2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Bayesian Fault Isolation in Multivariate Statistical Process Monitoring

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***Abstract*— Consider a set of multivariable input/output process data. Given a new observation we ask the following questions: is the new observation normal or abnormal? is one of the inputs or outputs abnormal (faulty) and which? Assuming a linear regression model of the process, the problem is solved through Bayesian hypothesis testing. The proposed formulation differs from existing multivariable statistical process control methods by taking uncertainty (variance) of the empirical regression model into account. The derived solution matches the established methods for anomaly detection and fault isolation in case there is no model uncertainty. Taking the model uncertainty into account, the proposed solution yields significant accuracy improvement compared to existing approaches. This is because ill-conditioned multivariable regression models can have large uncertainty even for large training data sets. The paper also demonstrates that isolating faults to a small ambiguity group works significantly better than the exact isolation.**

I. Introduction

This section starts from describing the problem informally. A rigorous formulation is presented in the next sections.

A. Problem

Consider data generated by a multivariable statistical process (plant)

$$D_N = \{x(t), y(t)\}_{t=1}^{N}, \tag{1}$$

where $x(t) \in \Re^n$ are the independent variables (plant inputs), $y(t) \in \Re^m$ are dependent variables (plant outputs, quality variables), and t is the observation number. There are N observations at all in D_N. We assume that D_N is generated by a random process (nominal process). The process is stationary and observations $\{x(t), y(t)\}$, $\{x(s), y(s)\}$ are independent for $s \neq t$. Given a new observation $\{x, y\}$ and the training set D_N we ask the following questions: is the new observation normal or abnormal? is one of the inputs x_j or outputs y_k abnormal (faulty) and which?

This paper considers an easy generalization of the above questions. We assume that the input and output faults have signatures defined by the two sets

$$F = \{f_1, ..., f_n\}, \tag{2}$$
$$G = \{g_1, ..., g_m\} \tag{3}$$

A fault in input channel j is described by $f_j = e_j$, where $e_j \in \Re^n$ is a unit vector. A fault in output channel k is described by $g_k = e_k$, where $e_k \in \Re^m$ is a unit vector.

This work was supported by NASA Grant NNX07AEIIA

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The following statistical hypotheses are evaluated to answer the question.

$\mathbf{H}_N$: Null hypothesis. Observation $\{x, y\}$ is generated by the nominal process.

$\mathbf{H}_A$: Anomaly. Observation $\{x, y\}$ is abnormal, deviates from the nominal process. This can be modeled as observing $y + h$ instead of y, where h is a nuisance vector.

$\mathbf{H}_{I,j}$: Input channel fault. An anomaly where $\{x, y\}$ is generated by the nominal process and $x + zf_j$ is observed instead of x, where z is a scalar nuisance parameter.

$\mathbf{H}_{O,k}$: Output channel fault. An anomaly where y is generated along with x by the nominal process and $y + zg_k$ is observed instead of y; z is a scalar nuisance parameter.

We consider the following anomaly detection and fault isolation problems:

1. Determine which hypothesis holds: $\mathbf{H}_N$ nominal or $\mathbf{H}_A$ anomaly.
2. If $\mathbf{H}_A$ holds, find a list of all likely $\mathbf{H}_{I,j}$ and $\mathbf{H}_{O,k}$ hypotheses (the ambiguity group).
3. Find the most likely hypothesis out of all the above listed hypotheses.

B. Baseline approach

A standard approach to solving the stated monitoring problem is to fit a linear regression model to the training data (1) and then use this model for statistical testing of the hypotheses. We recite it in this section and later use as a baseline for presentation of the proposed approach.

The regression model assumes that process inputs $x(t)$ are deterministic and the outputs $y(t)$ are i.i.d. (independent identically distributed) random variables such that

$$y = Bx + v, \quad v \sim N(0, S), \tag{4}$$

where matrices B and S define the regression model. Linear regression models are commonly used for process monitoring. Using regressors x that are nonlinear functions of the original independent variables allows modeling nonlinear maps in the form (4).

