2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

A Distributed Control Law with Guaranteed LQR Cost for Identical Dynamically Coupled Linear Systems

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Abstract— **This paper considers a collection of agents performing a shared task making use of relative information communicated over an information network. A two step design procedure for distributed state feedback control of such systems is proposed. The control law is guaranteed to provide a certain level of performance in terms of an LQR cost at a network level. An analysis of the proposed control law in the presence of delays in the relative information is carried out to obtain a bound on the maximum delay that can be accommodated.**

I. INTRODUCTION

In the last decade, research in formation control of multi-agent systems and consensus problems has gained a good deal of attention. These dynamical systems are often interconnected over an information network, and are supposed to operate in agreement, i.e., in a synchronised manner. According to [1], broadly speaking, state agreement, synchronisation and consensus problems can be viewed from an identical point of view. Central to these problems is the graph describing the topology of the interconnections. Algebraic graph theory has been widely employed in a variety of research work dealing with such systems. Many of the novel results have made use of systems and control theory, along with graph theory. Many researchers have contributed in this area. For details and examples, see [2]-[7] and the references therein.

Multiple identical dynamical systems interacting with each other over an information network are often studied from the perspective of graph theory. In [2], the connection between stability of the network dynamical system and the Laplacian eigenvalues of the underlying graph topology has been studied. Consensus problems for the case of multiple systems are studied in [4], where an analysis is made of switching variations and the effects of delays in the relative sensing (associated with the edges of the graph). For a detailed review on consensus and coordination of multiple systems, see [8] - [10] and the references therein. In the case of a network of dynamical systems, there are three possible control methods: centralised, decentralised, and distributed control. Comparisons of all these methods have been undertaken in [6] and the conclusions tend to favour distributed control.

Recently there has been progress in addressing performance issuues associated with stabilization and agreement problems in a network of dynamical systems. Specifically, research effort has been put into optimal control for a network of dynamical systems ensuring a certain level of overall LQR performance. In [11], a distributed LQR design for identical dynamically coupled systems is presented. The work illustrates how the stability of a network is related to the robustness of local controllers and the topology of the network. In [12], a LQR based decentralized receding horizon control (RHC) scheme is used to achieve coordination of a network of air vehicles. In [13], the influence of the interconnection graph on closed loop performance is discussed in an LQR framework. In [14], $\mathcal{H}_2$ performance has been analysed for a complex network system. In [15], an approximation method to solve large-scale LQR optimal control problems for spatially distributed systems is presented. In [16], the consensus problem has been investigated from an LQR perspective.

In this paper, the focus is on identical linear time invariant plants interconnected over an information network, exchanging relative measurements. Bidirectional communication is assumed. An LQR control design method, using the relative information available at node level to stabilize a network, is presented. The novelty of this paper, along with the control design, is the use of an augmented cost function, incorporating the Laplacian of the network topology, to guarantee performance. An analysis of the proposed distributed control laws in the presence of time delays is carried out to provide an estimate of the maximum bound on the time delays that can be accommodated.

II. PRELIMINARIES

The notation used in this paper is standard. The set of real numbers is denoted by $\mathbb{R}$. The set of real-valued vectors of length m is given by $\mathbb{R}^m$. The set of arbitrary real-valued $m \times n$ matrices is given by $\mathbb{R}^{m\times n}$. The expression $\mathcal{C}ol(.)$ denotes a column vector and $\mathcal{D}iag(.)$ denotes a diagonal matrix. For a symmetric positive definite (s.p.d) matrix $P = P^{\mathrm{T}} > 0$. I_n denotes an identity matrix of dimension $n \times n$.

