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Optimal Hedging for Multivariate Derivatives Based on Additive Models

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Abstract— **In this paper, we consider optimal hedges for a class of derivative securities whose underlyings are untraded, using the additive sum of smooth functions of traded assets that minimizes the mean square error. Based on the necessary and sufficient condition, we derive a methodology to compute optimal smooth functions efficiently by solving a system of linear equations. Moreover, we extend the idea to basket options consisting of a portfolio of stocks, where individual payoff functions of traded assets are optimally computed. We also provide numerical experiments to illustrate our methodology.**

I. Introduction

To explain the motivation of this work, let us consider the (standard) minimum variance hedging problem given as

$$\min_{\alpha} \mathrm{Var}\left(\Delta Y - \alpha \Delta S\right), \tag{1}$$

where ΔY and ΔS are price changes of two assets, Y and S, in a certain time period, and $\alpha \in \Re$ is a hedge ratio. In a typical situation, the asset Y can not be traded frequently in the market, whereas S may be a liquidly traded asset.

It is known that the optimal hedge ratio, denoted by α^*, provides the minimizer of

$$\min_{\alpha, c} \mathbb{E}\left[\left\{\Delta Y - (\alpha \Delta S + c)\right\}^2\right] \tag{2}$$

which may be solved as an ordinary linear regression given empirical observation data. In this sense, the standard minimum variance hedge of (1) is equivalent to the simple ordinary linear regression problem.

The standard minimum variance hedge may be generalized for nonlinear case, in which a nonlinear smooth function f is searched to minimize the following mean square error:

$$\mathbb{E}\left[\left\{\Delta Y - f\left(\Delta S\right)\right\}^2\right]. \tag{3}$$

In the case where multiple assets are available, the problem may be formulated as follows,

$$\min_{f_i \in \mathscr{S}} \ \mathbb{E}\left[\left\{\Delta Y - \sum_{i=1}^{m} f_i\left(\Delta S_i\right)\right\}^2\right], \tag{4}$$

where $\Delta S_i,\ i = 1, \ldots, m$ are price changes of asset S_i and $\mathscr{S}$ is a set of smooth functions. Obviously, the problem addresses the standard minimum variance hedge (or more generally, the multivariate minimum variance hedge using multivariate linear regression) as a special case when f_i is linear, and therefore, we can expect to get the better hedge effect. This is our basic idea in our previous work [10], [11] that we applied the generalized additive model (GAM; see [5], [9]) for constructing optimal payoff functions of weather derivatives given empirical observation data.

The objective of this paper is to provide a theoretical framework for nonlinear minimum variance hedging in continuous time setting. For this objective, we first formulate the nonlinear minimum variance hedging problem, and provide a necessary and sufficient condition for the optimal smooth functions. Then, we derive an algorithm to compute the optimal smooth functions based on the suitable discretization, and demonstrate the optimal hedges. Moreover, we extend the idea to the basket options case, whose underlying is defined as the weighted average of many stocks.

II. Problem formulation

Let Y_t ($t \in [0,\ T]$) be the value of an asset (being nontraded or illiquid), and $S_{i,t},\ i = 1, \ldots, m$ the values of liquidly traded m assets, under a probability space (Ω, $\mathscr{F}$, $\mathbb{P}$) and filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$. We consider the problem of hedging the payoff of derivative security on Y_t using the additive sum of smooth functions of $S_{i,t},\ i = 1, \ldots, m$. To this end, we define the nonlinear minimum variance hedging problem as follows:

$$\min_{f_i \in \mathscr{S}} \ \mathbb{E}\left[\left\{g\left(Y_T\right) - \sum_{i=1}^{m} f_i\left(S_{i,T}\right)\right\}^2\right], \tag{5}$$

where $g\left(Y_T\right)$ stands for the terminal payoff of a derivative security with a given payoff function g. Note that Y_t may be the value of portfolio as in Section V. Also, note that the problem formulation of (5) is slightly different from that of (4) as it minimizes the mean square error between the terminal payoff $g\left(Y_T\right)$ and the sum of $f_i\left(S_{i,T}\right),\ i = 1, \ldots, m$.

Although we do not know how to find the optimal smooth functions yet, there exists a necessary and sufficient condition for optimal smooth functions, $f_1^*, \ldots, f_m^*$, as follows:

Lemma 1: Smooth functions $f_1^*, \ldots, f_m^*$ provide minimizers of the problem (5), if and only if the following conditions are satisfied:

$$\mathbb{E}\left[g\left(Y_T\right)\middle| S_{i,T}\right] - \sum_{j=1}^{m} \mathbb{E}\left[f_j^*\left(S_{j,T}\right)\middle| S_{i,T}\right] = 0, \ \ i = 1, \ldots, m \tag{6}$$

Proof: For the proof, see pp. 108 in [5].

In this paper, we demonstrate how to compute the smooth functions, $f_1^*, \ldots, f_m^*$, satisfying Lemma 1.