Training data (1) can be formed into two data matrices

$$X = [x(1), ..., x(N)] \in \Re^{n,N}, \tag{5}$$
$$Y = [y(1), ..., y(N)] \in \Re^{m,N} \tag{6}$$

Maximum Likelihood Estimates (MLE) of the parameters of regression model (4) are well known to be

$$B_N = YX^T \left(XX^T\right)^{-1}, \tag{7}$$
$$S_N = (Y - B_N X)(Y - B_N X)^T / N \tag{8}$$

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These estimates are biased (because of division by N rather than by $N-1$ in (8)), but asymptotically accurate. We assume that matrices XX^T and VV^T, where $V = Y - B_N X$, are invertible. Non-invertible matrices XX^T and VV^T, in (7), (8) can be handled by introducing shrinkage / regularization / priors and inverting $XX^T + aI$ and $VV^T + bI$ instead, where I denotes identity matrix of appropriate size. Scalar regularization parameters a and b are determined by numerical accuracy requirements.

Assuming that the estimates (7), (8) provide true parameters of the model leads to the following indexes for process monitoring

$$M_1(x, y) = \|y - B_N x\|^2_{S_N^{-1}}, \tag{9}$$

$$M_2(x, y; h) = \min_z \|y - B_N x - zh\|^2_{S_N^{-1}}, \tag{10}$$

$$z = (y - B_N x)^T S_N^{-1} h / \|h\|^2_{S_N^{-1}}, \tag{11}$$

where the h is the assumed fault signature; notation $\|h\|^2_{S_N^{-1}} = h^T S_N^{-1} h$ is used. Anomaly is monitored through M_1. Fault isolation is performed using M_2. The optimal fault amplitude estimate z in (10) is given by (11). For the input fault in channel j, the assumed signature is $h = B_N f_j$; for the output fault k, the signature is $h = g_k$. Usually, the fault hypothesis providing the smallest index M_2 is assumed to hold.

C. Prior work

The formulated problems are in the well established area of the statistical process monitoring known as Multivariable Statistical Process Control (MSPC). The monitoring decisions are based on past nominal data only. This is a desirable feature, since in practice, faulty operation data are hard to come by.

The prior MSPC work on fault isolation is based on geometrical argument of subspace projections, distances, and angles for the monitored data. A historical perspective on such work can be found in [1], [5], [7], [11]. The recent paper [1] argues that the "reconstruction-based contribution" index, essentially the same as (10), (11), provides the best results.

The main novelty of this work is that the indexes (9), (10), are extended to explicitly include model uncertainty. The model uncertainty (variance) is proportional to the condition number of XX^T and inverse in the training data set size N. MSPC data often has ill-conditioned design matrix X. For such data, the model uncertainty can be large even with all the statistical averaging provided by a large number N of samples in the training set. In [9], confidence intervals, which are related to the model uncertainty, are used to test significance of linear combinations of regression coefficients. Though motivated by a different problem, and derived differently, mathematically the results of [9] are related to intermediate steps of the proposed solution.

The special case of anomaly detection for $n = 0$ (no independent parameters x) is well known. In that case, assuming perfect model (7), (8) yields χ^2_m statistics for (9). The Bayesian formulation including the uncertainty (variance) of the model yields Hotelling T^2 statistics for (9) that is broadly used in MSPC.

The described setup is related to the Partial Least Squares (PLS) formulations in MSPC that separate the process data into independent variables x and dependent variables (quality parameters) y. The PLS handles the model uncertainty implicitly and suboptimally by truncating smaller singular values of the input covariance. We keep the small singular values of the covariance, and deal with uncertainty explicitly as a part of the optimal Bayesian formulation of anomaly detection and fault isolation with independent variables x. In a sense, the motivation of our approach is related to the original underlying idea of the Hotelling T^2 monitoring approach.

Estimation problems with random model uncertainty have been considered earlier in a different context. Optimal Bayesian estimation taking into account gaussian uncertainty of linear models has been recently considered in [8], [13]. These nonconvex problems require sufficiently complex algorithms. The problems in this paper are much easier to solve because just a single scalar fault intensity is estimated for each fault hypothesis.

The proposed approach differs from other MSPC work in explicitly pursuing a set of suspect faults (ambiguity group) rather that trying to isolate a single root cause fault. This appears to be new for MSPC. Though in related earlier work, e.g., see [6], several candidate faults are evaluated to find the root cause fault, this is done in implicit and interactive way. To the author's best knowledge, the earlier MSPC work does not explicitly pursue the ambiguity group as the end result of the fault isolation. Fault ambiguity groups are commonly considered in the literature on circuit testing and discrete fault diagnostics from discrete data.

D. Contributions

The contributions of this paper are as follows.