In this section, concepts from graph theory are quoted, which are quite standard. For a detailed understanding of the notions of graph theory, readers are encouraged to refer to texts such as [17]. A graph $\mathcal{G}$ consists of a set of vertices,

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This work is supported by a UKIERI Grant.

denoted $\mathcal{V}$ and a set of edges $\mathcal{E} \subset \mathcal{V}^2$ where $e = (\alpha, \beta) \in \mathcal{V}^2$, i.e, an unordered pair, denotes an edge. A network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, represents a simple, finite graph consisting of N vertices and k edges. The graphs are assumed to be undirected. It is also assumed that the graph contains no multiple identical edges between two nodes and no loops. For the graph $\mathcal{G}$, the adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are adjacent nodes of the graph, and $a_{ij} = 0$ otherwise. This is a symmetric matrix. The symbol $\Delta(\mathcal{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathcal{G}$, $\mathcal{L}$, is defined as the difference $\Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. The smallest eigenvalue of $\mathcal{L}$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1} = \mathcal{C}ol(1, \ldots 1)$. The Laplacian $\mathcal{L}$ is always rank deficient and positive semi-definite. Moreover, the rank of $\mathcal{L}$ is $n-1$ if and only if $\mathcal{G}$ is connected.

III. Linear System Model

This paper considers a collection of N identical dynamical systems indexed as $1, 2, ..., N$. It is assumed that each system has access to its own state measurements together with relative external measurements with respect to the other dynamical systems which it can sense or interact with. Bi-directional communication is assumed. Such an interconnected system can be represented as a graph, with N vertices or nodes, each representing an $n-$dimensional dynamical system. An edge in this graph indicates the existence of relative sensing among the dynamical systems.

Suppose the dynamics of the i^{th} individual node are given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the states and the control inputs. The constant matrices $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$ and it is assumed that the pair (A, B) is controllable. The signals representing the exchange of relative information are assumed to have the form

$$z_i(t) = \sum_{j\in\mathcal{J}_i} (x_i(t) - x_j(t)) \tag{2}$$

for $i = 1 \ldots N$. They represent the external measurements relative to the other dynamical systems which the i^{th} vehicle can sense. The nonempty set $\mathcal{J}_i \subset \{1, 2, \ldots N\}/\{i\}$ denotes the dynamical systems, for which the i^{th} dynamical system has information. At a network level, the system given in (1) is represented by

$$\dot{X}(t) = (I_N \otimes A)X(t) + (I_N \otimes B)U(t) \tag{3}$$

where

$$X(t) = \mathcal{C}ol(x_1(t), ..., x_N(t)) \tag{4}$$

$$U(t) = \mathcal{C}ol(u_1(t), ..., u_N(t)) \tag{5}$$

At network level, (2) can be represented as

$$Z(t) = (\mathcal{L} \otimes I_n)X(t) \tag{6}$$

where $Z(t) = \mathcal{C}ol(z_1(t), \ldots, z_N(t))$. Here, an assumption is made that each dynamical system has information about at least one other system which ensures $rank(\mathcal{L}) = N - 1$.

IV. Problem Definition

The problem which will be considered is the design of state feedback control laws of the form

$$u_i(t) = -Kx_i(t) - \Phi K z_i(t) \tag{7}$$

for $i = 1, \ldots, N$, where $K \in \mathbb{R}^{m\times n}$ and $\Phi \in \mathbb{R}^{m\times m}$, to minimise, the cost function

$$J = \int_0^\infty (X(t)^{\mathrm{T}}((I_N \otimes Q_1) + (\mathcal{L} \otimes Q_2))X(t) + U(t)^{\mathrm{T}}(I_N \otimes R)U(t))dt \tag{8}$$

where $Q_1 = Q_1^{\mathrm{T}} \in \mathbb{R}^{n\times n} \geq 0$, $Q_2 = Q_2^{\mathrm{T}} \in \mathbb{R}^{n\times n} \geq 0$, $R = R^{\mathrm{T}} \in \mathbb{R}^{m\times m} > 0$. This problem will be tackled in a sub-optimal way via a two step optimization process. Details are given below:

- **Step 1**: First the control gain matrix K in (7) will be synthesized to optimize the LQR performance at a decoupled node level by solving a classical LQR problem: find a control law $u_i = -Kx_i$ which stabilizes

