Omega_\rho / \Omega_{\rho_s}$:

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_2^{-1}(\rho_s)) + L_x|x(t) - x(t_k)|.$$

Taking into account Eq. 3 and the continuity of $x(t)$, the following bound can be written for all $t \in [t_k, t_{k+1})$, $|x(t) - x(t_k)| \leq M\Delta$. Using this expression, we obtain the following bound on the time derivative of the Lyapunov function for $t \in [t_k, t_{k+1})$, for all initial states $x(t_k) \in \Omega_\rho / \Omega_{\rho_s}$:

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_2^{-1}(\rho_s)) + L_x M\Delta.$$

If the condition of Eq. 7 is satisfied, then there exists $\epsilon_w > 0$ such that the following inequality holds for $x(t_k) \in \Omega_\rho / \Omega_{\rho_s}$:

$$\dot{V}(x(t)) \leq -\epsilon_w/\Delta, \ \forall t = [t_k, t_{k+1}).$$

Integrating this bound on $t \in [t_k, t_{k+1})$, we obtain that:

$$\begin{aligned}& V(x(t_{k+1})) \leq V(x(t_k)) - \epsilon_w \\ & V(x(t)) \leq V(x(t_k)), \ \forall t \in [t_k, t_{k+1})\end{aligned} \tag{11}$$

for all $x(t_k) \in \Omega_\rho / \Omega_{\rho_s}$. Using Eq. 11 recursively, it is proved that, if $x(t') \in \Omega_\rho / \Omega_{\rho_s}$, the state converges to $\Omega_{\rho_s}$ in a finite number of sampling times without leaving the stability region. Once the state converges to $\Omega_{\rho_s} \subseteq \Omega_{\rho^*}$, it remains inside $\Omega_{\rho^*}$ for all times. This statement holds because of the definition of $\rho^*$. This proves that the closed-loop system under the LEMPC of Eq. 5 is always bounded in $\Omega_\rho$ and is ultimately bounded in $\Omega_{\rho^*}$. ■

*Remark 1:* Instead of requiring that the closed-loop system state settles to a steady-state at the end of the prediction horizon as in [7], in the proposed design, the LEMPC of Eq. 5 has two different operation modes. In the first operation mode, the LEMPC optimizes the economic cost function within the stability region $\Omega_\rho$. When the proposed LEMPC is in the second operation mode, it drives the closed-loop system state to the steady-state quickly. The proposed LEMPC also possesses a stability region which can be explicitly characterized.

*Remark 2:* Note that in order to achieve optimal performance, in general, the prediction horizon of the LEMPC of Eq. 5 should be long enough to cover the period in which the process operation should be optimized. However, long prediction horizon may not be practical for a real-time implementation of an MPC algorithm (especially when nonlinear systems with a large number of manipulated inputs are considered) because of the high computational burden. For certain applications, we may overcome this issue by driving part of the system states to certain economic optimal set-points and operating the rest of the system states in a time-varying manner to further maximize the economic cost function. This implies that we operate part of the system in the second operation mode and part of the system in the first operation mode simultaneously. Please see Section IV for an application of this approach to a chemical process example.

TABLE I
PARAMETER VALUES

| | | | |
|---|---|---|---|
| $T_0 = 300$ | $K$ | $F = 5$ | $\frac{m^3}{hr}$ |
| $V = 1.0$ | $m^3$ | $E = 5 \times 10^4$ | $\frac{kJ}{kmol}$ |
| $k_0 = 8.46 \times 10^6$ | $\frac{1}{hr}$ | $\Delta H = -1.15 \times 10^4$ | $\frac{kJ}{kmol}$ |
| $C_p = 0.231$ | $\frac{kJ}{kgK}$ | $R = 8.314$ | $\frac{kJ}{kmolK}$ |
| $\sigma = 1000$ | $\frac{kg}{m^3}$ | $C_{As} = 2$ | $\frac{kmol}{m^3}$ |
| $T_s = 400$ | $K$ | $C_{A0s} = 4$ | $\frac{kmol}{m^3}$ |
| $Q_s = 0$ | $\frac{KJ}{hr}$ | | |

