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Adaptive Control with Loop Transfer Recovery: A Kalman Filter Approach

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Abstract— **In this paper we develop a Kalman filter based adaptive controller for multivariable uncertain systems with loop transfer recovery of an associated reference system. This approach increases the level of confidence of adaptive control systems by providing a means for preserving stability margins even under uncertainty and failures. In addition, it results in an optimization based time-varying adaptation gain. An example is provided to illustrate the efficacy of the proposed approach.**

I. Introduction

Direct adaptive controllers require less modeling information than robust controllers and can address system uncertainties and system failures. These controllers directly adapt controller parameters in response to system variations for the purpose of canceling the effect of modeling uncertainty, without necessarily estimating the parameters of the unknown system. This property distinguishes them from adaptive controllers which employ an estimation algorithm to estimate the unknown system parameters, and employ a controller that depends on the estimated parameters. This paper presents a direct Kalman filter based adaptive controller for multivariable uncertain systems with loop transfer recovery of an associated reference system.

In the recent years, there have been a number of efforts focused on improving direct adaptive controllers [1]–[12]. These approaches impose constraints on the weights to improve an existing adaptive law, and are commonly referred to as composite adaptation [13]. In general, the gradient of a norm of the error in the constraints produces terms that are used to modify an existing direct adaptive control law. However, using a gradient method can result in slow parameter convergence towards a local minimum [14]. In addition, gradient based modification terms have a fixed adaptation gain that often has to be chosen high to obtain satisfactory results. However, this choice can interact negatively with unmodeled dynamics and amplify the effect of sensor noise. A Kalman filter modification approach to adaptive control has been proposed to overcome these problems in Refs. 7 and 8 that approximately enforces constraints on the weights. When compared with gradient based modification terms, this approach to modification improves both adaptive stabilization and command following. The Kalman filter based adaptive control proposed in this paper is an extension of the previous work in that we not only enforce constraints on the weights, but also optimize the adaptation gain.

From the perspective of verification and validation of adaptive control systems, there is a need to address two problems. The first problem is to design an adaptive controller so that the closed-loop system can guarantee stabilization and command following while maintaining the stability margins of the reference system. The second problem is to preserve the reference system margins to the degree possible even under uncertainty and failures. While the aspect of stabilization and command following can be dealt with in a nonlinear setting by employing Lyapunov stability analysis, issues related to margins can be addressed using the adaptive loop recovery (ALR) modification term proposed in Ref. 5. By employing ALR modification, the loop transfer properties of a reference system associated with a nominal control design are preserved in an adaptive system. As a result, this term increases the level of confidence one has in the use of adaptive control systems.

In this paper, we show that Kalman filter based adaptive controllers can be obtained by imposing a constraint on the weight estimates, and formulating and solving an associated minimization problem. To address the problem of maintaining a measure of stability margin in the adaptive system, we also use a Kalman filtering approach to approximately enforce the ALR constraint. The resulting adaptive control law is shown to guarantee that the closed loop error signals are uniformly ultimately bounded (UUB). An illustrative example is provided to demonstrate the efficacy of the proposed approach.

The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, and $(\cdot)^{-1}$ denotes inverse. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, $\|\cdot\|$ for the Euclidian vector norm or for the Frobenius matrix norm, $\mathrm{vec}(\cdot)$ for the column stacking operator, and $\otimes$ for Kronecker product.

II. Preliminaries

Consider the controlled uncertain system given by

$$\dot{x}(t) \quad = \quad Ax(t) + B[u(t) + \Delta(x(t))], \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices, and $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ is matched uncertainty. We assume that the full state vector is available for feedback and the control input is restricted to the class of admissible controls consisting of measurable functions.

In order to achieve trajectory tracking, define a reference system given by

$$\dot{x}_{\mathrm{m}}(t) \quad = \quad A_{\mathrm{m}}x_{\mathrm{m}}(t) + B_{\mathrm{m}}r(t), \tag{2}$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$ is the reference state vector, $r(t) \in \mathbb{R}^r$ is a bounded piecewise continuous reference input, $A_{\mathrm{m}} \in \mathbb{R}^{n\times n}$ is Hurwitz, and $B_{\mathrm{m}} \in \mathbb{R}^{n\times r}$, $r \leq m$.

Assumption 1. The unknown matched uncertainty in (1) can be linearly parameterized as

$$\Delta(x) \quad = \quad W^{\mathrm{T}}\beta(x), \tag{3}$$

where $W \in \mathbb{R}^{s\times m}$ is the unknown constant weight matrix and $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is the basis function vector of the form

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$\beta(x(t)) = [\beta_1(x(t)),\ \beta_2(x(t)),\dots,\ \beta_s(x(t))]^{\mathrm{T}} \in \mathbb{R}^s$.