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{9}$$

  subject to minimizing

$$J_i = \int_0^\infty (x_i(t)^{\mathrm{T}} Q_1 x_i(t) + u_i(t)^{\mathrm{T}} R u_i(t))dt \tag{10}$$

  where Q_1 and R are associated with the LQR cost functional given in (8). The system considered here is identical and decoupled. At this point, no interactions between the agents are considered and hence the cost considered can be viewed as the case when $Q_2 = 0$ in the cost function in (8).
- **Step 2**: Once K has been synthesized, choose a design matrix $\Phi \in \mathbb{R}^{m\times m}$ so that the collection of systems

$$\dot{x}_i(t) = Ax_i(t) + Bu_i \tag{11}$$

  for $i = 1...N$ are stabilized by the distributed control laws

$$u_i(t) = -Kx_i(t) - \Phi K z_i(t) \tag{12}$$

  for $i = 1, ..., N$, where $z_i(t)$ is given in (2). Using (5) and (6), the control law at the network level is represented by

$$U(t) = -(I_N \otimes K)X(t) - (\mathcal{L} \otimes \Phi K)X(t) \tag{13}$$

  and the objective is to minimize

$$J = \int_0^\infty (X(t)^T((I_N \otimes Q_1) + (\mathcal{L} \otimes Q_2))X(t) + U(t)^T(I_N \otimes R)U(t))dt \tag{14}$$

Remark: In (8), the term $\mathcal{L} \otimes Q_2$ penalizes relative information and tries to ensure simultaneous convergence. Since this work considers undirected or bidirectional graphs for the inter-agent communication, the Laplacian $\mathcal{L}$ is symmetric positive semi-definite with row sum equalling zero. This row sum property is preserved by $\mathcal{L} \otimes Q_2$. At convergence if $x_i = x_s$ for $i = 1, \ldots, N$ then $X^T(\mathcal{L} \otimes Q_2)X = 0$ and the individual decoupled node level LQR performance as in (10) is recovered.

Remark : A similar LQR cost function is also considered in [11]. However, the approach to solve the associated LQR problem is different. In [11], the suboptimal distributed LQR problem is posed as a single LQR problem exploiting the properties of the graph associated with the communication topology. The order of the LQR problem to be solved depends on the maximum vertex degree (plus one), but not on the total number of nodes in the network. Whereas in this paper, the sub-optimal distributed LQR problem is solved systematically in two steps: the first step involves solving a node level LQR problem; and the second step involves obtaining a scaling matrix, addressing the distributed control part, which creates an optimization problem which depends on the number of nodes of the graph.

In this paper, the first part of control design directly solves a node level LQR problem to obtain a gain matrix K. A further LQR optimization problem is subsequently solved to obtain the scaling matrix Φ. The controller at node level is then given by (7). The following section tackles each of these steps in detail.

V. Control Design Procedure

A. Details of Step 1

First assume without loss of generality that the input distribution matrix from (1) has the form

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}$$

where $B_2 \in \mathbb{R}^{m \times m}$. This is so-called regular form [18]. Solving the standard LQR problem posed in Step 1, the optimal feedback gain

$$K = -R^{-1}B^T P \tag{15}$$

where the symmetric positive definite Lyapunov matrix P satisfies the Algebraic Riccati equation

$$PA + A^T P + Q_1 - PBR^{-1}B^{\mathrm{T}}P = 0 \tag{16}$$

B. Details Step 2

In order to solve the optimization problem in Step 2, first introduce a change of coordinates $x \mapsto \hat{T}x = \hat{x}$ where

$$\hat{T} := \begin{pmatrix} I_{(n-m)\times(n-m)} & 0 \\ P_{22}^{-1}P_{12}^T & I_m \end{pmatrix} \tag{17}$$

and P_{12} and P_{22} are obtained by decomposing the Lyapunov matrix P from (16) as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \tag{18}$$