- We propose a Bayesian approach to statistical process monitoring taking into account the uncertainty (variation) of the model estimated from the past process data. The proposed method matches known MSPC approaches if there is no model uncertainty.
- The proposed Bayesian formulation of fault isolation problem can be solved efficiently.
- The paper demonstrates that computing a small set of likely faults (ambiguity group) provides a critical improvement of fault isolation performance. The isolation accuracy can be made uniformly high for all single channel faults despite the model uncertainty.
- The paper provides simulation results demonstrating that the proposed approach can improve fault isolation accuracy where the known approach fails.

The paper outline is as follows. Section II introduces the optimal Bayesian log-posteriors needed for the hypothesis testing. Section III describes the proposed hypothesis testing approach to fault diagnostics. Finally, Section IV presents a numerical example illustrating the performance of the proposed method.

II. Bayesian formulation

The hypotheses introduced in Section I can be tested by computing their likelihoods. The hypotheses assume that the new data $\{x, y\}$ are modified by applying the fault signatures (2), (3). Subsection II-A derives the expression for posterior likelihood for $\{x, y\}$. In Subsection II-B the posterior expression is modified for each hypothesis.

A. Log-likelihood index

The probability density for $y \sim N(Bx, S)$ in (4) yields the single-point likelihood index

$$\begin{aligned} L(y, x, B, S) &= -\log p(y, x, B, S) - c \\ &= \frac{1}{2}(y - Bx)S^{-1}(y - Bx) + \frac{1}{2}\det(S), \end{aligned} \tag{12}$$

where $c = \frac{m}{2}\log(2\pi)$. MLE estimation of B and S from i.i.d. data D_N (1) is given by

$$L_N = \min_{B,S} \sum_{t=1}^{N} L(y(t), x(t), B, S) \tag{13}$$

Solution of (13) is textbook material. Optimal B, S are given by (7), (8); the optimal index (13) is

$$L_N = \frac{N}{2}\det\left(\frac{1}{N} Y \left[I - X^T(XX^T)^{-1}X\right] Y^T\right) \tag{14}$$

It is known that

$$L_N(B_N, S_N) = \frac{N}{2}\det(S_N) \tag{15}$$

Substituting S_N from (8) and B_N from (7) into (15) yields (14).

A possible approach would be to compute the uncertainty (distributions) of the matrices B_N, S_N estimated from the training data D_N. The uncertainty then can be used in the Bayesian formulation for the new observations $\{x, y\}$. It is much more convenient, however, to consider the joint Bayesian formulation for all observations. By considering the new observation $\{x, y\}$ to be a part of the training set (1), we get the extended log-posterior index

$$L_+ = \min_{B,S}\left(L(y, x, B, S) + \sum_{t=1}^{N} L(y(t), x(t), B, S)\right) \tag{16}$$

The solution to (16) can be expressed through the solution (15) to old problem (13) by expressing matrix S_+ in the extended problem (16) through the old matrix S_N (8). The extended matrix S_+ is described by (8), (7) with X (5) and Y (6), augmented by the new data $\{x, y\}$.

Using rank-1 update expressions for YY^T, YX^T, XX^T, and $(XX^T)^{-1}$ and then applying Matrix Determinant Lemma yields

$$\begin{aligned} \det(S_+) = [1 + N^{-1}(u - y)^T S_N^{-1}(u - y)/(1 + r_x)] \\ \cdot \det(S_N) \cdot [N/(N + 1)]^m, \end{aligned} \tag{17}$$

where $u = B_N x$. Substituting (17) into the expression $L_+ = ((N + 1)/2) \cdot \det(S_+)$ leads to

$$L_+(x, y) = C_N N + C_N M_+(x, y), \tag{18}$$

$$M_+(x, y) = \|y - B_N x\|^2_{S_N^{-1}} \cdot \left[1 + x^T Q_N^{-1} x\right]^{-1}, \tag{19}$$

where B_N (7), S_N (8), L_N (14), along with C_N and Q_N, are computed through training data (5), (6),

$$\begin{aligned} C_N &= N^{-1} L_N [N/(N - 1)]^{m-1}, \\ Q_N &= XX^T, \quad B_N = YX^T Q_N^{-1}. \end{aligned} \tag{20}$$

Note, that the second multiplier in (19) attenuates the first least-squares term. This is a manifestation of regression dilution effect, also known as "attenuation". The effect is well discussed in connection to errors-in-variables statistical models. In the errors-in-variable framework predictor variable x could be considered normally distributed and Q_N its empirical covariance.

B. Hypothesis likelihoods

The posterior likelihoods for the hypotheses introduced in Section I for fault signatures (2), (3) can be expressed using (18)–(20).

