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# Distributed $H_\infty$ filtering over randomly switching networks

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*Abstract*— **The paper considers a distributed robust estimation problem over a network with randomly changing topology. The objective is to deal with these changes locally, by switching observer gains at affected nodes only. We propose sufficient conditions which guarantee a suboptimal $H_\infty$ level of disagreement of estimates in such observer networks, both in the mean-square sense and with probability 1. When the status of the network is known globally these sufficient conditions enable the network gains to be computed by solving certain LMIs. When the nodes are to rely on a locally available information about the network topology, additional rank constraints are used to condition the gains, given this information.**

## I. Introduction

One of the motivations for using distributed multisensor networks is to make the network resilient to loss of communication. This has led to an extensive research into the development of Kalman filtering techniques over unreliable communication networks [13] and networks with limited channel capacity [9]. Another emerging approach is concerned with distributed Kalman filtering over networks with time-varying and switching topology [14].

Recently, there has been a growing interest in Markovian switching models to describe estimator networks, where the probability of communication links to be active is governed by a Markovian switching rule [15], [12], [8]. Such models are widely used in the analysis of communication channels with random data loss.

Unless the governing Markov process is composed from independent two-state Markov processes describing the status of individual links [6], [12], [15], the majority of references employing Markovian network models rely on the assumption that the complete state of the underlying Markov chain is known to every controller or filter [7], [1]. In the context of distributed filtering, this assumption requires each node of the network to know the graph of the entire network in order to deploy suitable gains.

On the other hand, it was noted in [6] that modelling communication dropouts as independent Markov processes leads to LMI problems whose complexity grows exponentially; see e.g. [12], [15]. The issue was addressed in [6] by further assuming that while the links are statistically independent of each other the dropouts are governed by unknown (hence *arbitrary*) Markovian probability distributions. However the assumption of independence between communication links is not satisfied in a number of practical situations, for example, when a communication path between two network hubs is lost and packets are being re-routed; this may trigger congestion and dropouts in other areas of the network. The random processes describing the status of each node are interdependent in this case. Furthermore, such processes may not be Markov even if the overall network model is Markovian. The objective of this paper is to develop distributed filtering technique which overcomes the need for broadcast of global communication topology and does not require Markovian segmentation of the network.

The main contribution of this paper is the methodology of distributed $H_\infty$ filter design which enables filters to be implemented in a truly distributed fashion, by utilizing only locally available information about the system's connectivity, without assuming independent status of communication links. This design constraint referred to as the 'locality constraint', is an important distinction of our methodology, compared with, e.g., [15], [6]. To overcome difficulties arising from partial knowledge of changing graph connectivity and non-Markovian nature of the process $\eta_i(t)$, we use the approach previously developed for problems of decentralized control of jump parameter systems [18]. It involves a two-step design procedure. First, an auxiliary problem is solved under simplifying assumption that the complete network topology instantaneously available at each node. At the second step, conditioning on the locally available information is carried out.

In comparison with [15], our approach enables the node estimators to reach relative $H_\infty$ consensus about the estimate of the reference plant. This allows each filter to track the reference state even when the reference is unobservable from node's measurements. To achieve this, we extend the vector dissipativity theory [5] and the related LMI technique [16] to Markovian jump parameter systems; see also [17]. This enables dissipativity of a large-scale Markovian jump parameter system describing evolution of estimation errors to be studied using vector storage functions and vector supply rates. We establish both mean-square convergence and convergence with probability 1 of the distributed filters under consideration.

## II. Problem formulation

### A. Networks with Markovian switching topology

To describe the class of switching networks of estimators under consideration we begin with a directed weakly connected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V} = \{1, \ldots, N\}$ is the set of nodes, and $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$ is the set of edges of $\mathbf{G}$. The edge originating at node $j$ and ending at node $i$, will be denoted $(j, i)$. The graph does not contain self-loops, i.e., $(i, i) \notin \mathbf{E}$.

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A subset $\mathbf{B}(t) \subseteq \mathbf{V}$ consisting of nodes supplying information to node $i$ at a particular time instant $t$ forms an instantaneous neighborhood of $i$. In this paper we consider switching networks, so the instantaneous neighbourhood of each node can change with time. Also, we allow for the situation where sensors at each node can operate in several regimes, which are chosen according to a probability law. Hence, at every time instant, the state of each node's neighbourhood is determined by both the communication links and the sensing regime active at this time instant. Formally, we say that two neighbourhoods $\mathbf{B}(t_1)$, $\mathbf{B}(t_2)$ of node $i$, are identical if $\mathbf{B}(t_1) = \mathbf{B}(t_2)$ and the sensing regime of node $i$ at time $t_1$ is the same as that at time $t_2$. Otherwise, we say that these neighbourhoods are distinct.

Suppose that node $i$ can have $M_i$ distinct neighbourhoods denoted $\mathbf{V}_i^1, \ldots, \mathbf{V}_i^{M_i}$. The dynamic of the neighbourhood set of $i$ can be described as a mapping $\eta_i : [0, \infty) \to \mathscr{I}_i$, where $\mathscr{I}_i \triangleq \{1, \ldots, M_i\}$ is the index set of the collection of distinct neighbourhoods.