where $P_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_{22} \in \mathbb{R}^{m\times m}$. Because P is symmetric positive definite, the sub-matrix P_{22} is symmetric positive definite and therefore nonsingular. Clearly the transformation $\hat{T}$ is nonsingular. Following the change of coordinates, $(A, B, K, P) \mapsto (\hat{A}, \hat{B}, \hat{K}, \hat{P})$ where the matrices $\hat{A} = \hat{T}A\hat{T}^{-1}$, $\hat{B} = \hat{T}B$, $\hat{K} = K\hat{T}^{-1}$ and finally $\hat{P} = (\hat{T}^{-1})^T P\hat{T}^{-1}$. It can easily be verified that the Lyapunov matrix in the new coordinates has the block diagonal form

$$\hat{P} = \begin{pmatrix} P_{11} - P_{12}P_{22}^{-1}P_{12}^T & 0 \\ 0 & P_{22} \end{pmatrix} \tag{19}$$

It is also easy to see that $\hat{B} = B$, i.e the input distribution matrix is invariant under the transformation $\hat{T}$. In these coordinates it can be verified that the feedback gain matrix $\hat{K}$ has the structure

$$\hat{K} = \begin{pmatrix} 0 & \hat{K}_2 \end{pmatrix} \tag{20}$$

where $\hat{K}_2 \in \mathbb{R}^{m\times m}$ and $\det(\hat{K}_2) \neq 0$. (This follows easily from the structures of $\hat{B}$ and $\hat{P}$ because $\hat{K} = -R^{-1}\hat{B}^T\hat{P}$) In the new coordinate system the node level LQR cost functions

$$J_i = \int_0^\infty (\hat{x}_i(t)^{\mathrm{T}}\hat{Q}_1\hat{x}_i(t) + u_i(t)^{\mathrm{T}}Ru_i(t))dt$$

where $\hat{Q}_1 := (\hat{T}^{-1})^T Q_1 \hat{T}^{-1}$.
Next design a distributed state feedback control law for the networked system

$$\dot{\hat{x}}_i(t) = \hat{A}\hat{x}_i + \hat{B}u_i \tag{21}$$

where

$$u_i(t) = -\hat{K}\hat{x}_i(t) - \Phi\hat{K}\hat{z}_i(t) \tag{22}$$

for $i = 1, \ldots, N$, where

$$\hat{z}_i(t) = \sum_{j\in\mathcal{J}_i}(\hat{x}_i(t) - \hat{x}_j(t))$$

represents the relative sensing measurement with respect to the new coordinate systems, to minimise an upper bound on the quadratic performance in (8). The states of the network in the transformed coordinates are $\hat{X}(t) = (I_N \otimes \hat{T})X(t)$ and the cost function becomes

$$J = \int_0^\infty (\hat{X}(t)^T((I_N \otimes \hat{Q}_1) + (\mathcal{L} \otimes \hat{Q}_2))\hat{X}(t) + U(t)^T(I_N \otimes R)U(t))dt \tag{23}$$

where $\hat{Q}_2 := (\hat{T}^{-1})^T Q_2 \hat{T}^{-1}$. In this step $\hat{K} \in \mathbb{R}^{m\times n}$ will be considered as fixed, and $\Phi \in \mathbb{R}^{m\times m}$ represents the available design freedom.

In the $\hat{X}$ coordinates, from (21), the system at network level is given by

$$\dot{\hat{X}}(t) = (I_N \otimes \hat{A})\hat{X}(t) + (I_N \otimes \hat{B})U(t) \tag{24}$$

The control law in (22) at a network level is given by

$$U(t) = -(I_N \otimes \hat{K})\hat{X}(t) - (\mathcal{L} \otimes \Phi\hat{K})\hat{X}(t) \tag{25}$$

Substituting (25) in (24), the closed loop system is

$$\dot{\hat{X}}(t) = [(I_N \otimes (\hat{A} - \hat{B}\hat{K})) - (\mathcal{L} \otimes \hat{B}\Phi\hat{K})]\hat{X}(t) \quad (26)$$