Before showing the solution method for smooth functions, we discuss how to replicate $f_i^*\left(S_{i,T}\right),\ i = 1, \ldots, m$ as cash values. Since each f_i^* $(i = 1, \ldots, m)$ is a smooth function,

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there are two approaches to attain $f_i^*(S_{i,T}),\ i=1,\dots,m$. The first approach is to use European type calls and puts with maturity T and any strikes, where any twice continuously differentiable function, $f(x)$, of the terminal stock price $S_T = x$, can be replicated by a unique initial position of $f(S_0) - f'(S_0)S_0$ unit discount bounds, $f'(S_0)$ shares, and $f''(K)\,\mathrm{d}K$ out-of-the money options of all strikes K [3]:

$$f(x) = \left[f(S_0) - f'(S_0)S_0\right] + f'(S_0)x + \int_0^{S_0} f''(K)(K-x)^+\,\mathrm{d}K + \int_{S_0}^{\infty} f''(K)(x-K)^+\,\mathrm{d}K \tag{7}$$

The advantage of this approach is that we do not have to estimate any parameters such as volatilities or mean rates of returns of the underlying assets once the target payoff function f is specified.

The second approach is to dynamically trade $S_{i,t}$ to replicate the terminal payoff $f_i^*(S_{i,T})$. For this approach to be applicable, we need to introduce price dynamics for Y_t and $S_{i,t},\ i=1,\dots,m$, namely the "dynamic hedging model." Note that, in this framework, although the total market is incomplete since Y_t is not tradable, each payoff $f_i^*(S_{i,T}),\ i=1,\dots,m$ may be replicated by trading $S_{i,t}$ dynamically. We further discuss this approach in Section IV.

III. Solution method for optimal smooth functions

Recall that, from Lemma 1, the problem reduces to finding a set of real-valued functions, $f_1^*,\dots,f_m^*$, satisfying

$$\sum_{j=1}^{m} \mathbb{E}\left[f_j^*(S_{j,T})\middle| S_{i,T}=x_i\right] = \mathbb{E}\left[g(Y_T)\middle| S_{i,T}=x_i\right], \tag{8}$$

where

$$\mathbb{E}\left[f_i^*(S_{i,T})\middle| S_{i,T}=x_i\right] = f_i^*(x_i),\quad i=1,\dots,m. \tag{9}$$

Assume that there exist joint PDFs of pairs, $(S_{i,T},\ S_{j,T})$ and $(Y_T,\ S_{i,T})$, denoted by

$$\phi_{S_i,S_j}(x_i,\ x_j),\quad i,j=1,\dots,m,\ i\neq j \tag{10}$$

and

$$\phi_{Y,S_i}(y,\ x_i),\quad i=1,\dots,m, \tag{11}$$

respectively. Also let $\phi_{S_j|S_i}(x_j|x_i)$ and $\phi_{Y|S_i}(y|x_i)$ be conditional PDFs defined as

$$\phi_{S_j|S_i}(x_j|x_i) := \frac{\phi_{S_i,S_j}(x_i,\ x_j)}{\phi_{S_i}(x_i)},\quad \phi_{Y|S_i}(y|x_i) := \frac{\phi_{Y,S_i}(y,\ x_i)}{\phi_{S_i}(x_i)}.$$

where $\phi_{S_i}(x_i),\ i=1,\dots,m$ are marginal PDFs. Then condition (8) may be rewritten as follows:

$$\begin{aligned} & f_i^*(x_i) + \sum_{j\neq i}\int_{\Re} f_j^*(x_j)\cdot\phi_{S_j|S_i}(x_j|x_i)\,\mathrm{d}x_j \\ = & \int_{\Re} g(y)\cdot\phi_{Y|S_i}(y|x_i)\,\mathrm{d}y,\quad i=1,\dots,m. \end{aligned} \tag{12}$$

We would like to find $f_i^*,\ i=1,\dots,m$ such that (12) holds for suitable domain of input variables. Here we provide a solution method which consists of the following three steps:

1) Discretize condition (12) for y, x_i and x_j $(i,j=1,\dots,m)$ dimensions to obtain a set of linear equations.
2) Solve the set of linear equations to find discretized points of smooth functions.
3) Construct smooth functions using cubic splines.

Note that the above method may be applied if the joint PDFs of pairs in Y_T and $S_{i,T},\ i=1,\dots,m$ are specified.

