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A Sequential Linear Quadratic Approach for Constrained Nonlinear Optimal Control

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***Abstract*— A sequential quadratic programming method is proposed for solving nonlinear optimal control problems subject to general path constraints including mixed state-control and state-only constraints. The proposed algorithm formulates linear quadratic optimal control subproblems with a solution that provides a descent direction for a non-differentiable exact penalty function. A set of conditions is given under which the minimization of the merit function produces a sequence of controls with limit points that satisfy the first order necessary conditions of the optimal control problem. The subproblems solved at each step of the algorithm inherit the structure of the nonlinear optimal control problem and can be solved efficiently via Riccati methods.**

I. Introduction

Optimal control problems provide an important class of tools for modeling and solving complex engineering problems. Numerous algorithms have been proposed for computing the solution of such problems. Indirect methods solve Continuous-Time Optimal Control Problems (CT-OCP) by considering the first-order necessary conditions and applying *Pontryagin's Maximum Principle* to reduce them into equivalent multi-point boundary value problems [1]. On the other hand, the strategy in direct methods is to solve the CT-OCP through an approximating Nonlinear Programming (NLP) problem and relate the optimality conditions of the CT-OCP to the Karush-Kuhn-Tucker (KKT) optimality conditions of the NLP [2].

In practice, direct methods are widely used for solving numerically OCP's because of their relative ease of formulation and desirable stability properties. An overview of direct methods is provided in [3] where such approaches are classified as *simultaneous* or *sequential*. Simultaneous methods tackle the NLP in the space of state and control variables; most notable are direct collocation methods [4]–[6] and direct multiple shooting methods [7]–[9]. Sequential methods (or Direct Single Shooting Methods) tackle the NLP in the space of control variables after first eliminating the state variables by simulating the dynamical equations [10], [11]. Thus sequential methods have fewer variables than simultaneous methods and also have the additional advantage that intermediate iterates are dynamically feasible, a property that is important in real-time applications such as Model Predictive Control (MPC). Unlike most sequential methods, simultaneous methods lead to sparse NLP's that can be exploited to speed up execution; examples are the use of sparse solvers [4], condensing [8], [12], and band-structure exploiting [13]. Popular methods for solving the NLP problems (either in sequential or simultaneous methods) are Sequential Quadratic Programming (SQP) and Interior Point (IP) methods.

In this paper, we propose a novel SQP approach for computing the solution to Discrete-Time Nonlinear Optimal Control Problems (DT-OCP) subject to general nonlinear path constraints on the states and controls. At each iteration, a constrained Linear-Quadratic Optimal Control (LQ-OCP) subproblem is obtained by taking a quadratic approximation of the cost function and linearizing the dynamical equations and state/control constraints about the current solution candidate. These LQ-OCP's can be solved efficiently by exploiting their structure with the algorithm of [14], [15] or with the interior point method of [13]. We show that the solution to these subproblems are descent directions for a non-differentiable merit function in the control variables and provide conditions under which convergence to a solution of the KKT conditions of the NL-OCP is achieved. The proposed algorithm extends the algorithm of [16] where no path constraints are allowed and thus we refer to it as the *Constrained Sequential Linear Quadratic* (CSLQ) algorithm. Our algorithm can be categorized as a single shooting SQP method but with the key characteristic that the QP subproblem is solved via a constrained LQ-OCP problem that preserves the inherit structure of the NL-OCP. In this manner, the CSLQ algorithm shares the advantages of sequential methods such dynamic feasibility while enjoys the computational efficiency of simultaneous methods due to sparsity. Indeed, [11] showed before that single shooting methods can in fact use LQ-OCP formulations for solving the SQP subproblems and exploit problem structure. However, we differ from [11] in several respects. In particular, our formulation allows for general nonlinear path constraints and nonconvex cost functions and there is no need to modify the solution of the LQ-OCP subproblem as in [11] to maintain feasibility. Our choice for the Hessian approximation is that of a Constrained Gauss-Newton method [17] but we introduce additional linear terms in the cost that prove to be instrumental in connecting the optimality conditions of each LQ-OCP subproblem to the those of the NL-OCP.

The remainder of the paper is organized as follows. Section II defines the problem treated, Section III describes the proposed CSLQ algorithm and Section IV presents a numerical example illustrating the results.