Since $\mathcal{L}$ is symmetric positive semi-definite, by spectral decomposition $\mathcal{L} = V\Lambda V^T$ where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathcal{L}$ and $\Lambda = diag(\lambda_1, \ldots \lambda_N)$ is the matrix of the eigenvalues of $\mathcal{L}$. Consider an orthogonal state transformation

$$\hat{X} \mapsto (V \otimes I_n)\hat{X} = \tilde{X} \quad (27)$$

In the new coordinates

$$\dot{\tilde{X}}(t) = [I_N \otimes (\hat{A} - \hat{B}\hat{K})]\tilde{X}(t) - (\Lambda \otimes \hat{B}\Phi\hat{K})\tilde{X}(t) \quad (28)$$

and the weighting matrices $(I_N \otimes \hat{Q}_1)$ and $(\mathcal{L} \otimes \hat{Q}_2)$ from (23) map to

$$(V^T \otimes I_N)^T(I_N \otimes \hat{Q}_1)(V^T \otimes I_N) = (I_N \otimes \hat{Q}_1) \quad (29)$$
$$(V^T \otimes I_N)^T(\mathcal{L} \otimes \hat{Q}_2)(V^T \otimes I_N) = (\Lambda \otimes \hat{Q}_2) \quad (30)$$

Using (29) and (30), the quadratic performance at the network level in (23) can be represented as

$$J = \int_0^\infty (\tilde{X}(t)^T((I_N \otimes \hat{Q}_1) + (\Lambda \otimes \hat{Q}_2))\tilde{X}(t) + U(t)^T(I_N \otimes R)U(t))dt \quad (31)$$

Since Λ is a diagonal matrix, (28) can be represented at a node level in the transformed coordinates as

$$\dot{\tilde{x}}_i = (\hat{A} - \hat{B}\hat{K} - \lambda_i\hat{B}\Phi\hat{K})\tilde{x}_i \quad (32)$$

for $i = 1, \ldots N$ and (31) as

$$J = \sum_{i=1}^{N}\int_0^\infty (\tilde{x}_i(t)^{\mathrm{T}}(\hat{Q}_1 + \lambda_i\hat{Q}_2)\tilde{x}_i(t) + u_i(t)^{\mathrm{T}}Ru_i(t))dt \quad (33)$$

For each of the decoupled node level systems in (32), consider a quadratic Lyapunov function $\tilde{V}_i = \tilde{x}_i^T P_i \tilde{x}_i$ where each P_i has the structure

$$P_i = \begin{pmatrix} P_{i11} & 0 \\ 0 & P_2 \end{pmatrix} \quad (34)$$

where $P_{i11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $P_2 \in \mathbb{R}^{m\times m}$ are s.p.d. In the $\hat{x}$ coordinates the Lyapunov matrix associated with the optimum LQR cost for the $(\hat{A}, \hat{B})$ pair has a block-diagonal form as shown in (19). In (34) the block-diagonal structure is retained for the subsequent optimization. Note that P_2 is defined to be the same for all $i = 1, \ldots, N$.

The objective of Step 2 is to solve the following optimization problem

Minimize $\sum_{i=1}^{N} trace(P_i)$ subject to the constraints

$$P_i(\hat{A} - \hat{B}\hat{K} - \lambda_i\hat{B}\Phi\hat{K}) + (\hat{A} - \hat{B}\hat{K} - \lambda_i\hat{B}\Phi\hat{K})^T P_i + (\hat{Q}_1 + \lambda_i\hat{Q}_2) + (\hat{K} + \lambda_i\Phi\hat{K})^T R(\hat{K} + \lambda_i\Phi\hat{K}) < 0 \quad (35)$$

and

$$P_i > 0 \quad (36)$$

for $i = 1, \ldots, N$. As argued in [19]

$$J \leq \sum_{i=1}^{N} trace(P_i)$$

and so the minimization problem outlined above constitutes minimizing an upper bound on the LQR cost at a network level.