First, we discretize condition (12) to approximate the integrals as

$$\begin{aligned} & f_i^*(x_i) + \sum_{j\neq i}\sum_{l=1}^{N}\left[f_j^*\left(x_j^{(l)}\right)\cdot\phi_{S_j|S_i}\left(x_j^{(l)}\middle| x_i\right)\delta_{x_j}\right] \\ = & \sum_{l=1}^{N}\left[g\left(y^{(l)}\right)\cdot\phi_{Y|S_i}\left(y^{(l)}\middle| x_i\right)\delta_y\right] \end{aligned} \tag{13}$$

for given $x_i,\ i=1,\dots,m$, where δ_{x_j} and δ_y are assumed to satisfy

$$\sum_{l=1}^{N}\phi_{S_j|S_i}\left(x_j^{(l)}\middle| x_i\right)\delta_{x_j} = \sum_{l=1}^{N}\phi_{Y|S_i}\left(y^{(l)}\middle| x_i\right)\delta_y = 1.$$

Note that δ_{x_j} and δ_y may depend on x_i as well, but we will omit to specify that dependence for brevity. We then discretize condition (13) for x_i dimensions, e.g., $x_i^{(k)},\ k=1,\dots,N$, as

$$\begin{aligned} & f_i^*\left(x_i^{(k)}\right) + \sum_{j\neq i}\sum_{l=1}^{N}\left[f_j^*\left(x_j^{(l)}\right)\cdot\phi_{S_j|S_i}\left(x_j^{(l)}\middle| x_i^{(k)}\right)\delta_{x_j}\right] \\ = & \sum_{l=1}^{N}\left[g\left(y^{(l)}\right)\cdot\phi_{Y|S_i}\left(y^{(l)}\middle| x_i^{(k)}\right)\delta_y\right]. \end{aligned}$$

Let $\boldsymbol{f}_i\in\Re^N$ $(i=1,\dots,m)$ and $\boldsymbol{g}\in\Re^N$ be vectors whose k-th entries are, respectively, given as

$$\begin{aligned} \boldsymbol{f}_i[k] &:= f_i\left(x_i^{(k)}\right), \\ \boldsymbol{g}_i[k] &:= g\left(y^{(k)}\right),\quad k=1,\dots,N. \end{aligned}$$

Also, let $\Phi_{i,y}\in\Re^{N\times N}$ $(i=1,\dots,m)$ and $\Phi_{i,j}\in\Re^{N\times N}$ $(i,j=1,\dots,m,\ i\neq j)$ be matrices whose $(k,\ l)$-entries are given as

$$\begin{aligned} \Phi_{i,j}[k,\ l] &:= \phi_{S_j|S_i}\left(x_j^{(l)}\middle| x_i^{(k)}\right)\delta_{x_j}, \\ \Phi_{i,y}[k,\ l] &:= \phi_{Y|S_i}\left(y^{(l)}\middle| x_i^{(k)}\right)\delta_y,\quad k,l=1,\dots,N \end{aligned}$$

With these definitions and notations, we have the following proposition:

Proposition 1: For each $i=1,\dots,m$, condition (12) may be discretized as

$$\boldsymbol{f}_i + \sum_{j\neq i}\Phi_{i,j}\boldsymbol{f}_j = \Phi_{i,y}\boldsymbol{g}. \tag{14}$$

Consequently, we obtain the following system of linear equations with respect to $\boldsymbol{f} := \left[\boldsymbol{f}_1^\top,\dots,\boldsymbol{f}_m^\top\right]^\top\in\Re^{mN}$:

$$\Phi\boldsymbol{f} = \hat{\boldsymbol{g}} \tag{15}$$

where

$$\Phi := \begin{bmatrix} I_{N\times N} & \Phi_{1,2} & \Phi_{1,3} & \cdots & \Phi_{1,m} \\ \Phi_{2,1} & I_{N\times N} & \Phi_{2,3} & \cdots & \Phi_{2,m} \\ \Phi_{3,1} & \Phi_{3,2} & I_{N\times N} & \ddots & \Phi_{3,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \Phi_{m,1} & \Phi_{m,2} & \Phi_{m,3} & \cdots & I_{N\times N} \end{bmatrix} \in \Re^{mN\times mN}$$

$$\hat{\boldsymbol{g}} := \begin{bmatrix} \Phi_{y,1} & 0 & \cdots & 0 \\ 0 & \Phi_{y,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Phi_{y,m} \end{bmatrix} \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{g} \\ \vdots \\ \boldsymbol{g} \end{bmatrix} \in \Re^{mN}.$$

Although the solution to (15) may not be unique, it can be expressed using the generalized inverse matrix as

$$\boldsymbol{f} = \Phi^{\top}\left[\Phi\Phi^{\top}\right]^{-1}\hat{\boldsymbol{g}}. \tag{16}$$