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II. Problem Formulation and Main Results

The nonlinear optimal control problems (NL-OCP) considered in this work are of the form

$$\min_{u(n),\ x(n)} \quad J = \psi(x(N)) + \sum_{n=0}^{N-1} L(x(n), u(n), n) \quad (1a)$$

$$\text{subject to} \quad x(n+1) = f(x(n), u(n), n); \quad x(0) = x_0. \quad (1b)$$

$$\phi(x(m), u(m), m) \le 0 \quad m \in \mathcal{C}_{xu} \quad (1c)$$

where J in (1a) is the performance index of the NL-OCP and $x(n) \in \mathbb{R}^{n_x}$ and $u(n) \in \mathbb{R}^{n_u}$ are the system state and control input vectors at time n, respectively. The evolution of the state sequence is governed by the first order ordinary difference equation in (1b) and the range of values that can be assumed by both the control and state vectors are restricted by the path constraints as defined in (1c), where $\mathcal{C}_{xu} \subset \{0, 1, \dots, N-1\}$ is the subset of indices at which the previous constraints are imposed. Moreover, the path constraint vector has dimension $\phi(x(n), u(n), n) \in \mathbb{R}^{p_n}$ where p_n is a positive integer. We assume that f, ψ, and ϕ are continuously differentiable and L twice continuously differentiable with respect to their arguments.

A. Formulation of the Constrained Sequential Linear Quadratic (CSLQ) Algorithm

In the following, we describe the proposed CLSQ algorithm for solving the nonlinear optimal control problem (NL-OCP) in (1). In this algorithm, a sequential quadratic programming (SQP) approach is used where the generally difficult NL-OCP problem is replaced with a sequence of more manageable linear quadratic optimal control subproblems subject to linear state/control constraints. The solution to each of these subproblems then provides a search direction that can be used to step towards the optimal solution of (NL-OCP). More specifically, given the current control $U_k = \left[u^T(0) \dots u^T(N-1)\right]^T$ and corresponding state $X_k = \left[x^T(1) \dots x^T(N)\right]^T$ trajectory sequences at step k, we consider control and state variations $\bar{U}_k = \left[\bar{u}^T(0) \dots \bar{u}^T(N-1)\right]^T$ and $\bar{X}_k = \left[\bar{x}^T(1) \dots \bar{x}^T(N)\right]^T$, respectively. We then formulate the following Linear Quadratic Optimal Control Problem (LQ-OCP) by taking a second order approximation of the performance index (1a) and linearizing the dynamic equations (1b) and constraints (1c) along the current solution with respect to the variations $\bar{U}_k$ and $\bar{X}_k$ (dependence on k is omitted for notational clarity):

$$\begin{aligned}\min_{\bar{x}(n), \bar{u}(n), \xi(n)} \bar{J} =& \frac{1}{2}\bar{x}(N)^T Q(N) \bar{x}(N) + \bar{x}(N)^T \bar{x}^o(N) + \\ &+ \frac{1}{2}\sum_{n=0}^{N-1} \left[\bar{x}^T(n) \ \ \bar{u}^T(n)\right] \begin{bmatrix} Q(n) & K(n) \\ K^T(n) & R(n) \end{bmatrix} \begin{bmatrix} \bar{x}(n) \\ \bar{u}(n) \end{bmatrix} \\ &+ \sum_{n=0}^{N-1} \left(\bar{x}^T(n)\bar{x}^o(n) + \bar{u}^T(n)\bar{u}^o(n)\right) + \rho\xi(0) \quad (2a)\end{aligned}$$

subject to

$$\bar{x}(n+1) = A(n)\bar{x}(n) + B(n)\bar{u}(n); \quad \bar{x}(0) = 0 \quad (2b)$$

$$\xi(n+1) = \xi(n); \quad (2c)$$

$$\text{Case (a)} : \xi(0) = 0 \ \text{ or} \quad (2d)$$

$$\text{Case (b)} : \xi(0) \ge 0 \text{ is unspecified} \quad (2e)$$

$$C(m)\bar{x}(m) + D(m)\bar{u}(m) + r(m) \le e_{p_m}\xi(m), \quad m \in \mathcal{C}_{xu} \quad (2f)$$

where e_{p_m} is a vector of ones of dimension p_m and we define:

$$\left.\begin{array}{rcl} Q(N) &=& \psi_{xx}(x(N)) \\ \overline{x}^o(N) &=& \psi_x^T(x(N)) \\ Q(n) &=& L_{xx}(x(n), u(n)) + \epsilon(n)I \\ R(n) &=& L_{uu}(x(n), u(n)) + \epsilon(n)I \\ K(n) &=& L_{xu}(x(n), u(n)) \\ \overline{x}^o(n) &=& L_x^T(x(n), u(n)) \\ \overline{u}^o(n) &=& L_u^T(x(n), u(n)) \\ A(n) &=& f_x(x(n), u(n)) \\ B(n) &=& f_u(x(n), u(n)) \\ C(m) &=& \phi_x(x(m), u(m)) \\ D(m) &=& \phi_u(x(m), u(m)) \\ r(m) &=& \phi(x(m), u(m)) \end{array}\right\} \quad (2g)$$