We assume that the hypotheses have prior likelihoods

$$\begin{aligned} P(\mathbf{H_N}) = p_N, \quad P(\mathbf{H_A}) = p_A, \\ P(\mathbf{H_{I,j}}) = p_I, \quad P(\mathbf{H_{O,k}}) = p_O, \end{aligned} \tag{21}$$

where $\mathbf{H_A}$ and $\mathbf{H_N}$ are complementary, hence, $p_N + p_A = 1$. The posterior log-likelihood of each hypothesis can be computed as

$$L(\mathbf{H}) = -\log\ P(\mathbf{H}|x, y) = L_+(x, y) - \log p_H, \tag{22}$$

where $\mathbf{H}$ is one of hypotheses in (21) and $p_H = P(\mathbf{H})$ is its prior probability. We used the Bayes rule: $P(\mathbf{H}|x, y) = P(x, y|\mathbf{H})P(\mathbf{H})$const, and ignored the const.

$\mathbf{H}_N$*: Null hypothesis:* The observation $\{x, y\}$ is generated by the nominal process. Hence the derivation of Subsection II-A holds without any modifications. From (18), (19), (21), and (22) we get

$$L(\mathbf{H_N}) = C_N N + C_N M_+(x, y) - \log p_N \tag{23}$$

$\mathbf{H}_A$*: Anomaly:* The anomaly hypothesis assumes that $y+h$ is observed instead of y, where h is a nuisance vector. From (18), (21), and (22) we get

$$\begin{aligned} L(\mathbf{H_A}) = \min_h \left(C_N N + C_N M_+(x, y + h) - \log p_A\right), \\ = C_N N - \log p_A, \end{aligned} \tag{24}$$

where in accordance with (19) $\min_h M_+(x, y + h) = 0$ is achieved for $h = -y + B_N x$.

$\mathbf{H}_{I,j}$*: Input channel fault:* The hypothesis of input fault j assumes that for the process data $\{x, y\}$ the observed input is $x + f_j z$ instead of x. The fault intensity (nuisance parameter) z is unknown, and $f = f_j \in F$ is a known input fault signature.

From (18), (21), and (22) we get

$$L(\mathbf{H_{I,j}}) = C_N N + C_N M_+(x + f_j z_*, y) - \log p_I, \tag{25}$$

where z_* is the most likely value of the nuisance parameter z. In accordance with (19),

$$z_* = \arg\min_z M_+[z], \tag{26}$$

$$M_+[z] = \frac{(y - B_N x - B_N f_j z)^T S_N^{-1} (y - B_N x - B_N f_j z)}{1 + (x - f_j z)^T Q_N^{-1} (x - f_j z)}$$

The function minimized in (26) has the form $M_+[z] = (az^2 + bz + c)/(dz^2 + ez + f)$. The derivative $dM_+[z]/dz$ has numerator that is quadratic in z. The minimum can be found by checking the two roots of this numerator.

$\mathbf{H}_{O,k}$*: Output channel fault:* The hypothesis of output fault k assumes that for the process data $\{x, y\}$, the observed output is $y + g_k z$ instead of y. The fault intensity (nuisance parameter) z is unknown, and $g = f_j \in G$ is a known output fault signature.

From (18), (21), and (22), we get

$$L(\mathbf{H_{O,k}}) = C_N N + C_N M_+(x, y + g_k z_*) - \log p_A, \quad (27)$$

where z_* is the most likely value of the nuisance parameter z. Differentiating to find minimum of (19) in z yields

$$\begin{aligned} z_* &= \arg\min_z M_+(x, y + g_k z_*) \\ &= (y - B_N x)^T S_N^{-1} g_k / \|g_k\|^2_{S_N^{-1}}. \end{aligned} \quad (28)$$

III. ANOMALY DETECTION AND FAULT ISOLATION

The problems formulated in Section I are solved by comparing the log-posteriors (23), (24), (25), and (27).

A. Monitoring for anomalies and the faults

We will consider three monitors for the process. Each monitor is a computational function that inputs a data point $\{x, y\}$ and produces discrete output: the accepted hypothesis or a list of several hypotheses.

Monitor 1 (ANO): Anomaly detection. To determine whether $\mathbf{H_N}$ nominal or $\mathbf{H_A}$ anomaly holds, $L(\mathbf{H_N})$ is compared to $L(\mathbf{H_A})$. If $L(\mathbf{H_N}) > L(\mathbf{H_A})$, the anomaly hypothesis $\mathbf{H_A}$ is accepted, otherwise the null hypothesis if $\mathbf{H_N}$ is accepted.