Then, the optimal smooth functions, f_i^* may be constructed using cubic splines,

$$f_i^*(x) = c_0 + c_1 x + \frac{1}{12}\sum_{k=1}^{N}\theta_k\left|x - x_i^{(k)}\right|^3, \quad i = 1,\ldots,m, \tag{17}$$

where c_0, c_1 and θ_k, $k = 1,\ldots,N$ are found to satisfy $f_i^*\left(x_i^{(k)}\right) = f_i^{(k)}$ and

$$\sum_{k=1}^{N}\theta_k = 0, \quad \sum_{k=1}^{N}\theta_k x_i^{(k)} = 0.$$

Remark 1: Although, we derived the set of linear equations based on the joint PDFs for Y_t and $S_{i,t}$, $i = 1,\ldots,m$, it is often the case in derivative pricing problems that the underlying stochastic processes, Y_t and $S_{i,t}$, $i = 1,\ldots,m$, are expressed as the following type of geometric processes:

$$Y_t = Y_0 e^{Z_t}, \quad S_{i,t} = S_{i,0}e^{X_{i,t}}, \ i = 1,\ldots,m, \tag{18}$$

where Z_t and $X_{i,t}$, $i = 1,\ldots,m$ are adopted to $\mathscr{F}_t$. In this case, we can work on joint PDFs (or corresponding conditional PDFs) for Z_T and $X_{i,T}$, $i = 1,\ldots,m$ instead of the ones for Y_T and $S_{i,T}$, $i = 1,\ldots,m$. Let joint PDFs of pairs, $(X_{i,T},\ X_{j,T})$ and $(Z_T,\ X_{i,T})$, be given as

$$\phi_{X_i,X_j}(x_i,\ x_j), \quad i,j = 1,\ldots,m,\ i \neq j$$

and

$$\phi_{Z,X_i}(z,\ x_i), \quad i = 1,\ldots,m,$$

respectively. Then, condition (12) is modified to

$$\begin{aligned} & f_i^*(S_{0,i}e^{x_i}) + \sum_{j\neq i}\int_{\Re} f_j^*\left(S_{0,j}e^{x_j}\right)\cdot\phi_{X_j|X_i}(x_j|x_i)\,\mathrm{d}x_j \\ = & \int_{\Re} g(Y_0 e^{z})\cdot\phi_{Z|X_i}(z|x_i)\,\mathrm{d}z, \quad i = 1,\ldots,m, \end{aligned} \tag{19}$$

where

$$\phi_{X_j|X_i}(x_j|x_i) := \frac{\phi_{X_i,X_j}(x_i,\ x_j)}{\phi_{X_i}(x_i)}, \quad \phi_{Z|X_i}(z|x_i) := \frac{\phi_{Z,X_i}(z,\ x_i)}{\phi_{X_i}(x_i)}, \tag{20}$$

and $\phi_{X_i}(x_i)$, $i = 1,\ldots,m$ are marginal PDFs. We see that the same approach can be applied by discretizing (19) for each dimension to derive the similar set of linear equations.

IV. Dynamic hedging model

In this section, we introduce price dynamics for $S_{i,t}$ that enable us to replicate the terminal payoff $f_i^*(S_{i,T})$ using dynamic trading strategy. Assume that, under the probability space $(\Omega,\ \mathscr{F},\ \mathbb{P})$, the values of liquidly traded assets $S_{1,t},\ldots,S_{m,t}$ and nontraded asset Y_t are governed by the following stochastic differential equations[1],

$$\mathrm{d}S_{i,t} = \mu_i S_{i,t}\mathrm{d}t + \sigma_i S_{i,t}\mathrm{d}W_{i,t}, \quad i = 1,\ldots,m \tag{21}$$

$$\mathrm{d}Y_t = \mu_{m+1}Y_t\mathrm{d}t + \sigma_{m+1}Y_t\mathrm{d}W_{m+1,t} \tag{22}$$

where $W_{1,t},\ldots,W_{m+1,t}$ are correlated Brownian motions with

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{d}t, \quad i,j = 1,\ldots,m+1,\ i\neq j.$$

For simplicity, let μ_i, σ_i and ρ_{ij} $(i,j = 1,\ldots,m+1,\ i\neq j)$ be constant parameters, although the result can readily be generalized for the case of deterministic functions of t. Note that the advantage of considering the above model is that there exists a dynamic trading strategy (see [2] and [6]) to replicate the terminal payoff $f_i^*(S_{i,T})$ once the optimal smooth functions are specified.

A. Case $m = 1$

We first derive the optimal smooth function for the case $m = 1$. The following proposition shows that f_1^* is expressed in a closed form for European call/put options with a strike price K:

Proposition 2: The optimal smooth function, f_1^*, is represented as

$$\begin{aligned} f_1^*(x) = & \ Y_0\exp\left\{\left(\mu_2 - \frac{\rho_{12}^2\sigma_2^2}{2}\right)T + \rho_{12}\sigma_2 b_1(x)\right\} \\ & \times\mathbf{N}(d_1(x)) - K\mathbf{N}(d_2(x)) \end{aligned} \tag{23}$$

when $g(y) = (y-K)^+$ for European call options, or

$$\begin{aligned} f_1^*(x) = & -Y_0\exp\left\{\left(\mu_2 - \frac{\rho_{12}^2\sigma_2^2}{2}\right)T + \rho_{12}\sigma_2 b_1(x)\right\} \\ & \times\mathbf{N}(-d_1(x)) + K\mathbf{N}(-d_2(x)) \end{aligned} \tag{24}$$