The subscripts on ψ, L, f, ϕ, and g denote the partials with respect to the state and control variables. $\rho > 0$ is a constant selected as discussed later in the paper. The auxiliary scalar state ξ is introduced to ensure the feasibility of (LQ-OCP). Indeed, two cases are included in the (LQ-OCP) formulation; first the solution of (LQ-OCP) is attempted with $\xi(0) = 0$—Case (a). However, if this problem is infeasible because of the constraints (2f), (LQ-LCP) is solved with leaving $\xi(0) \ge 0$ unspecified, as an optimization variable—Case (b), guaranteeing the feasibility of the optimal control problem. Furthermore, we select $\epsilon(n) > 0$ sufficiently big such that the weighting matrices in the cost function (2a) satisfy

$$R(n) = R^T(n) \ge \epsilon_0 I > 0, \quad (3)$$

$$Q(n) - K(n)R^{-1}(n)K^T(n) \ge 0 \qquad n = 0, \dots, N-1 \quad (4)$$

$$Q(n) = Q^T(n) \ge 0 \qquad n = 1, \dots, N \quad (5)$$

for a fixed $\epsilon_0 > 0$. The linear quadratic optimal control problem (2) is equivalent to a quadratic program (QP) and as such it can be solved using a general-purpose QP solver; however for problems with a long time horizon, this approach may prove computationally prohibitive due to the large-scale of the problem. Efficient algorithms to solve (2) that take advantage of the underlying structure of the problem have been reported in [15], [14], and [13].

Next, we consider the non-differentiable penalty function:

$$M(U) = J(X, U) + \rho\, V(X, U), \qquad X = F(U; x_0) \quad (6)$$

where $X = F(U; x_0)$ denotes that X is the state trajectory obtained from (1b) by applying the input U and with $x(0) =$

x_0. $J(X,U)$ is defined in (1a) and $V(X,U)$, representing the maximum constraint violation, is defined as

$$V(X,U) = \max_{i,n} \left\{ [\phi_i(x(n),u(n),n)]^+ \right\} \quad (7)$$

where $[\phi_i(x(n),u(n),n)]^+ \equiv \max\{0, \phi_i(x(n),u(n),n)\}$ with $\phi_i(x(n),u(n),n)$ denoting the i^{th} component of ϕ. The CSLQ algorithm is more precisely stated next.

Step 0: Set $k=0$, let $U_k = \left[u^T(0) \ \ldots \ u^T(N-1)\right]^T$ be an arbitrary control sequence and select a penalty parameter $\rho_k > 0$.

Step 1: Compute the state trajectory $X_k = \left[x^T(1) \ \ldots \ x^T(N)\right]^T$ corresponding to U_k by recursively computing the nonlinear dynamic equations in (1b).

Step 2: Formulate an (LQ-OCP) problem (2) by taking $\rho = \rho_k$ and the matrices in (2a) to (2f) as in (2g) and evaluated about the current solution (X_k, U_k). Then, **solve Case (a) of (LQ-OCP)** (i.e. by setting $\xi(0)=0$) and if feasible, let $\bar{U}_k = \bar{U}$ be the solution of the (LQ-OCP), $cv_k = 0$, and $\rho_{k+1} = \max\{\rho_k, \ \sum_{n=0}^{N} e_{p_n}^T \bar{\mu}(n) + \epsilon_0\}$. Else, **solve Case (b) of (LQ-OCP)** and take $\bar{U}_k = \bar{U}$, $cv_k = \xi_0$, and $\rho_{k+1} = \rho_k$. Here, cv_k is the maximum linearized constraint violation at step k. If $\|\bar{U}_k\| < tol$ and $cv_k < tol$, where tol is a termination tolerance, then **EXIT** with $U^* = U_k$, $X^* = X_k$, $\Lambda^* = \left[\bar{\lambda}^T(0) \ldots \bar{\lambda}^T(N)\right]$, and $\mu^* = \left[\bar{\mu}^T(0) \ldots \bar{\mu}^T(N)\right]$ being control, state, and Lagrange multiplier sequences that approximately satisfy the FONC optimality conditions for the (NL-OCP). Else if $k > k_{max}$ or $\rho_k > \rho_{max}$, then **EXIT** since the maximum number of iterations has been exceeded or the (NL-OCP) appears to be infeasible. Otherwise, proceed with Step 3.