Remarks:

- The block diagonal structure of P_i in (34) is enforced to help with the formulation of a convex representation of the problem. This does induce conservatism in the solution that will be obtained.
- For $\Phi = 0$ choosing $P_i = \hat{P}$ from (19), obtained from Step 1, is a feasible solution to the optimization problem. Thus the minimization problem is guaranteed to have a meaningful solution.

To develop a convex representation define

$$W_i := P_i^{-1} = \begin{pmatrix} W_{i1} & 0 \\ 0 & W_2 \end{pmatrix} \quad (37)$$

where $W_{i1} = P_{i1}^{-1}$ and $W_2 = P_2^{-1}$. Then pre and post multiplying (35) by W_i yields

$$(\hat{A} - \hat{B}\hat{K} - \lambda_i\hat{B}\Phi\hat{K})W_i + W_i(\hat{A} - \hat{B}\hat{K} - \lambda_i\hat{B}\Phi\hat{K})^T + W_i(\hat{Q} + \lambda_i\hat{Q}_1)W_i + W_i(\hat{K} + \lambda_i\Phi\hat{K})^T R(\hat{K} + \lambda_i\Phi\hat{K})W_i < 0 \quad (38)$$

for $i = 1 \ldots N$. Because of the structures inherent in the terms $\hat{K}$, and W_i, it follows

$$\Phi\hat{K}W_i = \begin{pmatrix} 0 & \Phi\hat{K}_2W_2 \end{pmatrix}$$

Define

$$\hat{Y} = \begin{pmatrix} 0 & \hat{Y}_2 \end{pmatrix}$$

where $\hat{Y}_2 = \Phi\hat{K}_2W_2$ then by the use of the Schur decomposition [19], inequality (38) can be written as

$$\begin{pmatrix} \Psi & W_i(\hat{Q} + \lambda_i\hat{Q}_1)^{1/2} & (\lambda_i\hat{Y} + \hat{K}W_i)^T \\ * & -I_n & 0 \\ * & * & -R^{-1} \end{pmatrix} < 0 \quad (39)$$

where $\Psi = \hat{A}_cW_i - \lambda_i\hat{B}\hat{Y} + W_i\hat{A}_c^T - \lambda_i\hat{Y}^T\hat{B}^T$ and the matrix $\hat{A}_c := \hat{A} - \hat{B}\hat{K}$. Define

$$Z_i := \begin{pmatrix} Z_{i1} & 0 \\ 0 & Z_2 \end{pmatrix} \quad (40)$$

conformably with the definition of P_i in (34) and W_i in (37). Then the objective minimize $\sum_{i=1}^{N} trace(Z_i)$ subject to

$$\begin{pmatrix} -Z_i & I_{n\times n} \\ I_{n\times n} & -W_i \end{pmatrix} < 0 \quad (41)$$

and (39) is equivalent to minimising

$$\sum_{i=1}^{N} trace(W_i^{-1}) = \sum_{i=1}^{N} trace(P_i)$$

subject to (39). Since (41) is equivalent to $W_i^{-1} < Z_i$.

The formal optimization problem associated with Step 2 can now be stated:

Minimize $\sum_{i=1}^{N} trace(Z_i)$ subject to (39)-(41) and $W_i > 0$

This is a convex optimization problem in terms of the LMI variables W_i, $\hat{Y}$ and Z_i. The design matrix Φ can then be obtained from $\hat{Y}$ as $\Phi = \hat{Y}_2 W_2^{-1} \hat{K}_2^{-1}$.