Monitor 2 (ISO): Fault isolation. Assume that anomaly hypothesis $\mathbf{H_A}$ holds and $L(\mathbf{H_N}) > L(\mathbf{H_A})$. We then find a set J of all input channel fault hypothesis $\mathbf{H_{I,j}}$ such that $L(\mathbf{H_A}) > L(\mathbf{H_{I,j}})$ and a set K of output channel fault hypothesis $\mathbf{H_{O,k}}$ such that $L(\mathbf{H_A}) > L(\mathbf{H_{O,k}})$. The combined set J, K (the ambiguity group) is the output of the fault isolation monitor. An empty ambiguity group means the fault is unknown.

Monitor 3 (MAP): Most likely hypothesis. Find the most likely hypothesis as the hypothesis that has the smallest likelihood. Note that if $L(\mathbf{H_N})$ is the smallest ($\mathbf{H_N}$ is the most likely), then Monitor 1 (ANO) would point at the null hypothesis. If $L(\mathbf{H_A})$ is the smallest, then Monitor 1 (ANO) would accept the anomaly hypothesis and Monitor 2 (ISO) would output an empty ambiguity group (the "unknown fault" case). If the smallest MAP likelihood is achieved for one of the input or output faults, then Monitor 1 (ANO) would accept the anomaly hypothesis and the ambiguity set of Monitor 2 (ISO) contains this fault.

Monitors 1–3 are based on computation of the likelihoods discussed in Section II-B and use the following parameters in the computations.

- Scalars and matrices B_N (7), S_N (8), L_N (14), and C_N, Q_N (20) computed from training data (5), (6).
- Known input fault signatures f_j in (2) and output fault signatures g_k in (3).
- Prior probabilities p_A, p_I, p_O in (21) that are the tuning parameters of the algorithm and are discussed below.

The claim of this paper is that Monitor 2 (ISO) for the Bayesian (robust) formulation provides superior performance. Other monitors including the baseline (non-robust) versions are discussed mainly for comparison purposes.

B. Anomaly detection

Consider the anomaly detection using Monitor 1 (ANO) first. Substituting (19), (23), and (24) into the decision condition $L(\mathbf{H_N}) > L(\mathbf{H_A})$ yields

$$\|y - B_N x\|^2_{S_N^{-1}} > [1 + x^T Q_N^{-1} x] \cdot C_N^{-1} \log(p_N/p_A) \quad (29)$$

Note that the first term in r.h.s of (29) gives the model uncertainty contribution and increases the anomaly detection threshold. The increase can be substantial if x is in the small singular value subspace of XX^T, i.e., is outside of the subspace covered by the training data D_N.

Assume that there is no model uncertainty, $Q_N^{-1} \ll 1$ and $x^T Q_N^{-1} x \ll 1$, or there are no input parameters x, i.e., $n = 0$. Recall that $p_N = 1 - p_A$. The anomaly detection condition (29) of Monitor 1 then becomes

$$\|y - B_N x\|^2_{S_N^{-1}} > R, \quad R = \frac{1}{C_N} \log \frac{1 - p_A}{p_A} \quad (30)$$

This has the same form as the standard MSPC anomaly detection condition based on Hotelling T^2 statistics.

C. Baseline fault isolation

Monitors 2 (ISO) and 3 (MAP) both require fault hypotheses to have likelihoods higher (negative log-likelihoods lower) than the likelihood of anomaly (unknown fault). This condition can be better understood in the baseline case. Assume there is no model uncertainty, $Q_N^{-1} \ll 1$. Then, the likelihood indexes (25) and (27) for faults can be obtained by setting $x^T Q_N^{-1} x = 0$ in (19). The likelihood indexes for the faults can be written in the common form that is an affine transformation of (10), (11)

$$L(\mathbf{H_F}) = C_N N + C_N \|r - r_h\|^2_{S_N^{-1}} - \log p_F, \quad (31)$$

where $r = y - B_N x$ is the prediction residual and $r_h = r^T S_N^{-1} h / \|h\|^2_{S_N^{-1}}$ is the projection of r on the fault signature line h in the residual space. This projection is orthogonal in the metrics based on the matrix S_N^{-1}. For an input fault, we have $\mathbf{H_F} = \mathbf{H_{I,j}}$, $p_F = p_I$, and $h = B_N f_k$. For an output fault, hypothesis $\mathbf{H_F} = \mathbf{H_{O,k}}$, $p_F = p_O$, and $h = g_k$.