Step 3: Let $\bar{U}_k$ from the previous step be a search direction and compute $U_{k+1} = U_k + \alpha \bar{U}_k$ where the step length $\alpha_k \in (0,1]$ is obtained by the Armijo rule, such as for the penalty function (6), it holds

$$M(U_k) - M(U_k + \alpha_k \bar{U}_k) \geq \sigma \alpha_k \epsilon_0 \|\bar{U}_k\|^2 \quad (8)$$

and $\alpha_k = \beta^{m_l}$ for some scalars $0 < \beta < 1$, $0 < \sigma < \frac{1}{2}$, and where m_l is the first nonnegative integer for which (8) is satisfied. (We remark that (23) below guarantees that α_k satisfying (8) can be found in a finite number of iterations by testing successively $l = 0, 1, \ldots$) Continue with Step 1. ∎

B. Properties of the CSLQ Algorithm

In the remainder of the paper, we elaborate on the properties of the proposed procedure.

Lemma 2.1: First-Order Necessary Conditions for (LQ-OCP)
Consider the linear quadratic optimal control problem defined by (2) and control and state trajectory sequences $\bar{U} = \left[\bar{u}^T(0) \ \ldots \bar{u}^T(N-1)\right]^T$ and $\bar{X} = \left[\bar{x}^T(1) \ \ldots \bar{x}^T(N)\right]^T$ respectively. Then, for $\bar{U}$ and $\bar{X}$ to be a global optimal solution of the (2), it is necessary that a co-state vector sequence $\bar{\Lambda} = \left[\bar{\lambda}^T(0) \ \ldots \bar{\lambda}^T(N)\right]^T$ and a multiplier vector sequence $\bar{\mu} = \left[\bar{\mu}^T(0) \ \ldots \bar{\mu}^T(N-1)\right]^T$ exist (with $\bar{\lambda}(n) \in \mathbb{R}^{N_x}$, $\bar{\mu}(n) \in \mathbb{R}^{p_n}$), such that for $n = 0, 1, \ldots, N-1$ the following conditions hold:

$$\bar{x}(n+1) = A(n)\bar{x}(n) + B(n)\bar{u}(n) \quad (9)$$

$$\xi(n+1) = \xi(n) = \xi_0 \quad (10)$$

$$\bar{\lambda}(n) = Q(n)\bar{x}(n) + A^T(n)\bar{\lambda}(n+1) + C^T(n)\bar{\mu}(n) + K(n)\bar{u}(n) + \bar{x}^o(n) \quad (11)$$

$$\lambda_\xi(n) = \lambda_\xi(n+1) - e_{p_n}^T \bar{\mu}(n) + \delta(n)(\rho - \eta) \quad (12)$$

$$0 = R(n)\bar{u}(n) + B^T(n)\bar{\lambda}(n+1) + D^T(n)\bar{\mu}(n) + K^T(n)\bar{x}(n) + \bar{u}^o(n) \quad (13)$$

and furthermore

$$\bar{x}(0) = 0 \quad (14)$$

$$\mathbf{C}ase\ (a)\ (\xi(0) = 0) : \xi_0 = 0 \ \text{ or} \quad (15)$$

$$\mathbf{C}ase\ (b)\ (\xi(0)\ is\ unspecified) :$$

$$\lambda_\xi(0) = 0, \quad \xi_0, \eta \geq 0, \quad \eta \cdot \xi_0 = 0 \quad (16)$$

$$\bar{\lambda}(N) = Q(N)\bar{x}(N) + C^T(N)\bar{\mu}(N) + \bar{x}^o(N) \quad (17)$$

$$\lambda_\xi(N) = -e_{p_N}^T \bar{\mu}(N) \quad (18)$$

$$C(m)\bar{x}(m) + D(m)\bar{u}(m) + r(m) \leq e_{p_m}\xi_0, \ m \in \mathcal{C}_{xu} \quad (19)$$

$$\bar{\mu}^T(m)\left[C(m)\bar{x}(m) + D(m)\bar{u}(m) + r(m) - e_{p_m}\xi_0\right] = 0, \quad m \in \mathcal{C}_{xu} \quad (20)$$

$$\bar{\mu}(m) \geq 0, \ m \in \mathcal{C}_{xu} \text{ and } \mu(m) = 0, \ m \notin \mathcal{C}_{xu}. \quad (21)$$

∎

We remark that η is the lagrange multiplier for the constraint $\xi_0 \geq 0$ in Case (b) and η can be taken to be arbitrary in Case (a); also $\delta(n)$ is defined by $\delta(0) = 1$, and $\delta(n) = 0, \ n \neq 0$.

The following result shows that the solution to the linear quadratic optimal control problem (2) is a descent direction for (6).