We now look at the entire network. At any time, it can be uniquely reconstructed by analysing the collection of neighbourhoods of all its nodes, $(\mathbf{V}_1^{m_1}, \ldots, \mathbf{V}_N^{m_N})$, $m_i \in \mathscr{I}_i$, $i = 1, \ldots, N$. Thus, the instantaneous network graph can be uniquely associated with an $N$-tuple $(m_1, \ldots, m_N)$, $m_i \in \mathscr{I}_i$. Owing to dependencies between the links, the number of admissible graph topologies, $M$, may be substantially lower than the maximum number $\prod_{i=1}^N M_i$ of all $N$-tuples. Let us consider all distinct *admissible* graph topologies $\mathbf{G}^m = (\mathbf{V}, \mathbf{E}^m)$, $m \in \mathscr{I} = \{1, \ldots, M\}$, $\mathbf{E}^m \subseteq \mathbf{E}$. Each digraph $\mathbf{G}^m$ is a subgraph of $\mathbf{G}$. The set of these subgraphs will be denoted $\mathscr{G} = \{\mathbf{G}^m, m \in \mathscr{I}\}$. Clearly there is a one-to-one mapping between the index set $\mathscr{I}$ and the set of all admissible $N$-tuples $(m_1, \ldots, m_N)$; it will be denoted $\Phi$: $(m_1, \ldots, m_N) = \Phi(m)$. An example illustrating how such a mapping can be constructed can be found in [18]. Also, we will write $m_i = \Phi_i(m)$, whenever $(m_1, \ldots, m_N) = \Phi(m)$.

Let $\mathbf{A}^m = [\mathbf{a}_{ij}^m]_{i,j=1,N}$ be the adjacency matrix of the digraph $\mathbf{G}^m$, i.e., $\mathbf{a}_{ij}^m = 1$ if $(j, i) \in \mathbf{E}^m$, otherwise $\mathbf{a}_{ij}^m = 0$. When the network is in configuration $\mathbf{G}^m$, the cardinality of the neighbourhood of node $i$, known as the in-degree of node $i$, is equal to $p_i^m = \sum_{j=1}^N \mathbf{a}_{ij}^m$, and the out-degree of node $i$ is $q_i^m = \sum_{j=1}^N \mathbf{a}_{ji}^m$. It is worth noting that

$$\mathbf{a}_{ij}^m = \begin{cases} 1 & \text{if } j \in \mathbf{V}_i^{\Phi_i(m)}; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Using the one-to-one mapping $\Phi$, the dynamics of the network topology can be described as the process $\eta(t) = \Phi^{-1}(\eta_1(t), \ldots, \eta_N(t))$, so that at each time instance, $\eta_i(t) = \Phi_i(\eta(t))$. Throughout this paper, it will be assumed that $\{\eta(t), t \geq 0\}$ is a stationary Markov random process taking values in the finite set $\mathscr{I}$; cf. [8]. It is defined in a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathsf{P})$, where $\mathscr{F}_t$ denotes the minimal right-continuous filtration with respect to which $\{\eta(t), t \geq 0\}$ is adapted [3]. Also, the $\sigma$-algebra $\mathscr{F}$ is the minimal $\sigma$-algebra which contains all measurable sets from the filtration $\{\mathscr{F}_t\}$. The transition probability rate matrix of the Markov chain $\{\eta(t), t \geq 0\}$ is denoted $\Lambda = [\lambda_{kl}]_{k,l=1}^M$, with $\lambda_{kl} \geq 0, k \neq l$ and $\lambda_{kk} = -\sum_{l \neq k} \lambda_{kl} \leq 0, \forall k \in \mathscr{I}$ [3]. In Section IV it will be further assumed that the Markov process $\eta(t)$ is irreducible[1] and has an invariant distribution $\bar{\lambda}$; i.e., there exists a row vector $\bar{\lambda}$, $\bar{\lambda}_l \geq 0$, $\sum_{l=1}^M \bar{\lambda}_l = 1$, such that $\bar{\lambda}\Lambda = \bar{\lambda}$. For irreducible Markov processes, if an invariant distribution exists, it is unique.

We will use the notation $(\eta, \mathscr{G}, \Phi)$ to refer to the switching network described above. Since $\eta(t)$ is stationary, then each process $\eta_i(t)$ is also stationary. However, in general $\eta_i(t)$ is not Markov, and the components of the multivariate process $(\eta_1(t), \ldots, \eta_M(t))$ may statistically depend on each other.

Throughout the paper $\mathsf{P}^{x_0,m_0}$, $\mathsf{E}^{x_0,m_0}$ denote, respectively, the conditional probability and conditional expectation, given $x(0) = x_0$, $\eta(0) = m_0$. Also, $L_2[0, \infty)$ denotes the Lebesgue space of vector-valued deterministic processes with the property $\|z\|^2 = \int_0^\infty \|z(t)\|^2 dt < \infty$.

### B. $H_\infty$ consensus estimation

Consider a plant described by the equation

$$\dot{x} = Ax + B_2\xi(t), \quad x(0) = x_0, \tag{2}$$

Here $x \in \mathbf{R}^n$ is the state, $\xi(t) \in \mathbf{R}^l$ is a deterministic disturbance, $\xi(\cdot) \in L_2[0, \infty)$. We assume $\xi(\cdot)$ to be such that $x(t)$ exists on any finite interval $[0, T]$.