when $g(y) = (K-y)^+$ for European put options, where and $\mathbf{N}$ is the standard normal distribution function, and

$$b_1(x) := \frac{1}{\sigma_1}\left\{\ln\left(\frac{x}{S_{1,0}}\right) - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T\right\} \tag{25}$$

$$\begin{aligned} d_1(x) := & \ \frac{1}{\sigma_2\sqrt{(1-\rho_{12}^2)T}}\times\left[\ln\left(\frac{Y_0}{K}\right)\right. \\ & \left. +\rho_{12}\sigma_2 b_1(x) + \left(\mu_2 + \frac{\sigma_2^2}{2} - \rho_{12}^2\sigma_2^2\right)T\right] \end{aligned}$$

$$\begin{aligned} d_2(x) := & \ \frac{1}{\sigma_2\sqrt{(1-\rho_{12}^2)T}}\times\left[\ln\left(\frac{Y_0}{K}\right)\right. \\ & \left. +\rho_{12}\sigma_2 b_1(x) + \left(\mu_2 - \frac{\sigma_2^2}{2}\right)T\right]. \end{aligned}$$

Proof: For the proof, see [12]. ■

[1] The problem setting in this paper addresses the one in [8] when $m = 1$. Also, the problem is closely related to the pioneering work of [4] for hedging the spot price using the self-financing portfolio of future price. Note that, in our formulation, we intend to hedge the payoff of illiquid asset derivatives using liquidly traded asset derivatives.

B. General case

In the case with m traded assets, $S_{i,T}$ and Y_T are given as

$$\begin{aligned} S_{i,T} &= S_{i,0}e^{\nu_i T+\sigma_i W_{i,T}}, \quad i=1,\ldots,m, \\ Y_T &= Y_0 e^{\nu_{m+1}T+\sigma_{m+1}W_{m+1,T}}, \end{aligned}$$

where $\nu_i := \mu_i - \sigma_i^2/2,\ i=1,\ldots,m+1$. Since the conditional expectation given $S_{i,t}=x_i$ corresponds to the one given $W_{i,t}=w_i$ with suitable parameter changes, condition (8) may be rewritten as

$$\sum_{j=1}^{m} \mathbb{E}\left[f_j^*(S_{j,t})\middle| W_{i,T}=w_i\right] = \mathbb{E}\left[g(Y_T)| W_{i,T}=w_i\right], \quad (26)$$

for $i=1,\ldots,m$. Note that the right hand side of (26) can be computed based on Proposition 2 as

$$\begin{aligned} \mathbb{E}\left[g(Y_T)| W_{i,T}=w_i\right] &= \mathbb{E}\left[g(Y_T)\middle| S_{i,T}=S_{i,0}e^{\nu_i T+\sigma_i w_i}\right] \\ &= \hat{g}_i(w_i), \quad i\in[1,\ m] \end{aligned} \quad (27)$$

using a smooth function $\hat{g}_i$. Also, since each $S_{i,t}$ is a function of $W_{i,t}$, we write

$$f_i^*(S_{i,t}) = \hat{f}_i(W_{i,t}),$$

and reformulate equation (26) as follows:

$$\sum_{j=1}^{m} \mathbb{E}\left[\hat{f}_j(W_{j,t})\middle| W_i(T)=w_i\right] = \hat{g}_i(w_i), \quad i=1,\ldots,m. \quad (28)$$

Let $p_{j|i}(w_j|w_i),\ j\neq i$ be the conditional probability density function of $W_{j,t}$ given $W_{i,t}$, i.e.,

$$p_{j|i}(w_j|w_i) := \frac{1}{\sqrt{2\pi(1-\rho_{ij}^2)T}}\exp\left\{-\frac{(w_j-\rho_{ij}w_i)^2}{2\left(1-\rho_{ij}^2\right)T}\right\}.$$

Then condition (28) may be written as follows:

$$\hat{f}_i(w_i)+\sum_{j\neq i}\int_{-\infty}^{\infty}\hat{f}_j(w_j)p_{j|i}(w_j|w_i)\,\mathrm{d}w_j = \hat{g}_i(w_i), \ \ i=1,\ldots,m.$$

With the similar argument to the derivation of condition (15), we can construct a set of linear equations by suitable discretization for w_i, w_j, $p_{j|i}(w_j|w_i)$, $\hat{f}_i(w_i)$, and $\hat{g}_i(w_i)$ as

$$\begin{bmatrix} I_{N\times N} & \Phi_{1,2} & \cdots & \Phi_{1,m} \\ \Phi_{2,1} & I_{N\times N} & \cdots & \Phi_{2,m} \\ \vdots & \ddots & \ddots & \vdots \\ \Phi_{m,1} & \Phi_{m,2} & \cdots & I_{N\times N} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{f}}_1 \\ \hat{\boldsymbol{f}}_2 \\ \vdots \\ \hat{\boldsymbol{f}}_m \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{g}}_1 \\ \hat{\boldsymbol{g}}_2 \\ \vdots \\ \hat{\boldsymbol{g}}_m \end{bmatrix}. \quad (29)$$

Optimal smooth functions $f_i^*,\ i=1,\ldots,m$ are then obtained using cubic splines.