## IV. APPLICATION TO A CHEMICAL PROCESS EXAMPLE

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where an irreversible second-order exothermic reaction $A \rightarrow B$ takes place [16]. $A$ is the reactant and $B$ is the desired product. The feed to the reactor consists of pure $A$ at flow rate $F$, temperature $T_0$ and molar concentration $C_{A0}$. Due to the non-isothermal nature of the reactor, a jacket is used to remove/provide heat to the reactor. The dynamic equations describing the behavior of the system, obtained through material and energy balances under standard modeling assumptions, are given below:

$$\frac{dC_A}{dt} = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT}} C_A^2 \tag{12a}$$

$$\frac{dT}{dt} = \frac{F}{V}(T_0 - T) + \frac{-\Delta H}{\sigma C_p} k_0 e^{\frac{-E}{RT}} C_A^2 + \frac{Q}{\sigma C_p V} \tag{12b}$$

where $C_A$ denotes the concentration of the reactant $A$, $T$ denotes the temperature of the reactor, $Q$ denotes the rate of heat input/removal, $V$ represents the volume of the reactor, $\Delta H, k_0$, and $E$ denote the enthalpy, pre-exponential constant and activation energy of the reaction, respectively and $C_p$ and $\sigma$ denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the process parameters used in the simulations are shown in Table I. The process model of Eq. 12 is numerically simulated using an explicit Euler integration method with integration step $h_c = 10^{-6}\ hr$.

The process model has one unstable steady-state and one stable steady-state in the operating range of interest. The control objective is to regulate the process in a region around the unstable steady-state ($C_{As}$, $T_s$) to maximize the production rate of $B$. There are two manipulated inputs. One of the inputs is the concentration of $A$ in the inlet to the reactor, $C_{A0}$, and the other manipulated input is the external heat input/removal, $Q$. The steady-state input values associated with the steady-state are denoted by $C_{A0s}$ and $Q_s$, respectively.

The process model of Eq. 12 belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f(x(t)) + g_1(x(t))u_1(t) + g_2(x(t))u_2(t)$$

where $x^T = [C_A - C_{As}\ T - T_s]$ is the state, $u_1 = C_{A0} - C_{A0s}$ and $u_2 = Q - Q_s$ are the inputs, $f = [f_1\ f_2]^T$ and $g_i = [g_{i1}\ g_{i2}]^T$ $(i = 1, 2)$ are vector functions. The inputs are subject to constraints as follows: $|u_1| \leq 3.5 kmol/m^3$ and $|u_2| \leq 5 \times 10^5\ KJ/hr$.

The economic measure that we consider in this example is as follows [16]:

$$L(x, u_1, u_2) = \frac{1}{t_f}\int_0^{t_f} k_0 e^{-\frac{E}{RT(\tau)}} C_A^2(\tau) d\tau \qquad (13)$$

where $t_f = 1\ hr$ is the final time of the simulation. This economic objective function is to maximize the average production rate over process operation for $t_f = 1\ hr$. We also consider that there is limitation on the amount of material which can be used over the period $t_f$. Specifically, the control input trajectory of $u_1$ should satisfy the following constraint:

$$\frac{1}{t_f}\int_0^{t_f} u_1(\tau) d\tau = 1\ kmol/m^3. \qquad (14)$$

This constraint means that the average amount of $u_1$ during one period is fixed. For the sake of simplicity and without loss of generality, we will refer to Eq. 14 as the integral constraint.

We will design an LEMPC following Eq. 5 to manipulate the two control inputs. We assume that the full system state $x$ is measured and sent to the LEMPC at synchronous time instants $t_k = k\Delta,\ k = 0, 1, \ldots$, with $\Delta = 0.01\ hr = 36\ sec$. The LMPC horizon is $N = 10$.

Since the LEMPC is evaluated at discrete-time instants during the closed-loop simulation, the integral constraint is enforced as follows:

$$\sum_{i=0}^{M-1} u_1(t_i) = \frac{t_f}{\Delta} \qquad (15)$$

where $M = 100$.