Also, consider a switching network $\{\eta, \mathscr{G}, \Phi\}$ of filters. Suppose each node takes measurements

$$y_i = \tilde{C}_i(\eta_i(t))x + \tilde{D}_i(\eta_i(t))\xi + \tilde{\bar{D}}_i(\eta_i(t))\xi_i, \quad y_i \in \mathbf{R}^r, \tag{3}$$

where $\xi_i(t) \in \mathbf{R}^{l_i}$ represents the measurement uncertainty at sensing node $i$, $\xi_i(\cdot) \in L_2[0, \infty)$. The coefficients of equation (3) take values in given sets of constant matrices of compatible dimensions, $\tilde{C}_i(\eta_i(t)) \in \mathbf{C}_i \triangleq \{C_i^k, k \in \mathscr{I}_i\}$, $\tilde{D}_i(\eta_i(t)) \in \mathbf{D}_i \triangleq \{D_i^k, k \in \mathscr{I}_i\}$, $\tilde{\bar{D}}_i(\eta_i(t)) \in \bar{\mathbf{D}}_i \triangleq \{\bar{D}_i^k, k \in \mathscr{I}_i\}$. It is assumed that $E_i^k = D_i^k(D_i^k)' + \bar{D}_i^k(\bar{D}_i^k)' > 0$ for all $i$ and $k \in \mathscr{I}_i$. We allow the coefficients in (3) to depend on $\eta_i$ to capture the situation where the node's sensors need to be adjusted in response to the changes in the node's communications. But there may be other situations which may require the sensing regime to change.

The measurements $y_i$ are processed at node $i$ according to the following estimation algorithm (cf. [15]):

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + \tilde{L}_i(\eta_i(t))(y_i(t) - \tilde{C}_i(\eta_i(t))\hat{x}_i) \\ &\quad + \sum_{j \in \mathbf{V}_i^{\eta_i(t)}} \tilde{K}_{ij}(\eta_i(t))(v_{ij} - H_{ij}\hat{x}_i), \quad \hat{x}_i(0) = 0, \end{aligned} \tag{4}$$

where $v_{ij}$ is the signal received at node $i$ from node $j$,

$$v_{ij} = H_{ij}\hat{x}_j + G_{ij}w_{ij}, \tag{5}$$

$w_{ij} \in L_2[0, \infty)$ is a disturbance affecting the information transmission from node $j$ to $i$, and $\tilde{L}_i(\cdot)$, $\tilde{K}_{ij}(\cdot)$ are the matrix valued functions defined on the index set $\mathscr{I}_i$. It is

[1] A continuous-time stationary Markov chain is irreducible if for any two states $m, k$ there exists $t > 0$ such that $\mathsf{P}(\eta(t) = m | \eta(0) = k) > 0$.

assumed that $F_{ij} = G_{ij}G'_{ij} > 0$ for all $i$ and $j \in \mathbf{V}_i^{m_i}$, $m_i \in \mathscr{I}_i$.

*Remark 1:* In equation (5), the matrices $H_{ij}$ and $G_{ij}$ do not depend on $\eta_i(t)$. On the other hand, the innovation and coupling gains $\tilde{L}_i(\cdot)$, $\tilde{K}_{ij}(\cdot)$ depend on the node's neighbourhood and its sensing regime at time $t$. This is to reflect a situation where node $j$ *always* broadcasts its information to node $i$, but node $i$ may fail to receive this information, or may choose not to accept it, e.g. due to congestion. Therefore, the summation in (4) is over *time-dependent* neighbourhoods. It is possible to consider a more general situation where the matrices $H_{ij}$ and $G_{ij}$ also depend on $\eta_i(t)$. Technically, this more general case is no different from the one pursued here.

At each time instance the node estimators (4) receive information only from the neighbours they are currently connected to. Therefore, the coupling and observer gains $\tilde{L}_i(\cdot)$, $\tilde{K}_i(\cdot)$ are sought to be functions of the state of the process $\eta_i$, rather than $\eta$. This enables the estimator nodes to operate by relying on a partial information about the topology of the communication graph encoded in the 'local' process $\eta_i$. This is an additional 'locality constraint', compared to the standard Markovian communication model, where the complete communication graph is assumed to be known at each node [15], [1].

We now formulate the distributed estimation problem with $H_\infty$ consensus of estimates under consideration. Associated with the system (2) and the set of filters (4) is the disagreement function (cf. [10])

$$\Psi(\hat{x}, m) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathbf{V}^{\Phi_i(m)}}\|\hat{x}_j - \hat{x}_i\|^2, \quad m \in \mathscr{I}, \quad (6)$$

$\hat{x} = [\hat{x}'_1 \ \dots \ \hat{x}'_N]'$. Clearly, $\Psi(\hat{x}, m) = \Psi(e, m)$, where $e = [e'_1 \ e'_2 \ \dots \ e'_N]'$, and $e_i = x - \hat{x}_i$ is the local estimation error at node $i$.

*Definition 1:* The distributed filtering problem under consideration is concerned with determining the sets of switching observer gains $\tilde{L}_i^k$ and interconnection coupling gains $\tilde{K}_i^k$, $k \in \mathscr{I}_i$, for the filters (4) which ensure that the following conditions are satisfied when we set $\tilde{L}_i(\eta_i(t)) = \tilde{L}_i^{\eta_i(t)}$, $\tilde{K}_{ij}(\eta_i(t)) = \tilde{K}_{ij}^{\eta_i(t)}$ for all $i = 1, \dots, N$ and $j \in \mathbf{V}^{\eta_i(t)}$:

(i) In the absence of the uncertainty, all node estimators converge exponentially in the mean-square sense and converge asymptotically with probability 1:

$$\mathsf{E}^{x_0,m_0}\|e_i(t)\|^2 \le ce^{-\epsilon t}, \quad (\exists c, \epsilon > 0),$$
$$\mathsf{P}^{x_0,m_0}(\lim_{t\to\infty}\|e_i(t)\|^2 = 0) = 1.$$