Consider the control signal

$$u(t) \quad = \quad u_{\mathrm{n}}(t) - u_{\mathrm{ad}}(t), \qquad (4)$$

where $u_{\mathrm{n}}(t)$ is the nominal control signal given by

$$u_{\mathrm{n}}(t) \quad = \quad -K_1 x(t) + K_2 r(t), \qquad (5)$$

$K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times r}$ are nominal control gains, and $u_{\mathrm{ad}}(t)$ is the adaptive control signal given by

$$u_{\mathrm{ad}}(t) \quad = \quad \hat{W}^{\mathrm{T}}(t)\beta(x(t)), \qquad (6)$$

where $\hat{W}(t) \in \mathbb{R}^{s\times m}$ is an estimate of W obtained from the adaptive weight update law

$$\dot{\hat{W}}(t) \quad = \quad \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big]. \qquad (7)$$

In (7) γ and σ are fixed positive adaptation and modification gains, respectively, $e(t) = x(t) - x_{\mathrm{m}}(t)$ is the error signal, and $P \in \mathbb{R}^{n\times n}$ is a positive-definite solution of the Lyapunov equation

$$0 \quad = \quad A_{\mathrm{m}}^{\mathrm{T}} P + P A_{\mathrm{m}} + L, \qquad (8)$$

for any $L = L^{\mathrm{T}} > 0$.

The system dynamics (1) can now be written as

$$\dot{x}(t) \quad = \quad A_{\mathrm{m}} x(t) + B_{\mathrm{m}} r(t) - B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)), \quad (9)$$

where $A_{\mathrm{m}} = A - BK_1$, $B_{\mathrm{m}} = BK_2$, and $\tilde{W}(t) \triangleq \hat{W}(t) - W$, with the system error dynamics and the error weight update given by

$$\dot{e}(t) \quad = \quad A_{\mathrm{m}} e(t) - B\tilde{W}^{\mathrm{T}}(t)\beta(x(t)), \qquad (10)$$

and

$$\dot{\tilde{W}}(t) \quad = \quad \gamma\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big]. \qquad (11)$$

Theorem 1. Consider the uncertain system given by (1) with the adaptive control law defined by (4)–(8), subject to Assumption 1. The corresponding errors given by (10) and (11) are UUB.

Proof. The proof of Theorem 1 is common in the literature [1], [13], and follows directly by considering the Lyapunov function candidate

$$\mathcal{V}(e(t), \tilde{W}(t)) = e^{\mathrm{T}}(t)Pe(t) + \frac{1}{\gamma}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)], \qquad (12)$$

and differentiating (12) along the closed-loop system trajectories of (10) and (11). □

III. Kalman Filter Based Adaptive Control Formulation

The weight update law given by (7) can be viewed as a gradient approach for enforcing the following linear constraint on the weight estimates

$$\hat{W}^{\mathrm{T}}(t)\Phi_1(\cdot) \quad = \quad \Phi_2(\cdot) \qquad (13)$$

where

$$\Phi_1(\cdot) \quad = \quad \sqrt{\sigma}\ I_s \qquad (14)$$
$$\Phi_2(\cdot) \quad = \quad \sqrt{\sigma}^{\ -1}\beta(x(t))e^{\mathrm{T}}(t)PB, \quad \sigma > 0. \qquad (15)$$

To see this, consider the cost function

$$\mathcal{J}(\hat{W}(t)) \quad = \quad \frac{1}{2}||\hat{W}^{\mathrm{T}}(t)\Phi_1 - \Phi_2(x(t))||^2. \qquad (16)$$

The negative gradient of $\mathcal{J}(\hat{W}(t))$ with respect to $\hat{W}(t)$ is given by

$$-\frac{\mathrm{d}\mathcal{J}(\hat{W}(t))}{\mathrm{d}\hat{W}(t)} \quad = \quad \beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t), \quad (17)$$

where we used (14) and (15). Multiplying both sides of (17) by $\gamma > 0$ gives (7).

The problem of estimating W so that it satisfies the the linear constraint in (13) can be formulated as an optimization problem. For this purpose, taking the transpose of (13) and then applying $\mathrm{vec}(\cdot)$ operator results in the following equivalent form

$$\bar{\Phi}_1^{\mathrm{T}}(\cdot)\omega(t) \quad = \quad \bar{\Phi}_2^{\mathrm{T}}(\cdot), \qquad (18)$$

where $\bar{\Phi}_1(\cdot) = I_m \otimes \Phi_1^{\mathrm{T}}(\cdot) \in \mathbb{R}^{ms\times ml}$, $\bar{\Phi}_2(\cdot) = \mathrm{vec}(\Phi_2^{\mathrm{T}}(\cdot)) \in \mathbb{R}^{ml}$, and $\omega(t) = \mathrm{vec}(\hat{W}(t)) \in \mathbb{R}^{ms}$. Define the stochastic process