Monitor 2 (ISO) includes fault in the ambiguity group if $L(\mathbf{H_A}) > L(\mathbf{H_F})$. From (24) and (31) we get the fault inclusion condition as

$$\|r - r_h\|^2_{S_N^{-1}} < W, \quad W = C_N^{-1} \log(p_F/p_A) \quad (32)$$

This condition is related to one discussed in [1].

D. Tuning rules

The tuning rules described below are based on the above analysis in the absence of uncertainty. We propose using the same tuning rules in case the uncertainty is present. This worked well in our numerical experiments.

The tuning is based on two parameters R in (30) and W in (32) that define prior probabilities (21) as

$$p_A = 1/(e^{C_N R} + 1),$$
$$p_O = p_I = p_A e^{C_N W}, \quad p_N = 1 - p_A \tag{33}$$

Selecting R is based on the acceptable false positive (false alarm) rate α in Monitor 1 (ANO) anomaly detection. Select R based on (30) such the complementary cumulative distribution function for T^2 gives the P-value α. This is the probability that the m-dof standard normal vector $S_N^{-1/2}(y - B_N x)$ is outside of the ball of radius R. (For a large data set size N, T^2 becomes the same as χ^2_m). In the example of Section IV we have $m = 4$ and $N = 200$. P-value of 0.02 for $T^2(m, N-1)$ is achieved for $R = 9.5$.

Selecting $W = R$ ensures that the false negative rate (the true fault is not included into the ambiguity group) is less than α. To prove this statement in the absence of model uncertainty consider a fault with signature h such that $r = y - Bx = z_* h + v$, where $v\ N(0, S)$. If there is no uncertainty, $B_N = B$, $S_N = S$. and the fault isolation monitor (32).

The P-value for the true fault not detected by Monitor 3 (MAP) satisfies the chain of inequalities

$$
\begin{aligned}
&P(\|r - zh\|^2_{S_N^{-1}} > W \cap \|r\|^2_{S_N^{-1}} > R) \\
<\ &P(\|r - r_h\|^2_{S_N^{-1}} > W \cap \|r\|^2_{S_N^{-1}} > R) \\
<\ &P(\|r - r_h\|^2_{S_N^{-1}} > W)\ <\ P(\|r\|^2_{S_N^{-1}} > W)
\end{aligned}
$$

The first inequality holds because r_h is an orthogonal projection of r on h and $\|r - zh\|^2_{S_N^{-1}} \geq \|r - r_h\|^2_{S_N^{-1}}$. The second inequality is a basic probability property. The third is a projection property. Since $\|r\|^2_{S_N^{-1}} \sim T^2$ and $W = R$ this proves that the P-value is less than α.

We followed the proposed simple tuning rules in the numerical example presented below. More accurate tuning of R and W could take into account magnitudes of the faults encountered with respect to the covariance S of the noise v (signal to noise ratio). This can be done by introducing prior distributions for the nuisance parameter into the formulations of the fault hypotheses. A possible practical disadvantage is that much more engineering effort would be required compared to the proposed simple tuning rules.

E. Computational Performance

The monitors described in Subsection III-A have excellent computational performance and can be scaled to very large numbers of points in N and large dimensions of x and y. The monitors are suitable for on-line real time implementation in process monitoring applications.

In on-line monitoring, the arriving data are processed by a monitor and then added to the historical data set D_N (1). As more data arrives, the data set size N could grow very large. The on-line monitoring computations could be split into two steps described below.

Step 1 is to compute the matrices B_N (7), S_N (8), Q_N (20), and Q_N^{-1} from data (5), (6). The scalars L_N (14) and C_N (20) are computed through these matrices. The computations can be carried iteratively as N grows by propagating rank-1 updates of the matrices. There is no need to store the entire data set D_N (1). The matrix Q_N^{-1} can be updated using the Sherman-Morrison formula without the need to do actual matrix inversion ($x = x_{n+1}$)

$$Q_{N+1}^{-1} = Q_N^{-1} - Q_N^{-1} x x^T Q_N^{-1} \cdot \left[1 + x^T Q_N^{-1} x\right]^{-1}.$$

Computational complexity of these updates is quadratic in the data vector sizes m, n; the most computationally expensive part is matrix-vector multiplication. The complexity and required memory do not depend on N. Step 1 updates are the same as in the well known MSPC and Recursive Least Squares algorithms; they are not specific to the proposed approach.