(ii) Given a constant $\gamma > 0$, the filter ensures the following mean-square $H_\infty$ consensus performance

$$\sup_{x_0,(\xi,\xi_i,w_{ij})\neq 0}\frac{\mathsf{E}^{x_0,m_0}\int_0^\infty \Psi(e(t),\eta(t))dt}{\mu(x_0,\xi,[\xi_i,w_{ij}]_{i,j=1,\dots,N})} \le \gamma^2, \quad (7)$$

$$\mu(x_0,\xi,[\xi_i,w_{ij}]_{i,j=1,\dots,N}) \triangleq \|x_0\|_P^2 + \|\xi\|_2^2 + \frac{1}{N}\sum_{i=1}^{N}\left(\|\xi_i\|_2^2 + \sum_{j=1}^{M}\|(\bar{\mathbf{a}}_{ij}(\cdot))^{1/2}w_{ij}(\cdot)\|_2^2\right).$$

Here, $\bar{\mathbf{a}}_{ij}(t) \triangleq \mathsf{E}^{m_0}\mathbf{a}_{ij}^{\eta_i(t)}$, $\|x_0\|_P^2 \triangleq x'_0Px_0$, $P = P' > 0$ is a fixed matrix to be determined.

(iii) All estimators converge in the mean-square and with probability 1:

$$\lim_{t\to\infty}\mathsf{E}^{x_0,m_0}\sum_{i=1}^{N}\|x(t) - \hat{x}_i(t)\|^2 = 0, \quad (8)$$

$$\mathsf{P}^{x_0,m_0}(\lim_{t\to\infty}\|x(t) - \hat{x}_i(t)\|^2 = 0) = 1. \quad (9)$$

To overcome difficulties arising from partial knowledge of the graph connectivity and non-Markovian nature of the process $\eta_i(t)$, we adopt the two-step approach recently proposed in [18]. First, in the next section we consider an auxiliary distributed estimation problem. This problem does not involve the 'locality constraint', but we will seek a solution to this problem using a network of *uncertain* estimators. Then in Section IV, we replace this uncertain estimator network with an estimator network which satisfies the 'locality constraint' and retains performance of the auxiliary design.

## III. An auxiliary global network dependent distributed estimator

In this section, we focus on the network governed by the Markov process $\eta(t)$. Let us define matrices $C_i^k$, $D_i^k$, $\bar{D}_i^k$, $k \in \mathscr{I}$, as follows

$$C_i^k = \tilde{C}_i^{\Phi_i(k)}, \quad D_i^k = \tilde{D}_i^{\Phi_i(k)}, \quad \bar{D}_i^k = \bar{\tilde{D}}_i^{\Phi_i(k)}. \quad (10)$$

By letting $C_i(\eta(t)) = C_i^{\eta(t)}$, $D_i(\eta(t)) = D_i^{\eta(t)}$, $\bar{D}_i(\eta(t)) = \bar{D}_i^{\eta(t)}$, the measurements at node $i$ can be expressed as the function of $\eta(t)$:

$$y_i = C_i(\eta(t))x + D(\eta_i(t))\xi + \bar{D}_i(\eta(t))\xi_i. \quad (11)$$

The auxiliary problem in this section is concerned with estimation of the state of the uncertain plant (2), (11) using a network of *uncertain* node estimators of the form

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + L_i(\eta(t))(y_i(t) - C_i(\eta(t))\hat{x}_i) \\ &\quad + \sum_{j\in\mathbf{V}^{\Phi_i(\eta(t))}} K_{ij}(\eta(t))(v_{ij} - \hat{x}_i) \\ &\quad + \sum_{j\in\mathbf{V}^{\Phi_i(\eta(t))}}(\omega_{ij}^{(1)} + \omega_{ij}^{(2)}) + \omega_i, \quad \hat{x}_i(0) = 0. \quad (12)\end{aligned}$$

Here, $L_i(\cdot)$, $K_i(\cdot)$ are matrix valued functions of the state of the Markov chain $\eta$ to be found, and $\omega_{ij}^{(1)}$, $\omega_{ij}^{(2)}$, and $\omega_i$ are estimator perturbations. It is assumed that these perturbations are random processes adapted to the filtration $\{\mathscr{F}_t, t \ge 0\}$ and such that the multivariate process $(\hat{x}_1, \dots, \hat{x}_N, \eta)$ is Markov with respect to that filtration. The introduction of these perturbations is a critical step to allow us to replace, at a later stage, the gains dependent on the global network configuration with localized ones. In this section, we assume that these perturbations satisfy the constraints:

$$\begin{aligned}&\|\omega_i(t)\|^2 \\ &\quad \le \alpha_i^2\left\|C_i(\eta(t))e_i(t) + D_i(\eta(t))\xi(t) + \bar{D}_i(\eta(t))\xi_i(t)\right\|^2, \\ &\|\omega_{ij}^{(1)}(t)\|^2 \le \beta_{ij}^2\left\|H_{ij}e_i(t) + G_{ij}w_{ij}\right\|^2, \\ &\|\omega_{ij}^{(2)}(t)\|^2 \le \beta_{ij}^2\left\|H_{ij}e_j(t)\right\|^2 \quad \text{a.s. } \forall t \ge 0, \quad (13)\end{aligned}$$

where $\alpha_i$, $\beta_{ij}$ are given constants, and $e_i = x - \hat{x}_i$ is the estimation error of the auxiliary estimator at node $i$. Since $x(t)$ is a deterministic process, then the multivariate process $(e_1, \ldots, e_N, \eta)$ is Markovian (since $(\hat{x}_1, \ldots, \hat{x}_N, \eta)$ is Markovian due to the assumption on $\omega_{ij}^{(1)}$, $\omega_{ij}^{(2)}$, $\omega_i$.