To ensure that the integral constraint is satisfied through the period $t_f$, at every sampling time in which the LEMPC obtains the optimal control input trajectory, it utilizes the previously computed inputs $u_1$ to constrain the first step value of the control input trajectory $u_1$ at the current sampling time. Based on the cost function formulation, for maximization purposes, it is expected that $C_A$ and $T$ should be increased which results in the fact that at the beginning of the closed-loop simulation $u_1$ should rise to its maximum value and after a while it will go down to its lowest value to satisfy the integral constraint. We assume that the decrease of the Lyapunov function starts from the beginning of the simulation (i.e., $t' = 0$) for part of the system state (i.e., temperature). To maximize the production rate, we pick a temperature set-point near the boundary of the stability region ($T = 430\ K$), considering the constraints on the control input $Q$. Due to the fact that the first ODE ($C_A$) of Eq 12 is input-to-state-stable (ISS) with respect to $T$, and the contractive constraint of Eq. 16g (see Eq. 16) ensures that the temperature converges to the set-point, the stability of the closed-loop system is guaranteed for all $x(0) \in \Omega_\rho$. To this end, we define $V_T(t_k) = (T(t_k) - 430)^2$. The LEMPC formulation for the chemical process example in question has the following form:

$$\max_{u_1, u_2 \in S(\Delta)} \frac{1}{N\Delta}\int_{t_k}^{t_{k+N}} [k_0 e^{-\frac{E}{RT(\tau)}} C_A^2(\tau)] d\tau \qquad (16a)$$

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t)) + \sum_{i=1}^{2} g_i(\tilde{x}(t))u_i(t) \qquad (16b)$$

$$u_1(t) \in g_\zeta,\ \forall\, t \in [t_k, t_{k+1}) \qquad (16c)$$

$$\tilde{x}(t_k) = x(t_k) \qquad (16d)$$

$$\tilde{x}(t) \in \Omega_\rho \qquad (16e)$$

$$u_i(t) \in U_i \qquad (16f)$$

$$\frac{dV_T(t_k)}{dT}(f_2(x(t_k)) + g_{22}(x(t_k))u_2(t_k)) \leq -\gamma V_T(t_k) \qquad (16g)$$

where $x(t_k)$ is the measurement of the process state at sampling time $t_k$, $\gamma = 9.53$ and the constraint of Eq. 16c implies that the first step value of $u_1$ should be chosen to satisfy the integral constraint where the explicit expression of $g_\zeta$ can be computed based on Eq.15 and the magnitude constraint on $u_1$. Also, the constraint of Eq. 16g enforces the Lyapunov function, based on the temperature, to decrease from the beginning of the simulation. The simulations were carried out using Java programming language in a Pentium 3.20 GHz computer. The optimization problems were solved using the open source interior point optimizer Ipopt.

The purpose of the following set of simulations is to demonstrate that: I) the proposed LEMPC design maximizes the economic measure $L(x, u_1, u_2)$; II) the proposed LEMPC design achieves asymptotic closed-loop stability under different initial conditions; and III) the proposed LEMPC design affords a higher cost function value compared to the steady-state operation.

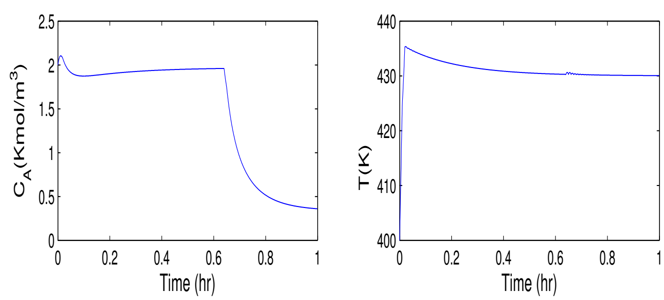


Fig. 1. State trajectories of the process under the proposed LEMPC design for initial condition $(C_A(0), T(0)) = (2\frac{kmol}{m^3}, 400K)$.