$$\dot{\omega}(t) \quad = \quad q(t), \qquad (19)$$
$$z(t) \quad = \quad \bar{\Phi}_1^{\mathrm{T}}(\cdot)\omega(t) + r(t), \qquad (20)$$

where $q(t)$ and $r(t)$ are zero-mean, Gaussian, white noise processes with covariances

$$\mathbb{E}\big\{q(t)q^{\mathrm{T}}(\tau)\big\} \quad = \quad \bar{Q}\delta(t-\tau),\ \bar{Q} \in \mathbb{R}^{ms\times ms} > 0, \ (21)$$
$$\mathbb{E}\big\{r(t)r^{\mathrm{T}}(\tau)\big\} \quad = \quad \bar{R}\delta(t-\tau),\ \bar{R} \in \mathbb{R}^{ml\times ml} > 0, \ (22)$$

and $z(t)$ is regarded as a measurement. The estimate of $z(t)$ is

$$\hat{z}(t) \quad = \quad \bar{\Phi}_1^{\mathrm{T}}(\cdot)\hat{\omega}(t), \qquad (23)$$

where $\hat{\omega}(t)$ is an estimate of $\omega(t)$. The Kalman filter associated with this problem formulation is given by [15]:

$$\dot{\hat{\omega}}(t) \quad = \quad \bar{S}(t)\bar{\Phi}_1(\cdot)\bar{R}^{-1}(z(t) - \hat{z}(t)),\ \hat{\omega}(0) = 0, \quad (24)$$
$$\dot{\bar{S}}(t) \quad = \quad -\bar{S}(t)\bar{\Phi}_1(\cdot)\bar{R}^{-1}\bar{\Phi}_1^{\mathrm{T}}(\cdot)\bar{S}(t) + \bar{Q}, \quad \bar{S}(0) = \bar{S}_0 > 0, \quad (25)$$

where $\bar{S}(t) \in \mathbb{R}^{ms\times ms}$. Since the objective is to approximately satisfy the constraint in (13), the logical choice for $z(t)$ when employing this estimator is $z(t) = \bar{\Phi}_2^{\mathrm{T}}(\cdot)$. Furthermore, choosing $\bar{R} = I_{m\times m} \otimes R$ with $R \in \mathbb{R}^{l\times l} > 0$ and $\bar{Q} = I_{m\times m} \otimes Q$ with $Q \in \mathbb{R}^{s\times s} > 0$, Appendix A of [8] shows that (24) and (25) with $z(t) = \bar{\Phi}_2^{\mathrm{T}}(\cdot)$ can equivalently be expressed as:

$$\dot{\hat{W}}(t) \quad = \quad -S(t)\Phi_1(\cdot)R^{-1}\big[\Phi_1(\cdot)^{\mathrm{T}}\hat{W}(t) - \Phi_2(\cdot)^{\mathrm{T}}\big], \quad \hat{W}(0) = 0, \quad (26)$$
$$\dot{S}(t) \quad = \quad -S(t)\Phi_1(\cdot)R^{-1}\Phi_1^{\mathrm{T}}(\cdot)S(t) + Q, \quad S(0) = S_0 > 0, \quad (27)$$

where $S(t) \in \mathbb{R}^{s\times s}$.

Using (14) and (15) in (26) and (27) and choosing $R = rI_l$, $r > 0$, leads to a Kalman filter based adaptive controller (KFAC) with weight update law given by:

$$\dot{\hat{W}}(t) \quad = \quad \Gamma(t)\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big], \qquad (28)$$
$$\dot{S}(t) \quad = \quad -(\sigma/r)S^2(t) + Q, \quad S(0) = S_0 > 0, \quad (29)$$

where $\Gamma(t) \triangleq S(t)/r$.

Remark 1. It follows from Proposition 1.1 in Ref. 16 that $S(t)$ exists and is symmetric and positive definite for all $t \geq 0$. Furthermore, it follows from Theorem A.2 in Ref. 8 that $S(t)$ is uniformly bounded. Hence, $\Gamma(t)$ is the positive definite, time-varying, and uniformly bounded Kalman filter gain for KFAC.

The weight error satisfies

$$\dot{\tilde{W}}(t) \;=\; \Gamma(t)\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\big]. \quad (30)$$

Theorem 2. Consider the uncertain system given by (1) with the adaptive control law defined by (4)–(6), (8), (28), and (29) subject to Assumption 1. The corresponding errors given by (10) and (30) are UUB.