*Definition 2:* The auxiliary distributed consensus estimation problem is to determine the sets of switching observer gains $L_i^m$ and interconnection coupling gains $K_i^m$, $m \in \mathscr{I}$, for the filters (12) which ensure that the following conditions are satisfied when we set $L_i(\eta(t)) = L_i^{\eta(t)}$, $K_{ij}(\eta(t)) = K_{ij}^{\eta(t)}$ for all $i = 1, \ldots, N$ and $j \in \mathbf{V}^{\Phi_i(\eta(t))}$.

(i) When $(\xi, \xi_i, w_{ij}) \equiv 0$, the large-scale system consisting of subsystems describing evolution of estimation errors in the auxiliary problem under consideration must be exponentially stable in the mean-square sense and asymptotically stable with probability 1 for all estimator perturbations $\omega_{ij}^{(1)}$, $\omega_{ij}^{(2)}$, and $\omega_i$ for which correspondingly modified constraints (13) hold.

(ii) In the presence of exogenous disturbances $\xi, \xi_i, w_{ij}$, the mean-square consensus performance condition in (7) is satisfied for all admissible estimator perturbations $\omega_{ij}^{(1)}$, $\omega_{ij}^{(2)}$, and $\omega_i$ subject to (13).

(iii) All estimators converge in the mean-square and with probability 1.

To justify the introduction of the auxiliary distributed consensus estimation problem, suppose a set of 'neighbourhood dependent' gains $\tilde{L}_i^l, \tilde{K}_{ij}^l$, $i = 1, \ldots, N$, $l \in \mathscr{I}_i$, $j \in \mathbf{V}_i^l$, is selected for each node and each distinct neighbourhood, and consider the network of estimators (4) constructed with these gains. Using (10) and (1), equation (4) can be rewritten as follows

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + L_i(\eta(t))(y_i(t) - C_i(\eta(t))\hat{x}_i) \\ &+ \sum_{j=1}^{N} \mathbf{a}_{ij}^{\eta(t)} K_{ij}(\eta(t))(v_{ij} - \hat{x}_i) \\ &+ (\tilde{L}_i(\eta_i(t)) - L_i(\eta(t)))(y_i(t) - C_i(\eta(t))\hat{x}_i) \\ &+ \sum_{j=1}^{N} \mathbf{a}_{ij}^{\eta(t)} (\tilde{K}_{ij}(\eta_i(t))) - K_{ij}(\eta(t))(v_{ij} - \hat{x}_i), \end{aligned} \quad (14)$$

where $K_i(\eta(t))$, $L_i(\eta(t))$ are random matrix-valued processes with values in a certain set of matrices $\{K_i^m, L_i^m, m \in \mathscr{I}\}$ to be determined. Equation (14) is a particular case of the uncertain estimator (12) in which perturbation processes are of particular form

$$\begin{aligned}\omega_i &= (\tilde{L}_i(\eta_i(t)) - L_i(\eta(t))) \\ &\quad \times (C_i(\eta(t))e_i(t) + D_i(\eta(t))\xi + \bar{D}_i(\eta(t))\xi_i), \\ \omega_{ij}^{(1)} &= (\tilde{K}_{ij}(\eta_i(t))) - K_{ij}(\eta(t))(H_{ij}e_i + G_{ij}w_{ij}), \\ \omega_{ij}^{(2)} &= -(\tilde{K}_{ij}(\eta_i(t))) - K_{ij}(\eta(t))H_{ij}e_j. \end{aligned} \quad (15)$$

Suppose now that for every node $i = 1, \ldots, N$ and every admissible neighbourhood configuration $\mathbf{V}_i^k$, $k \in \mathscr{I}_i$, 'local' gains $\tilde{L}_i^k$, $\tilde{K}_{ij}^k$, $j \in \mathbf{V}_i^k$, are chosen which satisfy the conditions

$$\begin{aligned}&\|\tilde{L}_i^{\Phi_i(m)} - L_i^m\|^2 \le \alpha_i^2, \\ &\|\tilde{K}_{ij}^{\Phi_i(m)} - K_{ij}^m\|^2 \le \beta_{ij}^2, \quad \forall m \in \mathscr{I}. \end{aligned} \quad (16)$$

It follows from (16) that the perturbations (15) satisfy (13). Therefore the estimator (4) in which this particular set of localized gains is employed, represents one instance of the auxiliary estimator (12), corresponding to the particular perturbation (15). Furthermore, if the matrices $K_i^m, L_i^m, m \in \mathscr{I}$ solve the auxiliary $H_\infty$ consensus estimation problem in Definition 2, then the distributed estimator (4) with the local gains selected above, satisfying (16), solves the robust consensus estimation problem in Definition 1, using only locally available information about the network topology.