V. Basket options

In the previous sections, we have assumed that Y_t stands for the value of nontraded asset and considered to hedge an option on Y_t using liquidly traded assets, $S_{i,t},\ i=1,\ldots,m$. Here we extend this idea to the problem of hedging basket options using payoffs of options on individual assets.

Let us replace Y_T in (5) by the weighted sum of traded assets, i.e.,

$$\min_{f_i\in\mathscr{S}}\ \mathbb{E}\left[\left\{g(Y_T)-\sum_{i=1}^{m}f_i(S_{i,T})\right\}^2\right], \quad Y_T := \sum_{i=1}^{m}\alpha_i S_{i,T}, \quad (30)$$

where $\alpha_i,\ i=1,\ldots,m$ are given weight parameters. Then the minimum variance hedging problem (30) is to find smooth payoff functions for terminal values of individual assets, $S_{i,T},\ i=1,\ldots,m$, that approximate the terminal payoff of basket option as close as possible in the minimum mean square sense.

Notice that the left hand side of equation (8) is indifferent even for basket options, and hence, the left hand side of (15) may be constructed similar to Proposition 1 if there are joint PDFs for $S_{i,T},\ i=1,\ldots,m$. using the conditional expectations of $g(Y_T)$ given $S_{i,T},\ i=1,\ldots,m$. We will demonstrate how to compute these conditional expectations when the value processes of $S_{i,t},\ i=1,\ldots,m$ are defined by (21).

Assume that $S_{i,t},\ i=1,\ldots,m$ follow the SDEs in (21), and let $\hat{g}_i$ be a function satisfying

$$\hat{g}_i(W_{i,T}) = E\left[g(Y_T)| W_{i,T}\right], \quad i=1,\ldots,m.$$

We would like to express $\hat{g}_i$ in a tractable form. The following proposition shows that $\hat{g}_i$ may be represented using unconditional expectation and thus be computed efficiently:

Proposition 3: For each $i\in[1,\ m]$ and a (nonrandom) dummy variable $w_i\in\Re$, there exist a function h_i and independent Brownian motions, $B_{2,t},\ldots,B_{m,t},\ t\in[0,\ T]$, satisfying

$$\hat{g}_i(w_i) = \mathbb{E}\left[h_i(w_i, B_{2,T},\ldots,B_{m,T})\right]. \quad (31)$$

Proof: Here we consider the case $i=1$, although the same technique may be applied for $i=2,\ldots,m$.

Let the covariance matrix of

$$\left[\frac{\mathrm{d}S_{1,t}}{S_{1,t}},\ldots,\frac{\mathrm{d}S_{m,t}}{S_{m,t}}\right]^\top$$

be decomposed as $LL^\top\mathrm{d}t$, where L is a lower triangular matrix defined by

$$L := \begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ \sigma_{21} & \sigma_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{bmatrix} \in \Re^{m\times m}, \quad \sigma_{11}=\sigma_1$$

based on the Cholesky decomposition. Then, we obtain the following equivalent representation to (21):

$$\begin{bmatrix} \mathrm{d}S_{1,t}/S_{1,t} \\ \vdots \\ \mathrm{d}S_{m,t}/S_{m,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}\mathrm{d}t + L\begin{bmatrix} \mathrm{d}B_{1,t} \\ \vdots \\ \mathrm{d}B_{m,t} \end{bmatrix}. \quad (32)$$

where $B_{1,t},\ldots,B_{m,t}$ are independent Brownian motions and $B_{1,t}\equiv W_{1,t}$. Since $S_{i,T}$ is expressed as

$$S_{i,T} = S_{i,0}\exp\left(\nu_i T + \sum_{j=1}^{i}\sigma_{ij}B_{j,T}\right), \quad i=1,\ldots,m,$$

there exists a function h_1 such that

$$g(Y_T) = g\left(\sum_{i=1}^{m} \alpha_i S_{i,T}\right) = h_1(W_{1,T}, B_{2,T}, \ldots, B_{m,T}).$$

Hence, we have

$$\begin{aligned} E[g(Y_T)|S_{1,T}] &= E[g(Y_T)|W_{1,T}] \\ &= \mathbb{E}[h_1(W_{1,T}, B_{2,T}, \ldots, B_{m,T})|W_{1,T}] \end{aligned} \quad (33)$$