Proof. Consider the Lyapunov function candidate

$$\mathcal{V}(\cdot) \;=\; e^{\mathrm{T}}(t)Pe(t) + \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\Gamma^{-1}(t)\tilde{W}(t)], \quad (31)$$

where $P > 0$ satisfies (8). Differentiating (31) along the closed-loop trajectories of (10) and (30) gives

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) \;&=\; 2e^{\mathrm{T}}(t)P\big[A_{\mathrm{m}}e(t) - B\tilde{W}^{\mathrm{T}}(t)\beta(x(t))\big] \\
&\quad +2\mathrm{tr}\Big[\tilde{W}^{\mathrm{T}}(t)\Big(\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t)\Big)\Big] \\
&\quad -\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t)\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t)\tilde{W}(t)\big] \\
&=\; -e^{\mathrm{T}}(t)Le(t) - 2\sigma\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t)\hat{W}(t)\big] \\
&\quad -\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t)\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t)\tilde{W}(t)\big] \\
&\leq\; -\lambda_{\min}(L)|e(t)|^2 - \sigma||\tilde{W}(t)||^2 + \sigma||W||^2 \\
&\quad -\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t)\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t)\tilde{W}(t)\big]. \quad (32)
\end{aligned}$$

Using $\Gamma^{-1}(t)\dot{\Gamma}(t)\Gamma^{-1}(t) = \big(rS^{-1}(t)\big)\big(-(\sigma/r^2)S^2(t) + Q/r\big)\big(rS^{-1}(t)\big) = rQS^{-1}(t) - \sigma I_s$ in (32) yields

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) \;&\leq\; -\lambda_{\min}(L)|e(t)|^2 - \sigma||\tilde{W}(t)||^2 + \sigma||W||^2 \\
&\quad -r\mathrm{tr}\big[\tilde{W}QS^{-1}(t)\tilde{W}(t)\big] + \sigma||\tilde{W}(t)||^2 \\
&=\; -\lambda_{\min}(L)|e(t)|^2 - r\lambda_{\min}\big(QS^{-1}(t)\big)||\tilde{W}(t)||^2 \\
&\quad +\sigma||W||^2 \\
&=\; -c_1|e(t)|^2 - c_2||\tilde{W}(t)||^2 + c_3, \quad (33)
\end{aligned}$$

where $c_1 \triangleq \lambda_{\min}(L) > 0$, $c_2 \triangleq r\lambda_{\min}\big(QS^{-1}(t)\big) > 0$, and $c_3 \triangleq \sigma||W||^2 > 0$. Either $|e(t)| \geq \Theta_e$ or $||\tilde{W}(t)|| \geq \Theta_w$ renders $\dot{\mathcal{V}}(\cdot) < 0$, where $\Theta_e \triangleq \sqrt{c_3/c_1}$ and $\Theta_w \triangleq \sqrt{c_3/c_2}$. Therefore, the closed loop signals $e(t)$ and $\tilde{W}(t)$ are UUB. □

Remark 2. If Assumption 1 is relaxed to

$$\Delta(x) \;=\; W^{\mathrm{T}}\beta(x) + \epsilon(x), \quad (34)$$

where $\epsilon : \mathbb{R}^n \to \mathbb{R}^m$ is the residual error satisfying $||\epsilon(x)|| < \bar{\epsilon}$, then (33) becomes

$$\begin{aligned}
\dot{\mathcal{V}}(\cdot) \;&\leq\; -c_1|e(t)|^2 - c_2||\tilde{W}(t)||^2 + c_3 \\
&\quad +2e^{\mathrm{T}}(t)PB\epsilon(x(t)) \\
&\leq\; -(c_1 - \xi)|e(t)|^2 - c_2||\tilde{W}(t)||^2 \\
&\quad (c_3 + ||PB||^2\bar{\epsilon}^2), \quad \xi > 0, \quad (35)
\end{aligned}$$

where Young's inequality ($x^{\mathrm{T}}y \leq \gamma x^{\mathrm{T}}x + y^{\mathrm{T}}y/4\gamma$, $\gamma > 0$) is used for $2e^{\mathrm{T}}(t)PB\epsilon(x(t))$. In this case, either $|e(t)| \geq \tilde{\Theta}_e$ or $||\tilde{W}(t)|| \geq \tilde{\Theta}_w$ renders $\dot{\mathcal{V}}(\cdot) < 0$, where $c_1 > \xi$, $\tilde{\Theta}_e \triangleq \sqrt{(c_3 + ||PB||^2\bar{\epsilon}^2)/(c_1 - \xi)}$, and $\tilde{\Theta}_w \triangleq \sqrt{(c_3 + ||PB||^2\bar{\epsilon}^2)/c_2}$. Hence, the closed loop signals $e(t)$ and $\tilde{W}(t)$ are UUB.

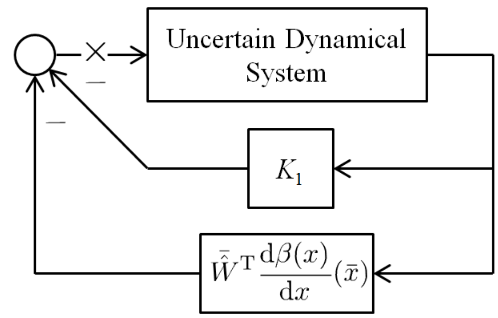

Fig. 1. Linearized adaptive system dynamics.