VI. Analysing Effects of Time-delays

Earlier it was assumed that the relative measurements were given by

$$\hat{z}_i(t) = \sum_{j \in \mathcal{J}_i} (\hat{x}_i(t) - \hat{x}_j(t)) \qquad (42)$$

This is very idealised and in reality delays will be present in the information network. Recently there has been interest in investigating the effects of delay in consensus problems. In [4], the effect of relative sensing delays in consensus problems has been studied with identical delays in the relative sensing signals. In [21], delay dependent stability criteria are obtained for a consensus problem with communication delays in a network. In [22], the effect of delays in coordinated motion of a network of multiple agents and oscillator synchronization is studied. In this section, the effect of a constant time delay across all relative measurements will be analyzed. In this scenario, the control law becomes

$$\hat{u}_i(t) \;=\; -\hat{K}\hat{x}_i(t) - \Phi\hat{K}\hat{z}_i(t-\tau) \qquad (43)$$

where

$$\hat{z}_i(t-\tau) \;=\; \sum_{j \in \mathcal{J}_i} (\hat{x}_i(t-\tau) - \hat{x}_j(t-\tau)) \qquad (44)$$

and τ is a fixed time delay. This is more realistic than the earlier formulation with no delay.

With the use of Kronecker algebra, the control law for the overall system can be written as

$$U(t) = -(I_N \otimes \hat{K})\hat{X}(t) - (\mathcal{L} \otimes \hat{B}\Phi\hat{K})\hat{X}(t-\tau) \qquad (45)$$

and the closed loop system is then given by

$$\dot{X}(t) = (I_N \otimes (\hat{A} - \hat{B}\hat{K}))\hat{X}(t) - (\mathcal{L} \otimes \hat{B}\Phi\hat{K})\hat{X}(t-\tau) \qquad (46)$$

After employing the transformation in (27), the delayed node level closed loop system is given by

$$\dot{\tilde{x}}_i(t) = \hat{A}_c\tilde{x}_i(t) - \lambda_i\hat{B}\Phi\hat{K}\tilde{x}_i(t-\tau) \qquad (47)$$

for $i = 1 \ldots N$, where $\hat{A}_c := \hat{A} - \hat{B}\hat{K}$. For convenience define

$$\hat{A}_i \;=\; -\lambda_i\hat{B}\Phi\hat{K} \qquad (48)$$

for $i = 1 \ldots N$. Then the system (47) can be represented as

$$\dot{\tilde{x}}_i(t) = \hat{A}_c\tilde{x}_i(t) + \hat{A}_i\tilde{x}_i(t-\tau) \qquad (49)$$

for $i = 1 \ldots N$, which is now in a standard time-delay form [20]. The systems in (47) will now be analysed to ascertain the maximum delay τ such that the system is stable for all $i = 1, \ldots, N$. A result from [20] which employs a Lyapunov-Krasovskii functional approach will be used to make this assessment.

Proposition 5.17 [20] *The system (49) is asymptotically stable for all fixed delays $\tau \in [\;0 \quad \hat{\tau}\;]$, where $\hat{\tau}$ represents the maximum acceptable time delay, if there exist real matrices $H_i = H_i^T$, M_i and*

$$\hat{P}_i > 0 \qquad (50)$$

$$S_i > 0 \qquad (51)$$

such that

$$\begin{pmatrix} \hat{W}_i & \hat{P}_i\hat{A}_i - M_i & -\hat{A}_c^T M_i^T \\ * & -S_i & -\hat{A}_i^T M_i^T \\ * & * & \frac{1}{\hat{\tau}}H_i \end{pmatrix} < 0 \qquad (52)$$

where

$$\hat{W}_i = \hat{P}_i\hat{A}_c + \hat{A}_c^T\hat{P}_i + S_i + \hat{\tau}H_i + M_i + M_i^T$$

for $i = 1 \ldots N$. Q.E.D

For given matrices $\hat{A}_c$ and $\hat{A}_i$ for $i = 1, \ldots N$, and a given value of $\hat{\tau}$, the matrix inequalities (50)-(52) are affine in S_i, H_i, M_i and $\hat{P}_i$. Considering these as the LMI variables, the problem formulation becomes

Maximize $\hat{\tau}$ subject to LMIs (50)-(52).

To find the largest possible value of the feasible delay $\hat{\tau}$ a bisection algorithm was employed over a suitable delay interval.