We discuss some properties of the conditional expectation in (33). First, we note that $S_{1,T}$ is a function of $W_{1,T}$ and is independent of the other factors, $B_{2,T}, \ldots, B_{m,T}$. This indicates that there exists a sigma algebra $\mathscr{G}_1 (\subset \mathscr{F})$ such that both $W_{1,T}$ and $S_{1,T}$ are $\mathscr{G}_1$-measurable and $B_{2,T}, \ldots, B_{m,T}$ are independent of $\mathscr{G}_1$. Then we can apply the Independence Lemma that a function $\hat{h}_1$ of a dummy variable $w_1 \in \Re$,

$$\hat{h}_1(w_1) := \mathbb{E}[h_1(w_1, B_{2,T}, \ldots, B_{m,T})], \quad (34)$$

satisfies the following condition:

$$\begin{aligned} \hat{h}_1(W_{1,T}) &= \mathbb{E}[h_1(W_{1,T}, B_{2,T}, \ldots, B_{m,T})|W_{1,T}] \\ &= E[g(Y_T)|W_{1,T}]. \end{aligned} \quad (35)$$

Clearly, conditions (34) and (35) indicate that the statement in the proposition holds with $i = 1$ and $\hat{g}_1 = \hat{h}_1$.

Similarly, we can obtain $h_i,\ i = 2, \ldots, m$ by reordering $S_{1,t}, \ldots, S_{m,t}$ so that $S_{i,t}$ is the first entry when applying the Cholesky decomposition. This completes the proof. ■

We see that, for any given real number $w_i \in \Re,\ i = 1, \ldots, m$, $\hat{g}_i(w_i)$ is computed by the unconditional expectation in (34). In general, this computation involves multiple integration, but usually executed efficiently based on the Monte Carlo method by generating independent Gaussian random numbers for independent Brownian motions. Once a set of random numbers is generated, we can compute $\hat{g}_i(w_i)$ for different values of $w_i = w_i^{(k)},\ k = 1, \ldots, N$ using the same set of random numbers to construct a real-valued vector $\hat{\boldsymbol{g}}_i \in \Re^N$ in the right hand side of equation (29). Then, we solve the set of linear equations for $\hat{\boldsymbol{f}}_i,\ i = 1, \ldots, m$ to find the optimal smooth functions using cubic splines. Note that other properties of basket option's hedge is discussed in [13].

VI. Numerical Experiment

In this numerical experiment, we first consider a problem of hedging an option whose underlying is a market index (being nontraded) using several stocks, where each asset dynamics is modeled as (21) and (22). We will formulate the problem as nonlinear minimum variance hedging and solve it by applying the proposed methodology.

We use the empirical data obtained from the Tokyo Stock Exchange (TSE) in the period of 2003-2005 for estimating the volatility and correlation parameters of stock returns, where the market index is assumed to be TOPIX and five stocks, $S_1, \ldots, S_5$, are chosen from those listed in the TSE. The correlation and volatility parameters of stock returns are estimated as in Table I, whereas we assume that each expected stock return corresponding to the drift parameter has the same sharp ratio $(= 0.25)$ with risk free interest rate $r = 0.05$. Then drift parameters are provided as in Table II.

TABLE I

Volatility and correlation of the stock returns

	INDEX	S_1	S_2	S_3	S_4	S_5
INDEX	1					
S_1	0.552	1				
S_2	0.636	0.298	1			
S_3	0.615	0.476	0.346	1		
S_4	0.557	0.291	0.406	0.341	1	
S_5	0.604	0.315	0.457	0.287	0.389	1
Volatility	0.176	0.549	0.227	0.421	0.307	0.232

TABLE II

Drift having the same sharp ratio $(= 0.25)$

	INDEX	S_1	S_2	S_3	S_4	S_5
Drift	0.094	0.187	0.107	0.155	0.127	0.108

We solve the problem (5) to find the minimizers $f_1^*, \ldots, f_5^*$ for hedging an at-the-money European call option with maturity $T = 1/4$, where the initial prices (or initial values) are set to be $Y_0 = 100$ and $S_{i,0} = 100,\ i = 1, \ldots, 5$. The correlation coefficient between $g(Y_T)$ and $\sum_{i=1}^{5} f_i^*(S_{i,T})$ may provide a hedge effect, and for this numerical experiment, it is obtained as

$$\mathrm{Corr}\left(g(Y_T),\ \sum_{i=1}^{5} f_i^*(S_{i,T})\right) = 0.805. \quad (36)$$

Next, we discuss the initial cost of the hedge. To compute the initial cost, we need to evaluate the initial price of the options under a risk neutral probability measure. Here we compute the minimal market price of risk (see e.g., [1]) to specify a risk neutral probability measure $\tilde{\mathbb{P}}$. Note that the market prices of risk for the traded assets $S_1, \ldots, S_5$ are the same and are given by their sharp ratios $(= 0.25)$, whereas the (minimal) market price of risk for the nontraded asset, denoted by $\hat{\theta}_y$, is found to be

$$\hat{\theta}_y = 0.3026.$$

We see that the market price of risk for the index is higher than those of traded assets, which may be interpreted as a risk premium for the nontradability of the index.