Step 2 is calculation and update of the hypothesis likelihoods using the results of Step 1 and the new data point $\{x, y\}$. Step 2 calculations are specific for the proposed method. The monitors of Subsection III-A compare the the log-posteriors (23), (24), (25), (27). For each of the three monitors, computation of the log-posteriors has quadratic complexity in sizes n of x and m of y; the most expensive part is again matrix-vector multiplication.

IV. Aircraft Performance Monitoring Example

As an example, we consider a simple linear model for monitoring of aircraft flight performance. A higher-fidelity simulation and problem motivation are discussed in [2]. A linearized model for Boeing 747 level flight at 40000 ft at Mach 0.8 (774 ft/s) and constant thrust from [4] is

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_z \\ \dot{\alpha} \\ \dot{r} \end{bmatrix} = A_a \begin{bmatrix} v_x - w_x \\ v_z - w_z \\ \alpha \\ r \end{bmatrix} + B_a u, \tag{34}$$

where v_x, v_z are longitudinal and normal aircraft speeds, w_x, w_z are respective wind gust speeds, α is angle of attack (AOA), r is the pitch rate; matrices A_a, B_a are

A_a				B_a
-0.003	0.039	0	-0.322	0.01
-0.065	-0.319	7.74	0	-0.18
0.020	-0.101	-0.429	0	-1.16
0	0	1	0	0

Model (34) was integrated by Euler method with time step of 0.1. The wind gust disturbances w_x, w_z were simulated as independent white gaussian noises with unit covariances. It was assumed that autopilot computes $u = 0.1\alpha + 0.1r$ to stabilize the flight regime.

Input vector $x \in \Re^4$ includes v_x (v-longitudinal), v_z (v-normal), α (AOA), and u (elevator). Output vector $y \in \Re^4$ includes $\dot{v}_x$ (a-longitudinal), $\dot{v}_z$ (a-normal), $\dot{\alpha}$ (r-pitch), and $\dot{r}$ (a-pitch). We assumed that the data are collected at interval of 1 to produce the historical data sets X and Y. The formulated problem roughly describes monitoring of FOQA (Flight Operations Quality Assurance) data.

Faults in one input or output channel were simulated. Fault signatures were scaled such that $\|B_* f_j\|_{S_*^{-1}} = 1$ and $\|g_k\|_{S_*^{-1}} = 1$. The 'true' matrices $B = B_*$ and S_* used in the scaling of the seeded faults were obtained by fitting the linear regression to a large simulated data set of 100,000 points. When inverting ill-conditioned matrices, we used regularization parameters 10^{-4} times the matrix norm. The applied faults had amplitude $R = 9.5$.

Table I shows averaged results of 18,000 Monte Carlo runs. Each fault was seeded in 1000 runs and 1000 runs had no fault. In each run, a single new data point was generated and processed by the monitors for the anomaly detection and fault isolation. The monitors used model identified from a data set D_{200} produced from simulation (34) prior to the Monte Carlo runs.

Table I shows the percentage of the simulation runs with various monitor reports. The upper part of the table shows results for optimal Bayesian monitors; the lower part, for the baseline version of the monitors described in Subsection III-C. The Bayesian monitors take model uncertainty into account. The baseline monitors neglect model uncertainty (variance).

The column labels in Table I mean:

NoFF - No fault found: hypothesis $\mathbf{H_N}$ holds, Monitor 1 (ANO) has not detected anomaly.

M1ANO - Monitor 1 (ANO) has detected anomaly: hypothesis $\mathbf{H_A}$ holds.

M2ISO - Accurate fault isolation: the seeded fault is in the Monitor 2 (ISO) hypothesis set.

M3MAP - Accurate MAP isolation: Monitor 3 (MAP) has correctly identified the seeded fault.

Ambig - Ambiguity set size: an average number of fault hypothesis in the set produced by Monitor 2 (ISO).

Seeded - The fault or no-fault condition simulated.

One can see a major difference in isolating faults in v-longitudinal and elevator channels. Table I shows that for the Bayesian (robust) ISO monitor the isolation error rate improves to less than 1% from respectively 94% and 38% error rates that the baseline ISO monitor has in these channels. Note the accuracy improvement from MAP to ISO monitors that is achieved at the expense of introducing the ambiguity. For most channels, the ambiguity set has about 2-3 faults on average.