Note, the value for the maximum permissible delay $\hat{\tau}$ which is obtained is conservative. In [20], Proposition 5.22 reduces the conservatism, and work is in progress to utilize this approach, as well as considering time varying delay.

VII. Numerical Example

Consider a network of 5 vehicles moving in a plane with the dynamics of each described by a double integrator in each of the directions x and y. The system is then represented by

$$\dot{\zeta}_i = A\zeta_i + Bu_i \qquad (53)$$

where ζ_i represents the states of the i^{th} vehicle, and consists of the x and y plane positions and velocities. The plant matrix and input distribution matrix are given by

$$A = \begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix} \quad B = \begin{pmatrix} 0_2 \\ I_2 \end{pmatrix} \qquad (54)$$

The matrices in the LQR cost function in (8) have been chosen as $Q_1 = 10I_4$, $Q_2 = 25I_4$ and $R = I_2$. The proposed two step design procedure has been followed. As a first step a control law of the form (7) is obtained by solving the standard LQR problem, using the Matlab command `'lqr'` giving the control gain matrix

$$K = \begin{pmatrix} 3.1623I_2 & 4.0404I_2 \end{pmatrix} \qquad (55)$$

Since the node level LQR problem has been solved, the matrices P_{12} and P_{22} from (18) can be isolated and the transformation $\hat{T}$ in (17) can be employed. In the transformed coordinates, $(\hat{A}, \hat{B}, \hat{K}, \hat{P})$, a block diagonal Lyapunov matrix is created which is exploited in the second step of the design procedure.

A cyclic nearest neighbour interconnection is assumed between the five agents. A distributed control law as in (8) is designed using the LMIs provided in (39) - (41), and $W_i > 0$ for $i = 1, \ldots, N$. Following the design procedure in Step 2, the scaling matrix Φ for the control gains is obtained as

$$\Phi = 0.6736 I_2 \tag{56}$$

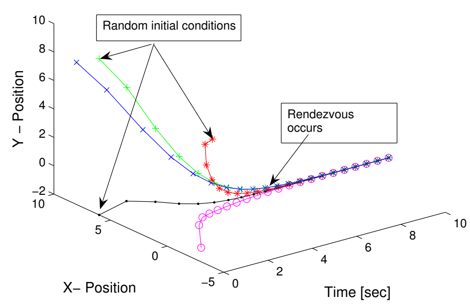

Fig. 1. Rendezvous with no delay

An estimate of the maximum permissible time delay for the systems in (49) is found by using a bisection algorithm while solving the LMIs (50)-(52). The optimization gives a value of $\hat{\tau}_{max} = 0.145$ seconds. However as suggested in [20] the maximum permissible delay bounds obtained is conservative.

Two sets of results are provided in this paper. Fig 1 shows that in the absence of delays a rendezvous of the five agents occurs at $4s$. Figure 2 shows the plots with a delay of $\tau = 0.22s$ for the same initial conditions. Clearly a rendezvous does not occur among the vehicles for this delay.

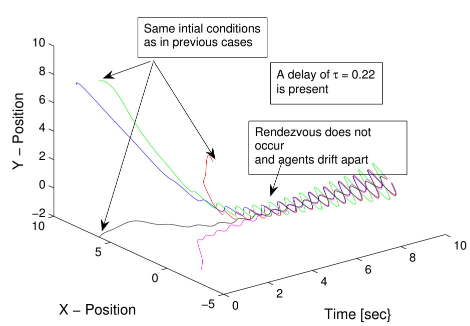

Fig. 2. Rendezvous greater than permissible delay

VIII. Conclusions

In this paper, a two step design procedure for distributed control of a collection of agents is proposed. The identical dynamical agents perform a shared task making use of relative information communicated over an information network. The proposed distributed control law is guaranteed to provide a certain level of performance in terms of an LQR cost at a network level. An analysis of the proposed control law in the presence of delays in the relative information is carried out and a bound on the maximum possible delay is obtained. In this paper the delays in the communication links are assumed to be fixed. However it is more realistic to consider delays which are time-varying.

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