This discussion motivates us to solve the auxiliary problem in Definition 2 as a stepping stone towards solving the problem posed in Section II. This solution is given in Lemma 1 below. The conditions of the lemma involve the following linear matrix inequalities in the variables $\tau_i^k > 0$, $\theta_{ij}^k > 0$, $\vartheta_{ij}^k > 0$, $X_i^k = (X_i^k)' > 0$, $i = 1, \ldots, N$, $k = 1, \ldots, M$, $j \in \mathbf{V}_i^{\Phi_i(k)}$:

$$\gamma^2 I - \tau_i^k \alpha_i^2 E_i^k > 0, \quad \gamma^2 I - \theta_{ij}^k \beta_{ij}^2 F_{ij} > 0, \quad (17)$$

$$\begin{bmatrix} Q_i^k & \star & \star & \star & \star \\ N_i^k & -\gamma^2 I & \star & \star & \star \\ S_i^k & 0 & -\gamma^2 I & \star & \star \\ \mathbf{1}_{1+2M_i} \otimes X_i^k & 0 & 0 & -\mathbf{T}_i & \star \\ \Xi_i' & 0 & 0 & 0 & -Z_i \end{bmatrix} < 0, \quad (18)$$

where $\mathbf{1}_s \triangleq [1 \ \ldots \ 1]' \in \mathbf{R}^s$, $\otimes$ denotes the Kronecker product, and $N_i^k \triangleq \left(I - (D_i^k)'(E_i^k)^{-1} D_i^k\right) B_2' X_i^k$, $S_i^k \triangleq -(\bar{D}_i^k)'(E_i^k)^{-1} D_i^k B_2' X_i^k$,

$$\mathbf{T}_i^k \triangleq \mathrm{diag}\left[\tau_i^k, \ \theta_{i,j_1}^k, \ \ldots, \ \theta_{i,j_{p_i^k}}^k, \ \vartheta_{i,j_1}^k, \ \ldots, \ \vartheta_{i,j_{p_i^k}}^k\right],$$

$$\begin{aligned} Q_i^k &\triangleq X_i^k (A + \delta_i I - B_2 (D_i^k)'(E_i^k)^{-1} C_i^k) \\ &+ (A + \delta_i I - B_2 (D_i^k)'(E_i^k)^{-1} C_i^k)' X_i^k + (p_i^k + q_i^k) I \\ &+ \sum_{j: i \in \mathbf{V}_j^{\Phi_j(k)}} \vartheta_{ji}^k \beta_{ji}^2 H_{ji}' H_{ji} + \sum_{l=1}^{M} \lambda_{kl} X_i^l \\ &- \gamma^2 (C_i^k)'(E_i^k)^{-1} C_i^k - \gamma^2 \sum_{j \in \mathbf{V}_i^{\Phi_i(k)}} H_{ij}' F_{ij} H_{ij}, \end{aligned}$$

$$\Xi_i = \left[\gamma^2 H_{ij_1}' F_{ij_1}^{-1} H_{ij_1} - I \ \ldots \ \gamma^2 H_{ij_{p_i^k}}' F_{ij_{p_i^k}}^{-1} H_{ij_{p_i^k}} - I\right],$$

$$Z_i = \mathrm{diag}\left[\frac{2\delta_{j_1}}{q_{j_1}^k + 1} X_{j_1}^k, \ \ldots, \ \frac{2\delta_{j_{p_i^k}}}{q_{j_{p_i^k}}^k + 1} X_{j_{p_i^k}}^k\right].$$

*Lemma 1:* Suppose the network $(\eta, \mathscr{G}, \Phi)$ and the constants $\gamma > 0$, $\alpha_i$, $\beta_{ij}$ and $\delta_i > 0$ are such that the coupled LMIs (17) and (18) in the variables $\tau_i^k > 0$, $\theta_{ij}^k > 0$, $\vartheta_{ij}^k > 0$, $X_i^k = (X_i^k)' > 0$, $j \in \mathbf{V}_i^{\Phi_i(k)}$, $i = 1, \ldots, N$, $k = 1, \ldots, M$, are feasible. Then the network of observers (12) with

$$\begin{aligned} K_{ij}^k &= \gamma^2 (X_i^k)^{-1} H_{ij}' F_{ij}^{-1}, \quad (19) \\ L_i^k &= \left[\gamma^2 (X_i^k)^{-1} (C_i^k)' + B_2 (D_i^k)'\right] (E_i^k)^{-1} \quad (20) \end{aligned}$$

solves the auxiliary estimation problem in Definition 2. The matrix $P$ in conditions (7) and (8) corresponding to this solution is $P = \frac{1}{N}\sum_{i=1}^{N} X_i^{m_0}$, where $m_0 = \eta(0)$.

As a by-product of Lemma 1 for the case where $\omega_i = \omega_{ij}^{(1)} = \omega_{ij}^{(2)} = 0$ and $\alpha_i = \beta_{ij} = 0$, we obtain the following result concerning distributed $H_\infty$ consensus estimation over networks with Markovian switching topology.

*Corollary 1:* Suppose the network $(\eta, \mathscr{G}, \Phi)$ and the constants $\gamma > 0$ and $\delta_i > 0$ are such that the coupled LMIs in the variables $X_i^k = (X_i^k)' > 0$, $j \in \mathbf{V}_i^{\Phi_i(k)}$, $i = 1, \ldots, M$, $k = 1, \ldots, M$,

$$\begin{bmatrix} \bar{Q}_i^k & \star & \star & \star \\ N_i^k & -\gamma^2 I & \star & \star \\ S_i^k & 0 & -\gamma^2 I & \star \\ \Xi_i' & 0 & 0 & -Z_i \end{bmatrix} < 0, \quad (21)$$

$$\begin{aligned} \bar{Q}_i^k &\triangleq X_i^k(A + \delta_i I - B_2(D_i^k)'(E_i^k)^{-1}C_i^k) \\ &+(A + \delta_i I - B_2(D_i^k)'(E_i^k)^{-1}C_i^k)'X_i^k + (p_i^k + q_i^k)I \\ &+\sum_{l=1}^{M}\lambda_{kl}X_i^l - \gamma^2(C_i^k)'(E_i^k)^{-1}C_i^k - \gamma^2\sum_{j\in\mathbf{V}_i^{\Phi_i(k)}} H_{ij}'F_{ij}H_{ij} \end{aligned}$$

are feasible. Then the network of observers (12) with $\omega_i = \omega_{ij}^{(1)} = \omega_{ij}^{(2)} = 0$ and $K_{ij}^k$, $L_i^k$ defined in (19), (20) solves the estimation problem in Definition 2. The matrix $P$ in conditions (7) and (8) corresponding to this solution is $P = \frac{1}{N}\sum_{i=1}^{N} X_i^{m_0}$, where $m_0 = \eta(0)$.