IV. Adaptive Loop Recovery

In this section we first summarize the ALR approach proposed in Ref. 5. The objectives of ALR are:

i. Design an adaptive controller so that the closed-loop system can achieve stabilization and command following while maintaining the margins of the reference system given by (2) in the absence of uncertainty ($\Delta(x(t)) = 0$).

ii. Preserve the reference system margins to the degree possible even under uncertainty.

While the aspect of stabilization and command following can be dealt with in a nonlinear setting by employing Lyapunov stability analysis, addressing issues related to margins requires linearization. Within the context of the adaptive control problem, this reduces to linearizing the adaptive weight update law. Fig. 1 illustrates the result of this linearization with weights frozen. When $\Delta(x(t)) = 0$ (objective *i*), the upper portion of this drawing represents the reference model dynamics. The margins calculated with the loop broken at $\times$ in this drawing with $\bar{\hat{W}} = 0$ correspond to the margins of the reference model. The bottom portion of this diagram shows the effect that the adaptive controller in steady state has on the loop properties of the reference model. However, this picture is fallacious because in reality the weights of the reference model are not frozen. But even if they were frozen it is apparent that even if the tracking error is zero, the margins of the reference model are not maintained, but instead they are modified in an unknown way by the lower feedback block. In the case of varying weights it is not possible to even calculate margins on the basis of Fig. 1, because the lower portion of this diagram is a time-varying matrix block with $\bar{\hat{W}}$ replaced by $\hat{W}(t)$. However, it might still be possible to achieve both objectives of ALR if one enforces the constraint $\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)) = 0$ for all $t \geq 0$ in the adaptive process, where $\beta_x(x(t)) \triangleq \frac{\mathrm{d}\beta(x(t))}{\mathrm{d}x(t)}$. In this direction, the following assumption is introduced in Ref. 5.

Assumption 2. There exists $W(t)$ such that

$$\Delta(x(t)) \;=\; W^{\mathrm{T}}(t)\beta(x(t)), \quad (36)$$

$$W^{\mathrm{T}}(t)\beta_x(x(t)) \;=\; 0. \quad (37)$$

In addition, $\beta_x(x(t))$ has full column rank.

Remark 3. Assumption 2 requires that there is sufficient redundancy in the choice of the basis function vector to allow simultaneous satisfaction of conditions (36) and (37). Including a bias term in the basis vector is one easy way to ensure the conditions are met.

Remark 4. It is necessary that the condition $\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)) = 0$ be maintained during both transitory and steady-state conditions.

It is shown in Ref. 5 that enforcement of the constraint $\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)) = 0$ in the adaptive weight update law can be approximated by considering a cost function

$$\mathcal{J}_A(t) = ||\hat{W}^{\mathrm{T}}(t)\beta_x(x(t))||^2, \tag{38}$$

and taking its negative gradient with respect to $\hat{W}(t)$, which leads to the ALR modification term given by

$$\dot{\hat{W}}_A(t) = -\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t). \tag{39}$$

In the context of the Kalman filter modification approach, the enforcement of the constraint $\hat{W}^{\mathrm{T}}(t)\beta_x(x(t)) = 0$ in the adaptive weight update law follows from (22) and (23) of Ref. 8 as

$$\dot{\hat{W}}_A(t) = -S_A(t)\beta_x(x(t))R_A^{-1}\beta_x^{\mathrm{T}}(x(t))\hat{W}(t), \tag{40}$$
$$\dot{S}_A(t) = -S_A(t)\beta_x(x(t))R_A^{-1}\beta_x^{\mathrm{T}}(x(t))S_A(t) + Q_A, \quad S_A(0) > 0, \tag{41}$$

where $R_A \in \mathbb{R}^{n\times n} > 0$ and $Q_A \in \mathbb{R}^{s\times s} > 0$. Remark 1 also applies to the solution of (41). Choosing $R_A > r_A I_n$, $r_A > 0$, yields

$$\dot{\hat{W}}_A(t) = -\Gamma_A(t)\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t), \tag{42}$$

where $\Gamma_A(t) \triangleq S_A(t)/r_A$. Note that (42) has a time-varying adaptation gain, whereas (39) has a fixed adaptation gain. The adaptive weight update law presented in the previous section with ALR modification is now given by

$$\dot{\hat{W}}(t) = \Gamma(t)\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t) - \Gamma_A(t)\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\big]. \tag{43}$$

The weight error $\tilde{W}(t) \triangleq \hat{W}(t) - W(t)$ satisfies

$$\dot{\tilde{W}}(t) = \Gamma(t)\big[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\hat{W}(t) - \Gamma_A(t)\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\big] - \dot{W}(t). \tag{44}$$

Theorem 3. Consider the uncertain system given by (1) with the adaptive control law defined by (4)–(6), (8), and (43), subject to Assumption 2 and $||\dot{W}(t)|| < \dot{w}^*$. The corresponding errors given by (10) and (44) are UUB.