Under the risk neutral probability measure, we computed the initial value of call option written on Y_t, which is given as

$$V_0 = e^{-rT}\tilde{\mathbb{E}}[g(Y_T)] \simeq 4.01. \quad (37)$$

On the other hand, initial values of options whose payoffs are determined by $f_i^*(S_{i,T})$ are given as

$$e^{-rT}\tilde{\mathbb{E}}[f_i^*(S_{i,T})], \quad i = 1, \ldots, 5. \quad (38)$$

Since each payoff may be hedged by the corresponding self-financing portfolio with the initial cost being equal to (38), the total cost of the replicating portfolio is obtained as

$$X_0 = \sum_{i=1}^{5} e^{-rT}\tilde{\mathbb{E}}[f_i^*(S_{i,T})]. \quad (39)$$

In our numerical experiment, the initial value of portfolio, X_0, is obtained as

$$X_0 \simeq 4.00, \tag{40}$$

which is almost the same as V_0 in this case.

When one takes a position to sell the option with V_0 at time $t = 0$ and construct a portfolio with X_0 to hedge the option, the initial cost of the hedged position is given by

$$X_0 - V_0. \tag{41}$$

If $X_0 - V_0$ is positive, then she has to pay an extra cost to construct the portfolio, and hence, $X_0 - V_0 > 0$ may be interpreted as a premium for the hedge. Here we evaluate the hedge cost using its ratio, and define the hedge cost ratio (HCR) as

$$HCR := \frac{X_0}{V_0}. \tag{42}$$

Note that $HCR > 1$ corresponds to $X_0 - V_0 > 0$.

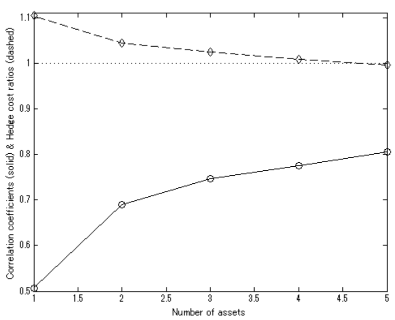

Fig. 1. Correlation coefficients (solid) & Hedge cost ratios (dashed)

To examine the relation of the hedge effect and the HCR with respect to the number of traded underlyings, we varied the number of traded underlying from $m = 1$ to $m = 5$ and obtained Fig. 1, where the solid line refers to the values of correlation coefficients with respect to $m = 1, \ldots, 5$, and the dashed to those of the HCRs. We see that both the hedge effect and the HCR are improved as the number of traded assets increases. Note that the hedge effect and the HCR are both one for the complete market case as shown by the dotted line in Fig. 1.

Next, we consider the basket option's hedge, in which the payoff depends on the weighted average of five stocks, $S_1, \ldots, S_5$, with the same parameter values in Tables I and II. We solve the nonlinear minimum variance hedging problem (30) to approximate the payoff of basket option,

$$g(Y_T), \; Y_T = \frac{1}{5}(S_{1,T} + \cdots + S_{5,T})$$

by the sum of individual options, $f_i^*(S_{i,T}), \; i = 1, \ldots, 5$.

Fig. 2 shows the scatter plot of $g(Y_T)$ vs. $\sum_{i=1}^5 f_i^*(S_{i,T})$. Similar to the first numerical experiment, we can evaluate the hedge effect by correlation coefficient between $g(Y_T)$ and $\sum_{i=1}^5 f_i^*(S_{i,T})$, which is obtained as

$$\mathrm{Corr}\left(g(Y_T), \; \sum_{i=1}^{5} f_i^*(S_{i,T})\right) = 0.98. \tag{43}$$

We see that the payoff of basket option may be approximated well using individual options in this example.

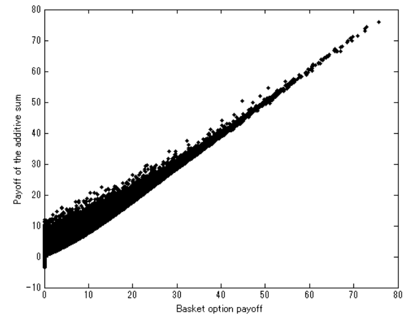

Fig. 2. Scatter plot for basket option's hedge in the ATM case

VII. Conclusion

In this paper, we demonstrated optimal hedges for a class of derivative securities whose underlyings are untraded, using the additive sum of smooth functions of traded assets that minimizes the mean square error. At first, we derived a methodology to compute optimal smooth functions efficiently by solving a system of linear equations based on the necessary and sufficient condition. Then, we extended the idea to basket options consisting of multiple stocks, where individual payoff functions of traded assets are optimally computed in the minimum variance hedging problem. We also provided numerical experiments to illustrate our proposed methodology.

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