It is worth noting that unlike similar LMIs in [15, Lemma 3], the LMIs (21) are partitioned in a way which makes possible to solve them in a fully distributed manner using gradient descent algorithms such as that proposed in [16].

## IV. The Main Result

In this section, the solution to the auxiliary distributed estimation problem developed in Section III will be used to obtain a distributed estimator whose nodes utilize only locally available information. This will be achieved by taking the expectation of the auxiliary node filters conditioned on the node neighbourhood as time approaches infinity. Computationally, our method is based on the following technical result of [18].

*Proposition 1:* Suppose the Markov process $\eta(t)$ is irreducible and has a unique invariant distribution $\bar{\lambda}$. Given a matrix-valued function $K(\cdot) : \mathscr{I} \to \{K^1 \ldots, K^M\} \subset \mathbf{R}^{n\times s}$, for every node $i$ and for all $k \in \mathscr{I}_i$ we have:

$$\lim_{t\to\infty} \mathsf{E}\left(K(\eta(t)) \mid \eta_i(t) = k\right) = \frac{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l K^l}{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l}. \quad (22)$$

Now let $K_{ij}^m$, $L_i^m$, $m \in \mathscr{S}$ be the coefficients of the auxiliary distributed estimator obtained in Lemma 1. Using Proposition 1, define $\tilde{K}_{ij}(\eta_i(t)) = \tilde{K}_{ij}^{\eta_i(t)}$, $\tilde{L}_i(\eta_i(t)) = \tilde{L}_i^{\eta_i(t)}$, where for each $i = 1, \ldots, N$ and $k \in \mathscr{I}_i$, we let

$$\tilde{K}_{ij}^k = \frac{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l K_{ij}^l}{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l}, \quad \tilde{L}_i^k = \frac{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l L_i^l}{\sum_{l:\Phi_i(l)=k}\bar{\lambda}_l}. \quad (23)$$

From Proposition 1, the processes $\tilde{K}_{ij}(\eta_i(t)), \tilde{L}_i(\eta_i(t))$ are the asymptotic minimum variance approximations of the corresponding processes $K_{ij}(\eta(t)), L_i(\eta(t))$ [2].

We have for all $i = 1, \ldots, N$, $m \in \mathscr{I}$, $m_i = \Phi_i(m)$, and $j \in \mathbf{V}_i^{m_i}$,

$$K_{ij}^m - \tilde{K}_{ij}^{m_i} = \frac{\sum_{l:l\neq m,\Phi_i(l)=m_i}\bar{\lambda}_l\left[K_{ij}^m - K_{ij}^l\right]}{\sum_{l:\Phi_i(l)=m_i}\bar{\lambda}_l}, \quad (24)$$

$$L_i^m - \tilde{L}_i^{m_i} = \frac{\sum_{l:l\neq m,\Phi_i(l)=m_i}\bar{\lambda}_l\left[L_i^m - L_i^l\right]}{\sum_{l:\Phi_i(l)=m_i}\bar{\lambda}_l}. \quad (25)$$

Note that the expressions on the right-hand side of (24) and (25) are linear matrix functions of $K_{ij}^m$ and $L_i^m$, respectively. This enables the bounds on $\|L_i(\eta(t)) - \tilde{L}_i(\eta_i(t))\|^2$, $\|K_{ij}(\eta(t)) - \tilde{K}_{ij}(\eta_i(t))\|^2$ to be established in the LMI form. Indeed, consider the collection of the rank-constrained LMIs in the variables $\tau_i^k$, $\theta_{ij}^k$, $\vartheta_{ij}^k$, $X_i^k$ and $Y_i^k$, consisting of the LMIs (17), (18), and the following additional LMIs,

$$\begin{bmatrix} \alpha_i^2 I & \Delta_i^{L,k} \\ (\Delta_i^{L,k})' & I \end{bmatrix} > 0, \quad \begin{bmatrix} \beta_{ij}^2 I & \Delta_{ij}^{K,k} \\ (\Delta_{ij}^{K,k})' & I \end{bmatrix} > 0, \quad (26)$$

$$\operatorname{rank}\begin{bmatrix} Y_i^k & I \\ I & X_i^k \end{bmatrix} \leq n, \quad (27)$$

where $\alpha_i$, $\beta_{ij}$ are the same constants as those employed in the LMIs (17), (18), and

$$\Delta_i^{L,k} \triangleq \frac{\sum_{\substack{l:l\neq k,\\ \Phi_i(l)=m_i}} \gamma^2\bar{\lambda}_l\left[Y_i^k(C_i^k)'(E_i^k)^{-1} - Y_i^l(C_i^l)'(E_i^l)^{-1}\right]}{\sum_{l:\Phi_i(l)=m_i}\bar{\lambda}_l},$$

$$\Delta_{ij}^{K,k} \triangleq \frac{\sum_{\substack{l:l\neq k,\\ \Phi_i(l)=m_i}} \gamma^2\bar{\lambda}_l\left[Y_i^k - Y_i^l\right]H_{ij}'F_{ij}^{-1}}{\sum_{l:\Phi_i(l)=m_i}\bar{\lambda}_l}.$$