Proof. Consider the Lyapunov function candidate given by (31). Differentiating (31) along the closed-loop trajectories of (10) and (44) gives

$$\dot{\mathcal{V}}(\cdot) \leq -c_1|e(t)|^2 - c_2||\tilde{W}(t)||^2 + \bar{c}_2||\tilde{W}(t)|| + c_3 - \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t)\Gamma_A(t)\beta_x(x(t))\beta_x^{\mathrm{T}}(x(t))\hat{W}(t)\big], \tag{45}$$

where $\bar{c}_2 \triangleq r\lambda_{\min}(S^{-1}(t))\dot{w}^* > 0$. Using Assumption 2 in the last term of (45) yields

$$\begin{aligned}\dot{\mathcal{V}}(\cdot) &\leq -c_1|e(t)|^2 - ||\tilde{W}(t)||\Big(c_2||\tilde{W}(t)|| - \bar{c}_2\Big) + c_3 \\ &= -c_1|e(t)|^2 - \Big(||\tilde{W}(t)||\sqrt{c_2} - \frac{\bar{c}_2}{2\sqrt{c_2}}\Big)^2 + \frac{\bar{c}_2^2}{4c_2} + c_3.\end{aligned} \tag{46}$$

Either $|e(t)| \geq \bar{\Theta}_e$ or $||\tilde{W}(t)|| \geq \bar{\Theta}_w$ renders $\dot{\mathcal{V}}(\cdot) < 0$, where $\bar{\Theta}_e \triangleq \sqrt{\bar{c}_3/c_1}$, $\bar{\Theta}_w \triangleq \big(\sqrt{\bar{c}_3} - \bar{c}_2/(2\sqrt{c_2})\big)/\sqrt{c_2}$, and $\bar{c}_3 \triangleq \bar{c}_2^2/(4c_2) + c_3$. Therefore, the closed loop signals $e(t)$ and $\tilde{W}(t)$ are UUB. □

Define $q(t) \equiv [e^{\mathrm{T}}(t), \mathrm{vec}(\tilde{W}(t))^{\mathrm{T}}]^{\mathrm{T}}$ and let $\mathcal{B}_r = \{q(t) : ||q(t)|| < r\}$, such that $\mathcal{B}_r \subset \mathcal{D}$ for a sufficiently large compact set $\mathcal{D}$. Then, we have the following corollary.

Corollary 1. Under the conditions of Theorem 3, an estimate for the ultimate bound is given by

$$r = \sqrt{\frac{\lambda_{\max}(P)\bar{\Theta}_e^2 + \lambda_{\max}(\Gamma^{-1}(t))\bar{\Theta}_w^2}{\lambda_{\min}(\bar{P})}}, \tag{47}$$

where $\bar{P} \triangleq \mathrm{diag}[P,\ \Gamma^{-1}(t)]$.

Proof. Denote $\Omega_\alpha = \{q(t) \in \mathcal{B}_r : q^{\mathrm{T}}(t)\bar{P}q(t) \leq \alpha\}$, $\alpha = \min_{||q(t)||=r} q^{\mathrm{T}}(t)\bar{P}q(t) = r^2\lambda_{\min}(\bar{P})$. Since

$$\begin{aligned}\mathcal{V}(\cdot) &= q^{\mathrm{T}}(t)\bar{P}q(t) \\ &= e^{\mathrm{T}}(t)Pe(t) + \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\Gamma^{-1}(t)\tilde{W}(t)],\end{aligned} \tag{48}$$

it follows that Ω_α is an invariant set as long as

$$\alpha \geq \lambda_{\max}(P)\bar{\Theta}_e^2 + \lambda_{\max}(\Gamma^{-1}(t))\bar{\Theta}_w^2. \tag{49}$$

Thus, the minimum size of $\mathcal{B}_r$ that ensures this condition has radius given by (47). The sets used in this proof are illustrated in Figure 2. □

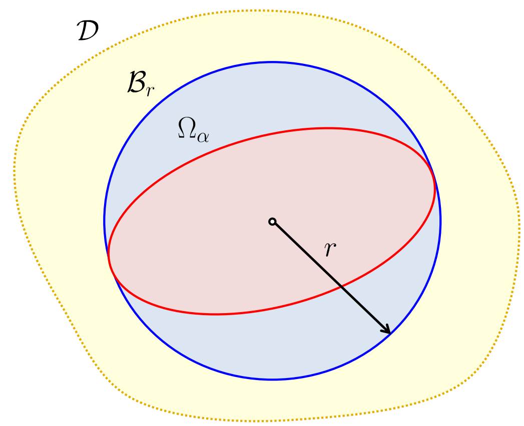

Fig. 2. Geometric representation of sets.