Theorem 2.2: Given the discrete-time nonlinear optimal control problem (1), consider a control sequence U and corresponding state trajectory X. Next, solve the constrained linear quadratic problem (2) obtained by taking a quadratic approximation of the cost index (1a) and a linear approximation of the constraints (1b)-(1c) about U and X. Then, if the solution $\bar{U}$ to (2) is not zero and if Case (a) is solved it holds

$$\rho \geq \sum_{n=0}^{N} \sum_{i=1}^{p_n} \bar{\mu}_i(n), \quad (22)$$

it follows that $\bar{U}$ satisfies

$$M(U + \alpha\bar{U}) \leq M(U) - \alpha\epsilon_0 \sum_{n=0}^{N-1} \bar{u}^T(n)\bar{u}(n) + o(\alpha). \quad (23)$$

Therefore, $\bar{U}$ is a descent direction for the non-differentiable penalty function

$$M(U) = J(X,U) + \rho\, V(X,U)$$

defined in (6).

Note that condition (22) is not required in Case (b); in Case (a), $\xi(0) = 0$ and the cost function of (2) does not depend on ρ.

Proof: Let $\widetilde{U} = U + \alpha\bar{U}$, $\widetilde{X} = X + \alpha\bar{X}$, with $\alpha > 0$. Then, the solution of (1b) corresponding to $\widetilde{U}$ is $\widetilde{X} + o(\alpha)$ and

$$M(\widetilde{U}) = J(\widetilde{X}, \widetilde{U}) + \rho\, V(\widetilde{X}, \widetilde{U}) + o(\alpha). \quad (24)$$

Next, notice that

$$J(\widetilde{X}, \widetilde{U}) = J(X, U) + \alpha \Delta J(X, U) + o(\alpha) \quad (25)$$

where

$$\Delta J(X,U) = \bar{x}^T(N)\bar{x}^o(N) + \sum_{n=0}^{N-1} \bar{x}^T(n)\bar{x}^o(n) + \bar{u}^T(n)\bar{u}^o(n) \quad (26)$$

using the definitions in (2g). Then, multiplying the terminal co-state condition (17) by $\bar{x}^T(N)$ and solving for $\bar{x}^T(N)\bar{x}^o(N)$ results in

$$\bar{x}^T(N)\bar{x}^o(N) = \bar{x}^T(N)\bar{\lambda}(N) - \bar{x}^T(N)Q(N)\bar{x}(N) + r^T(N)\bar{\mu}(N) - \xi_0 e_{p_N}^T \bar{\mu}(N) \quad (27)$$

(using (20) for $m = N$ and noting that $D(N) = 0$.)

Now, to express $\bar{x}^T(n)\bar{x}^o(n) + \bar{u}^T(n)\bar{u}^o(n)$ first multiply the costate equation (11) and the control equation (13) by $\bar{x}^T(n)$ and $\bar{u}^T(n)$, respectively. Then, summing these results and using the first-order necessary conditions (9) and (20) yields

$$\begin{aligned}\bar{x}^T(n)\bar{\lambda}(n) = {}& \bar{x}^T(n+1)\bar{\lambda}(n+1) + \bar{x}^T(n)\bar{x}^o(n) + \\ & + \bar{u}^T(n)\bar{u}^o(n) + \bar{x}^T(n)Q(n)\bar{x}(n) + \\ & + 2\bar{x}^T(n)K(n)\bar{u}(n) + \bar{u}^T(n)R(n)\bar{u}(n) \\ & - r^T(n)\bar{\mu}(n) + \xi_0 e_{p_n}^T \bar{\mu}(n). \end{aligned} \quad (28)$$

Solving for $\bar{x}^T(n)\bar{x}^o(n) + \bar{u}^T(n)\bar{u}^o(n)$ and summing the above equation from $n = 0$ to $n = N-1$ and noting from (14) that $\bar{x}_0 = 0$ gives:

$$\begin{aligned}\sum_{n=0}^{N-1} \bar{x}^T(n)\bar{x}^o(n) + \bar{u}^T(n)\bar{u}^o(n) = {}& \\ - \sum_{n=0}^{N-1} \big[\bar{x}^T(n)Q(n)\bar{x}(n) + 2\bar{x}^T(n)K(n)\bar{u}(n) + & \\ + \bar{u}^T(n)R(n)\bar{u}(n)\big] + & \\ + \sum_{n=0}^{N-1} r^T(n)\bar{\mu}(n) - \bar{x}^T(N)\bar{\lambda}(N) - \xi_0 \sum_{n=0}^{N-1} e_{p_n}^T \bar{\mu}(n). & \end{aligned} \quad (29)$$

Then, substituting (27) and (29) in (26) gives:

$$\begin{aligned}\Delta J = {}& -\bar{x}^T(N)Q(N)\bar{x}(N) - \sum_{n=0}^{N-1} \big[\bar{x}^T(n)Q(n)\bar{x}(n) + \\ & + 2\bar{x}^T(n)K(n)\bar{u}(n) + \bar{u}^T(n)R(n)\bar{u}(n)\big] + \\ & + \sum_{n=0}^{N} r^T(n)\bar{\mu}(n) - \xi_0 \sum_{n=1}^{N} e_{p_n}^T \bar{\mu}(n). \end{aligned} \quad (30)$$