By comparing Bayesian and baseline monitor results in Table I, one can notice that for many channels the Bayesian (robust) MAP yields slightly worse accuracy than the baseline MAP. This is because the robust MAP effectively has a higher threshold for isolating the fault. This means some of the correct faults hypothesis might be missed.

The results shown for this example support our claim that Bayesian (robust) ISO monitor outperforms other monitors.

NoFF	M1ANO	M2ISO	M3MAP	Ambig	Seeded
Robust Fault Isolation					
98.6	1.4	0.0	0.0		No fault
0.4	99.6	99.6	14.2	2.5	v-longit
0.0	100.0	100.0	72.7	3.2	v-normal
0.0	100.0	99.4	90.2	2.6	AOA
0.0	100.0	100.0	78.9	2.2	elevator
0.0	100.0	99.1	17.1	3.1	a-longit
0.0	100.0	99.4	87.5	2.0	a-normal
0.0	100.0	99.4	86.6	2.0	r-pitch
0.0	100.0	98.7	86.3	2.6	a-pitch
Baseline Fault Isolation					
98.6	1.4	0.0	0.0		No fault
0.4	99.6	5.8	0.1	1.5	v-longit
0.0	100.0	99.1	69.0	2.0	v-normal
0.0	100.0	99.3	94.3	1.6	AOA
0.0	100.0	62.3	62.3	1.0	elevator
0.0	100.0	99.1	55.6	2.0	a-longit
0.0	100.0	99.4	99.4	1.0	a-normal
0.0	100.0	99.4	99.4	1.0	r-pitch
0.0	100.0	98.5	91.1	1.6	a-pitch

TABLE I

FAULT ISOLATION FOR THE AIRCRAFT MONITORING EXAMPLE

REFERENCES

[1] Alcala, C.F. and Qin, S.J., "Reconstruction-based contribution for process monitoring," *Automatica*, Vol. 45, No. 7, 2009, pp. 1593–1600.

[2] Chu, E., Gorinevsky, D. and Boyd, S, "Detecting aircraft performance anomalies from cruise flight data," *AIAA Infotech @Aerospace Conference*, April 2010, Atlanta, GA, AIAA-2010-3308.

[3] Dunia, R. and Qin, S.J., "Subspace approach to multidimensional identification and reconstruction of faults," *AIChE Journal*, Vol. 44, No. 8, August 1998, pp. 1813–1831.

[4] Etkin, B. and Reid L.D., *Dynamics Of Flight: Stability And Control*, 3rd Ed, Whiley, 1996

[5] Kourti, T. and MacGregor, J.F., "Multivariate SPC methods for process and product monitoring," *Journal of Quality Technology*, Vol. 28, No. 4, 1996, pp. 409-427.

[6] Li, G., Alcala, C. F., Qin, S. J., and Zhou, D., "Generalized reconstruction-based contributions for output-relevant fault diagnosis with application to the Tennessee Eastman process," *IEEE Trans. on Control Systems Technology*, (to appear), doi: 10.1109/TCST.2010.2071415

[7] Montgomery D.C. *Introduction to Statistical Quality Control*, (4th edn). Wiley: New York, 2000.

[8] Nevat, I., Peterst, G.W., and Yuan, J., "Bayesian inference in linear models with a random gaussian matrix: algorithms and complexity," *IEEE 19th Internat. Symp. on Personal, Indoor and Mobile Radio Comm.*, Sept. 2008, Cannes, France.

[9] Nomikos, P and MacGregor J. F., "Multi-way partial least squares in monitoring batch processes," *Chemometrics and Intelligent Laboratory Systems*, Vol. 30, 1995, pp. 97-108.

[10] Piovoso, M.J., Kosanovich, K.A., and Yuk, J.P., "Process data chemometrica," *IEEE Trans. on Instrumentation and Measurement*, Vol. 41, No. 2, 1992, pp. 262–268.

[11] Raich, A. and Cinar, A., "Diagnosis of process disturbances by statistical distance and angle measures," *Computers and Chemical Engineering*, Vol. 21, No. 6, 1997, pp. 661–673.

[12] Runger, G.C., Alt, F.B., and Montgomery, D.C., "Contributors to a multivariate statistical process control chart signal", *Comm. in Stat. - Theory and Methods,* Vol. 25, No. 10, 1996, pp. 2203–2213.

[13] Wiesel, A., Eldar, Y.C., and Yeredor, A., "Linear regression with Gaussian model uncertainty: algorithms and bounds," *IEEE Trans. on Signal Processing*, Vol. 56, No. 6, June 2008, pp. 2194-2205