*Theorem 1:* Given a Markovian switching network $(\eta, \mathscr{G}, \Phi)$ and a collection of constants $\gamma$, $\alpha_i$, $\beta_{ij}$ and $\delta_i > 0$, $i = 1, \ldots, N$, associated with each node and its admissible neighbourhoods. Suppose there exist matrices $X_i^k = (X_i^k)' > 0$, $Y_i^k = (Y_i^k)' > 0$, and positive scalars $\tau_i^k$, $\theta_{ij}^k$, $\vartheta_{ij}^k$, $i = 1, \ldots, N$, $k \in \mathscr{I}$, $j \in \mathbf{V}_i^{\Phi_i(k)}$ which satisfy the matrix inequalities (17), (18), (26), and the rank constraint (27). Using the solution matrices $Y_i^k$, construct the auxiliary interconnection and innovation gains

$$K_{ij}^k = \gamma^2 Y_i^k H_{ij}' F_{ij}^{-1}, \quad (28)$$

$$L_i^k = \left[\gamma^2 Y_i^k (C_i^k)' + B_2(D_i^k)'\right](E_i^k)^{-1}, \quad (29)$$

Then, using (23) and (28), (29), construct the estimator network (4). The resulting distributed estimatior network solves the distributed robust estimation problem in Definition 1.

*Proof* The result follows from Lemma 1 in manner similar to the proof of Theorem 4 in [18]. □

*Remark 2:* Due to the rank constraints (27), the solution set to the matrix inequalities in Theorem 1 is non-convex. In general, it is difficult to solve such problems. Fortunately, several numerical algorithms have been proposed for this purpose [4], [11].

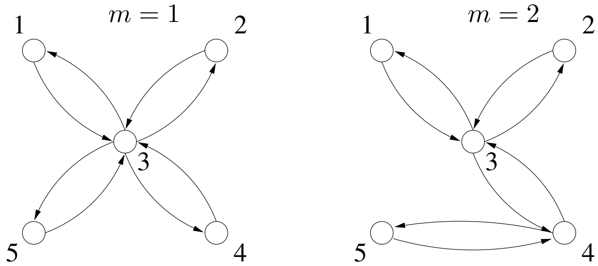


Fig. 1. Switching graph topology for the example.

TABLE I

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ |
|---|---|---|---|---|---|
| $m=1$ | $C_{02}$ | $C_{02}$ | $C_{03}$ | $C_{02}$ | $C_{03}$ |
| $m=2$ | $C_{02}$ | $C_{03}$ | $C_{03}$ | $C_{02}$ | $C_{03}$ |

## V. Example

Consider a plant of the form (2), with $A = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} -0.1246 \\ -0.4461 \\ 0.3350 \end{bmatrix}$. The plant is observed by a 5-node switching observer network which operates intermittently in two regimes, whose graph topologies are shown in Figure 1. The evolution of the network is therefore modelled as a two-state Markov chain with the transition probability rate matrix $\Lambda = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$. The matrices $C_i^m$ of the measurement equations are given in Table I, $C_{02} = [3.1923 \ -4.6597 \ 1]$, $C_{03} = [-0.8986 \ 0.1312 \ -1.9703]$, and we let $D_i = 0$, $\bar{D}_i = 0.01$ for all nodes. In addition, we let $H_{ij} = I_{3\times 3}$, and generated the matrices $G_{ij}$ randomly.

From Figure 1 and the above list of parameters, nodes 3, 4, and 5 have varying neighbourhoods, and node 2 has varying sensor parameters. Therefore in this example, we seek to obtain nonswitching observer gains for node 1 only. To this end, the auxiliary distributed estimation problem was solved numerically using Matlab, with additional norm-bounded uncertainty constraints of the form (16) defined for the communication link $(3,1)$ at node 1. This led to three uncertainty constraints of the form (13), where we set $\alpha_{13}^2 = 500$, $\beta_{13}^2 = 21$. These constants, as well as $\delta = 3.65$ were chosen by trial and error, to ensure that the corresponding minimum variance gains $\tilde{K}_{13}$ and $\tilde{L}_1$ computed according to (28), (29) with $Y_1^m = (X_i^m)^{-1}$, satisfy conditions (26) of Theorem 1 with $\gamma^2 = 7.3206$. From Theorem 1 we conclude that the constructed estimator solves the distributed estimation problem under consideration.

It is worth noting a significant difference between the gains obtained for the two modes at nodes 2 and 4. At the same time, the gains of the observers 3 an 5 vary to a substantially lesser extent. This suggests a possibility to use nonswitching gains for these observers as well, even though this was not the design requirement in this example. Such a phenomenon needs to be further investigated.

## VI. Conclusions

The paper has presented sufficient conditions for the synthesis of robust distributed consensus estimators connected over a Markovian switching graph. The proposed estimator provides a guaranteed suboptimal $H_\infty$ disagreement of estimates, while using only locally available information about connectivity of the network. Our conditions allow a robust filter network to be constructed by solving an LMI feasibility problem. These LMIs are partitioned in a way which makes possible to solve them in a fully decentralized fashion using gradient descent algorithms such as that proposed in [16]. When the entire network graph is available at every node, this feasibility problem is convex. Broadcast of the network status has been eliminated by conditioning on the locally available information. This has led to the introduction of rank constraints additional to the LMI conditions.


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