We next seek to bound

$$V(\widetilde{X}, \widetilde{U}) = V(X + \alpha\bar{X}, U + \alpha\bar{U}) = \max_{i,n}\{\theta_i(n)\} \quad (31)$$

where we defined for brevity

$$\theta_i(n) \equiv [\phi_i(x(n) + \alpha\bar{x}(n), u(n) + \alpha\bar{u}(n), n)]^+. \quad (32)$$

Then, it readily follows

$$\theta_i(n) \le [r_i(n)(1-\alpha) + \alpha\xi_0 + o(\alpha)]^+ \quad (33)$$

using definitions (2g) and with $r_i(n)$ denoting the i^{th} component of $r(n)$. Since $\xi_0 \ge 0$, we readily obtain

$$[r_i(n)(1-\alpha) + \alpha\xi_0 + o(\alpha)]^+ = \begin{cases} o(\alpha), & r_i(n) < 0 \\ \alpha\xi_0 + o(\alpha), & r_i(n) = 0 \\ (1-\alpha)r_i(n) + \\ \quad + \alpha\xi_0 + o(\alpha), & r_i(n) > 0 \end{cases} \quad (34)$$

and further

$$\theta_i(n) \le (1-\alpha)[r_i(n)]^+ + \alpha\xi_0 + o(\alpha). \quad (35)$$

Now substitute (35) in (31) to obtain

$$\begin{aligned} V(\widetilde{X}, \widetilde{U}) & \le (1-\alpha)\max_{i,n}\{[r_i(n)]^+\} + \alpha\xi_0 + o(\alpha) \\ & \le V(X, U) - \alpha\max_{i,n}\{[r_i(n)]^+\} + \alpha\xi_0 + o(\alpha). \end{aligned} \quad (36)$$

Then, combining (25), (30), and (36) in (24), we find

$$\begin{aligned} M(\widetilde{U}) \le {}& M(U) - \alpha\bar{x}^T(N)Q(N)\bar{x}(N) - \\ & - \alpha\sum_{n=0}^{N-1} \big[\bar{x}^T(n)Q(n)\bar{x}(n) + 2\bar{x}^T(n)K(n)\bar{u}(n) + \\ & + \bar{u}^T(n)R(n)\bar{u}(n)\big] + \\ & + \alpha\left(\sum_{n=0}^{N} r^T(n)\bar{\mu}(n) - \xi_0\sum_{n=0}^{N} e_{p_n}^T\bar{\mu}(n) - \right. \\ & \left. - \rho\max_{i,n}[r_i(n)]^+ + \rho\xi_0\right) + o(\alpha). \end{aligned} \quad (37)$$

From (21) and (37), we obtain

$$\begin{aligned} M(\widetilde{U}) \le {}& M(U) - \alpha\bar{x}^T(N)Q(N)\bar{x}(N) - \\ & - \alpha\sum_{n=0}^{N-1} \big[\bar{x}^T(n)Q(n)\bar{x}(n) + 2\bar{x}^T(n)K(n)\bar{u}(n) + \\ & + \bar{u}^T(n)R(n)\bar{u}(n)\big] - \\ & - \alpha\left(\rho - \sum_{n=0}^{N}\sum_{i=1}^{p_n}\bar{\mu}_i(n)\right)\max_{i,n}\{[r_i(n)]^+\} - \\ & - \alpha\xi_0\left(\sum_{n=0}^{N} e_{p_n}^T\bar{\mu}(n) - \rho\right) + o(\alpha). \end{aligned} \quad (38)$$

Consider first Case (a). Since in this case $\xi_0 = 0$ and by assumption $\rho \ge \sum_{n=0}^{N}\sum_{i=1}^{p_n}\bar{\mu}_i(n)$, it follows

$$M(\widetilde{U}) \le M(U) - \alpha\epsilon_0\sum_{n=0}^{N-1}\bar{u}^T(n)\bar{u}(n) + o(\alpha)$$

that is, (23) is satisfied. Next consider Case (b). In this case (12) and (18) yield

$$\sum_{n=0}^{N}\sum_{i=1}^{p_n}\bar{\mu}_i(n) = \sum_{n=0}^{N} e_{p_n}^T\bar{\mu}(n) = -\lambda_\xi(0) + \rho - \eta = \rho - \eta \tag{39}$$

since from (16) $\lambda_\xi(0) = 0$. Furthermore, (39) gives

$$\rho - \sum_{n=0}^{N}\sum_{i=1}^{p_n}\bar{\mu}_i(n) = \eta \geq 0 \quad \text{and}$$
$$\xi_0\left(\sum_{n=0}^{N} e_{p_n}^T\bar{\mu}(n) - \rho\right) = -\eta\xi_0 = 0$$

where we used again (16). Then from the above equations, (23) follows again. Thus under the assumptions of the theorem and for α sufficiently small, $M(U+\alpha\bar{U}) < M(U)$ and $\bar{U}$ is a descent direction for the merit function (6). ■

The following result gives conditions under which the CSLQ algorithm produces a sequence of controls that satisfy the FONC for the nonlinear optimal control problem (1). Its proof is omitted due to space limitations.