In the first set of simulations, we illustrate the application of the proposed LEMPC design starting from different initial conditions. The initial conditions are below and above the temperature set-point $430\ K$ to evaluate different controller

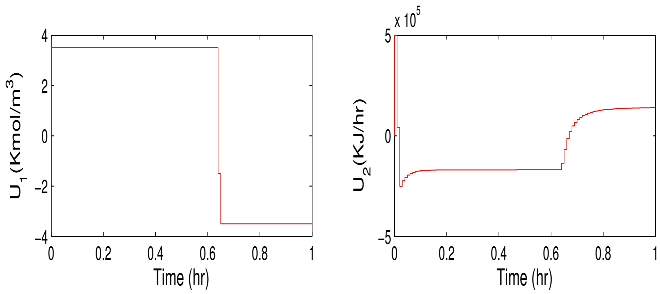


Fig. 2. Manipulated input trajectories under the proposed LEMPC design for initial condition $(C_A(0), T(0)) = (2\frac{kmol}{m^3}, 400K)$.

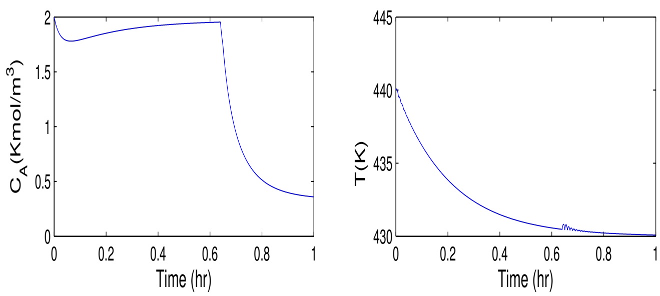


Fig. 3. State trajectories of the process under the proposed LEMPC design for initial condition $(C_A(0), T(0)) = (2\frac{kmol}{m^3}, 440K)$.

behavior. Figs. 1-4 depict the corresponding concentration and temperature (states) profiles, and the manipulated input profiles, respectively. As expected, in both scenarios, $u_1$ goes up to its allowable maximum value to increase reactant concentration as much as possible early on (given the second-order reaction rate) and after a while it drops to its minimum value to satisfy the integral constraint ($\frac{1}{t_f}\int_0^{t_f} u_1(\tau)d\tau = 1$). On the other hand, the temperature rises as fast as possible when the temperature initial condition is below $430\ K$ to maximize the reaction rate, and it decreases as slow as possible when the initial temperature is above $430\ K$ to maintain the maximum possible reaction rate while satisfying the stability constraint; in both cases, the temperature finally settles at $T = 430\ K$. For both initial condition sets, the proposed control design achieves practical stability.

Also, we have carried out a set of simulations to confirm that the application of the LEMPC design with the integral constraint on $u_1$ improves the economic objective function compared to the case that the system operates at a steady-state satisfying the integral constraint ($C_{As} = 1.55\frac{kmol}{m^3}, T_s = 430K, C_{A0s} = 5\frac{kmol}{m^3}, Q_s = -4.88 * 10^4\frac{KJ}{hr}$). This steady-state is computed by assuming that the reactant material amount is equally distributed in the interval $[0, t_f]$. To carry out this comparison, we have computed the total cost of each scenario based on the index of the following form:

$$J = \frac{1}{t_M}\sum_{i=0}^{M}[k_0 e^{-\frac{E}{RT(t_i)}} C_A^2(t_i)]$$

where $t_0 = 0\ hr$, $t_M = 1\ hr$ and $M = 100$. By comparing the cost function values, we find that in the proposed LEMPC design via time-varying operation (starting from $(C_A, T) = (2\frac{kmol}{m^3}, 400K)$), the cost function achieves a higher value

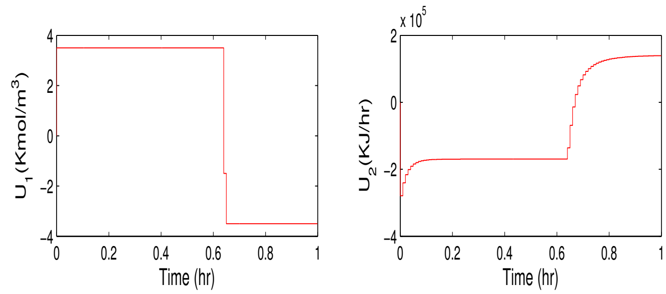


Fig. 4. Manipulated input trajectories under the proposed LEMPC design for initial condition $(C_A(0), T(0)) = (2\frac{kmol}{m^3}, 440K)$.

(1932.2) compared to the case of steady-state operation (1740.2).

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