V. ILLUSTRATIVE EXAMPLE

In this section, we apply KFAC in (28), (29) and KFAC with Kalman filter based ALR (KFAC–ALR) in (29), (41), (43) to a model of wing rock dynamics [17] given by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\big[u(t) + \Delta(x(t))\big], \tag{50}$$

where $\Delta(x(t)) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3|x_1(t)|x_2(t) + \alpha_4|x_2(t)|x_2(t) + \alpha_5 x_1^3(t)$ with $\alpha_1 = 0.9814$, $\alpha_2 = 1.5848$, $\alpha_3 = -0.6245$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0215$. In (50), $x_1(t)$ represents the roll angle, and $x_2(t)$ represents the roll rate. In this example, the control objective is to track a square wave reference command. The reference system is selected to be second order with a natural frequency of 0.5 rad/s, and a damping of 0.707, and to have unity low frequency gain from $r(t)$ to $x_1(t)$. This corresponds to choosing $K_1 = [0.25,\ 0.707]$ and $K_2 = 0.25$. We chose $\beta(x(t)) = \left[1/(1+\mathrm{e}^{-x_1}),\ 1/(1+\mathrm{e}^{-x_2}),\ 0.1\right]$. The design parameters for KFAC are chosen to be $L = I_3$, $\sigma = 0.005$, $Q = 1000I_3$, $r = 1$, and $S(0) = 0.001I_3$. Furthermore, the design parameters for Kalman filter based ALR modification are chosen to be $Q = 6000I_3$, $r_A = 60$, and $S_A(0) = 0.001I_3$, when it is employed.

Figures 3 and 4 show the closed loop responses when the nominal controller is applied without uncertainty ($\Delta(x) = 0$). Note that the difference between $x(t)$ and $x_{\mathrm{m}}(t)$ is not distinguishable at this scale. Figures 5 and 6 show the closed loop responses when the nominal controller is applied to the uncertain system. In this case, the system response is unstable. Figures 7 and 8 present the results when KFAC is employed. From these results we were able to obtain satisfactory system performance in terms of tracking the square wave reference command. Figures 9 and 10 present the results when KFAC is applied to the uncertain system with 0.010 seconds of input time delay. In this case, we were not able to achieve a reasonable response. The responses obtained when KFAC–ALR is employed for the same case but with 0.025 seconds of input time delay are shown in Figures 11 and 12. Clearly, the sensitivity of KFAC–ALR to time delay is less than that of KFAC.

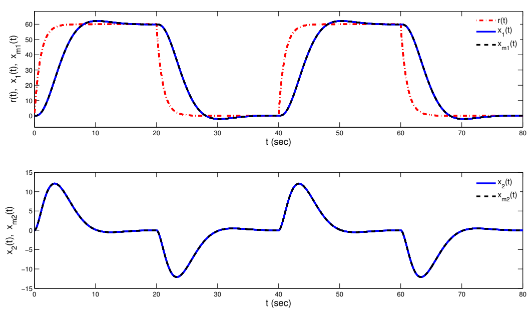

Fig. 3. Responses of reference input, state vector, and reference state vector with nominal controller without uncertainty.

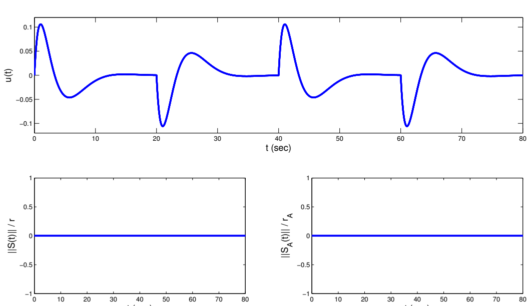

Fig. 4. Responses of control input, $\Gamma(t)$, and $\Gamma_A(t)$ with nominal controller without uncertainty.

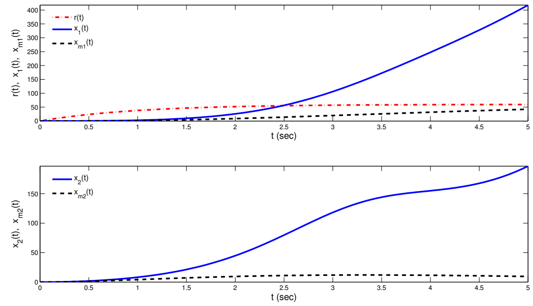

Fig. 5. Responses of reference input, state vector, and reference state vector with nominal controller for uncertain system.

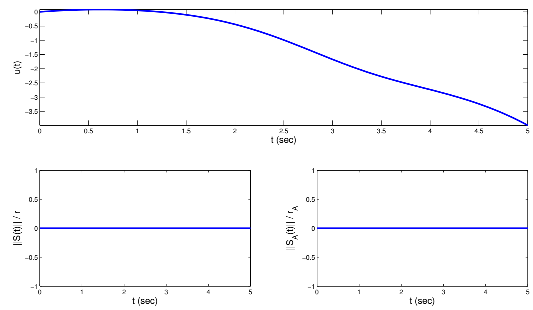

Fig. 6. Responses of control input, $\Gamma(t)$, and $\Gamma_A(t)$ with nominal controller for uncertain system.