Theorem 2.3: Let U_k be the sequence of controls generated by the CSLQ algorithm. Also assume that there exists $\bar{k}$ such that $\rho_k = \bar{\rho}$ for all $k \geq \bar{k}$ and that the (LQ-OCP) Case (a) ($\xi(0) = 0$) subproblems solved during Step 2 are feasible for an infinite set of times $\mathcal{K}$. Then every limit point U^* of U_k, $k \in \mathcal{K}$ is a stationary point of (NL-OCP), in the sense that there exist multiplier sequences Λ^*, and μ^* such that along with U^* and the corresponding state trajectory X^* obtained from (1b) satisfy the first order necessary optimality conditions for the nonlinear optimal control problem of (1). ■

We remark that if the nonlinear optimal control problem is feasible, the CSLQ algorithm typically resorts in solving Case (a) (LQ-OCP) subproblems after k is large enough, therefore, the conditions of Theorem 2.3 are satisfied and convergence is achieved. Theorem 2.3 shows that the algorithm converges to a stationary point of the merit function. It can be shown that the stationary points of the NL-OCP are stationary points of the merit function for $\rho > \sum_{n=0}^{N}\sum_{i=1}^{p_n}\mu_i^*(n)$ where the $\mu_i^*(n)$ are the lagrange multipliers of the NL-OCP. The converse is not true unless additional assumptions on the constraints such as convexity are invoked [18]; thus if either Case (b) subproblems are solved after some $k > K$ or $\rho_k \to \infty$, convergence to a stationary point of the merit function but not of the NL-OCP occurs and the algorithm may be restarted. However, the latter case typically signifies that the NL-OCP is infeasible or does not have lagrange multipliers.

III. Numerical Example: Acrobot

Consider the underactuated two-link robot depicted in Fig. 1 with a single actuator positioned on the second joint. This dynamical system is commonly known in the literature as the ‘acrobot’ for its resemblance to a gymnast on parallel bars. In this example, we will determine the necessary control effort to achieve a *swing-up* task consisting of completely inverting the starting position of the acrobot. To accomplish this maneuver, we must actuate the second link just as a gymnast would swing his legs to achieve this position. The problem definition is borrowed from [19]. This *swing-up* task can be formulated as an optimal control as:

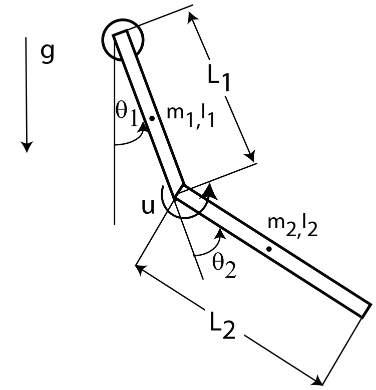

Fig. 1: Acrobot geometry

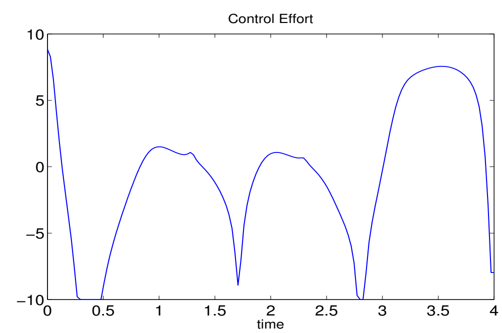

Fig. 2: Optimal control for the Acrobot *swing-up* task

$$\min_{u(t),\theta(t),\dot{\theta}(t)} \quad J = \int_0^{t_f} u^2(t)dt. \tag{40}$$

$$\text{subject to} \quad M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = Bu \tag{41}$$
$$[\theta^T(0) \quad \dot{\theta}^T(0)] = [0\ 0\ 0\ 0] \tag{42}$$
$$[\theta^T(t_f) \quad \dot{\theta}^T(t_f)] = [\pi\ 0\ 0\ 0] \tag{43}$$
$$-10 \leq u(t) \leq 10 \tag{44}$$

with

$$M(\theta) = \begin{bmatrix} I_2 + d + 2a\cos(\theta_2) & I_2 + a\cos(\theta_2) \\ I_2 + a\cos(\theta_2) & I_2 \end{bmatrix},$$
$$C(\theta,\dot{\theta}) = \begin{bmatrix} -2a\sin(\theta_2)\dot{\theta}_2 & -a\sin(\theta_2)\dot{\theta}_2 \\ a\sin(\theta_2)\dot{\theta}_1 & 0 \end{bmatrix}, \quad B^T = [0\ 1]$$
$$G(\theta) = \begin{bmatrix} b\sin(\theta_1) + c\sin(\theta_1+\theta_2) \\ c\sin(\theta_1+\theta_2) \end{bmatrix} \tag{45}$$

where $a = m_2L_1L_2/2$, $b = L_1(m_1 + 2m_2)g/2$, $c = m_2gL_2$, and $d = I_1 + m_2L_1^2$ with $u(t)$ representing the torque applied at the second joint and $\theta(t) = [\theta_1(t)\,\theta_2(t)]^T$ and $\dot{\theta}(t) = [\dot{\theta}_1(t)\,\dot{\theta}_2(t)]^T$ specifying the angle and angular velocity of each of the two links respectively. More

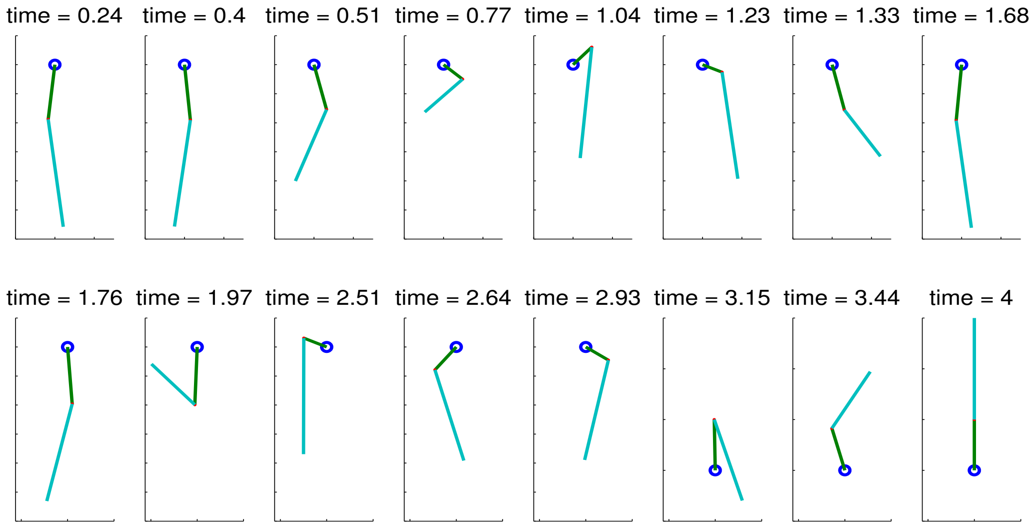

Fig. 3: Optimal Acrobot poses for *swing-up* task.

specifically, the *swing-up* task consists of starting from a hanging static position $\left([\theta^T(0)\ \dot{\theta}^T(0)] = 0\right)$ at $t = 0\,sec$, and reaching an inverted position with zero velocity $\left([\theta^T(t_f)\ \dot{\theta}^T(t_f)] = [\pi\ 0\ 0\ 0]\right)$ within $t_f = 4\,sec$. Additionally, we required that the task be performed with minimum actuation effort as expressed by the performance index (40) and without exceeding the actuation bounds stated in (44).We approximate the continuous-time problem in discrete-time as follows. The time horizon is partitioned into $N = 150$ equal subintervals of length T_s and the control $u(t)$ is assumed to be piecewise constant over each of the subintervals; that is $u(t) = u(nT_s) \equiv u(n)$ for $t_n = nT_s \leq t < t_{n+1} = (n+1)T_s$, $n = 0, \ldots, N-1$. Then, a 4^{th}-order Runge-Kutta integration rule is used to discretize the system dynamics and cost function. The continuous-time control constraints where enforced at the discretization points as $-10 \leq u(n) \leq 10 \quad n = 0, \ldots, N-1$. Using the physical parameters: $m_1 = 1$, $m_2 = 1$, $L_1 = 1$, $L_2 = 2$, $I_1 = 1/3$, $I_2 = 1$, and $g = 9.81$, the proposed algorithm terminated with a cost $J^D = 51.4$ and with a constraint violation $\sum \left| [\theta^T(t_f)\ \dot{\theta}^T(t_f)] - [\pi\,0\,0\,0] \right| = 0.001$. The optimal control is plotted in Fig. 2, while some acrobot poses of the optimal swing-up motion are shown in Fig. 3.

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