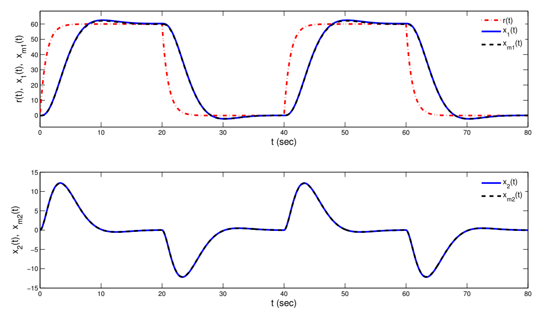

Fig. 7. Responses of reference input, state vector, and reference state vector with KFAC for uncertain system.

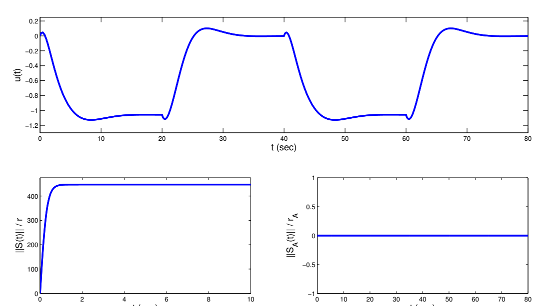

Fig. 8. Responses of control input, $\Gamma(t)$, and $\Gamma_A(t)$ with KFAC for uncertain system.

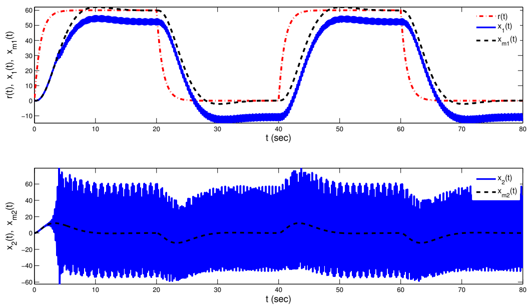

Fig. 9. Responses of reference input, state vector, and reference state vector with KFAC for uncertain system under 0.010 seconds of input time delay.

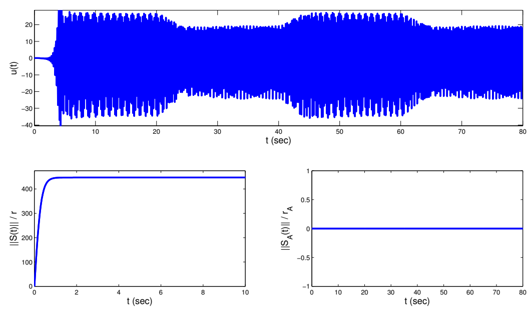

Fig. 10. Responses of control input, $\Gamma(t)$, and $\Gamma_A(t)$ with KFAC for uncertain system under 0.010 seconds of input time delay.

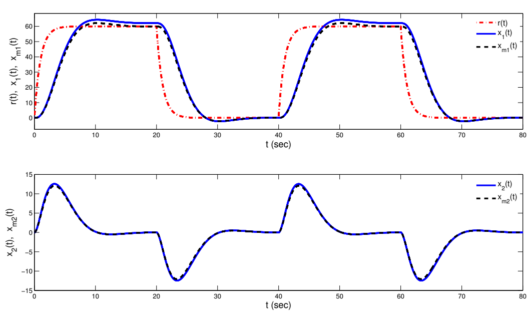

Fig. 11. Responses of reference input, state vector, and reference state vector with KFAC–ALR for uncertain system under 0.025 seconds of input time delay.

VI. Conclusion

The intent of this paper has been to present a Kalman filter based adaptive controller with loop transfer recovery of an associated reference system. A model of wing rock dynamics illustrates the presented theory. The new controller showed significant improvement when ALR modification is employed with respect to robustness to time-delay. The key properties are that the adaptation gain is optimization based, time-varying, does not require tuning, and increases the level of confidence of adaptive control systems.

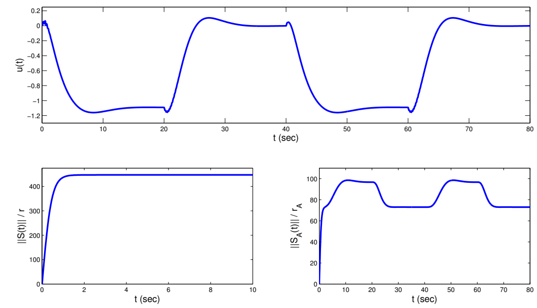

Fig. 12. Responses of control input, $\Gamma(t)$, and $\Gamma_A(t)$ with KFAC–ALR for uncertain system under 0.025 seconds of input time delay.

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