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Synergistic Potential Functions for Hybrid Control of Rigid-Body Attitude*

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Abstract— **Achieving global asymptotic stabilization of rigid-body attitude is impossible using smooth feedback; however, this obstruction can be overcome using a hybrid controller that coordinates a "synergistic" family of potential functions and their corresponding feedbacks. In this paper, we show that it is impossible to construct a synergistic family of potential functions from the widely used class of "modified trace functions," despite the fact that one can choose a subset of these functions whose only common critical point is the identity element. With this as motivation, we introduce a parametrized diffeomorphism that is capable of altering the critical behavior of potential functions and generating a synergistic family, paving the way for global asymptotic attitude stabilization of rigid-body attitude by hybrid feedback.**

I. Introduction

It is well known that global asymptotic stabilization of rigid-body attitude is a fundamentally difficult task due to the topological complexity of the underlying state space, $\mathsf{SO}(3)$. This is because $\mathsf{SO}(3)$ is not a vector space, but a compact manifold without boundary, which implies that $\mathsf{SO}(3)$ does not have the topological property of contractibility [1]. This fact precludes the existence of a continuous, time-invariant, state-feedback control law that *globally* asymptotically stabilizes a particular attitude [2]–[4].

The best achievable result with smooth feedback is *almost* global asymptotic stability, where the attraction basin for a desired attitude excludes a nowhere dense set of Lebesgue measure zero. An almost global asymptotic stability result was obtained in [5] using "total energy" as a Lyapunov function. The control paradigm in [5] is based on shaping the potential energy of the closed-loop system by feedback and removing energy from the system with an additional damping-injection term. As in the recent work [6], this leads the system to converge to critical points of an appropriate potential energy (termed a "navigation function" in [5] and an "error function" in [6], [7]), selected by the control designer to have a global minimum at the desired attitude and no other local minima (which lead to asymptotically stable equilibria). Due to topological obstructions, smooth functions on $\mathsf{SO}(3)$ have at least four critical points and the so-called "modified trace function" studied in [5], [7]–[11] is one such example that achieves this minimal number of critical points.

While smooth feedback strategies based on total energy and a single potential function are inherently limited by the topology of $\mathsf{SO}(3)$, a hybrid feedback that coordinates a "synergistic" family of potential functions and their associated feedbacks can overcome such topological obstructions. Inspired by using multiple Lyapunov functions in feedback [12] and analysis [13], synergism, as defined in this work, is a condition on a family of potential functions requiring that at each critical point (that is not the desired attitude) of each potential function in the family, there exists another potential function of lower value. When this condition is satisfied, it leads to a hybrid controller providing for robust global asymptotic stability of a desired attitude [14]. In this work, we show that the class of modified trace functions is not large enough to generate a synergistic family. With this negative result as motivation, we introduce a parametrized diffeomorphism of $\mathsf{SO}(3)$ that is capable of generating a synergistic family of potential functions. In fact, we provide an explicit construction of a synergistic family of only two potential functions, generated by composing the proposed diffeomorphism with a modified trace function. In previous works, the authors have applied the same strategy to planar rotations [15] and reduced attitude stabilization for the "3D pendulum" [16], [17].

The remainder of this paper is organized as follows. Section II provides several preliminary definitions and notation used in this work. Section III examines potential functions in the context of $\mathsf{SO}(3)$. Section IV introduces the notion of synergism and shows that any number of modified trace functions cannot generate a synergistic family. Section V introduces a particular diffeomorphism of $\mathsf{SO}(3)$ and uses it to generate a synergistic family of potential functions. Finally, Section VI provides some concluding remarks and the Appendix provides the proof of some of the main results.

II. Preliminaries

We denote the special orthogonal group of order three as

$$\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I,\ \det(R) = 1\}.$$

Given two vectors $y, z \in \mathbb{R}^3$, their cross product can be represented by a matrix multiplication: $y \times z = [y]_\times z$, where

$$[y]_\times = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}$$

constitutes an isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times 3} : S^\top = -S\}$, the Lie algebra of $\mathsf{SO}(3)$. We denote

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the inverse operation of $[\cdot]_\times$ as $\operatorname{vec}_\times : \mathfrak{so}(3) \to \mathbb{R}^3$, defined implicitly as

$$\operatorname{vec}_\times [y]_\times = y,$$

for all $y \in \mathbb{R}^3$. By defining $\operatorname{skew} : \mathbb{R}^{3\times 3} \to \mathfrak{so}(3)$ as the map $\operatorname{skew} A = (A - A^\top)/2$, we can extend the definition of $\operatorname{vec}_\times$ to all of $\mathbb{R}^{3\times 3}$ by taking its composition with skew [5]. In this direction, we define $\psi : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ as

$$\psi(A) = \operatorname{vec}_\times(\operatorname{skew} A) = \frac{1}{2}\begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

The attitude of a rigid body, denoted R, represents a rotation of coordinates that express a vector in a body-fixed frame to coordinates in an inertial frame. Given a Lebesgue measurable angular velocity $\omega \in \mathbb{R}^3$, expressed in the body-fixed frame and defined on an open interval, the rigid body obeys the kinematic equation

$$\dot{R} = R\,[\omega]_\times \qquad R \in \mathsf{SO}(3).$$

The n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$ is

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}.$$

Given a rotation angle $\theta \in \mathbb{R}$ and an axis $u \in \mathbb{S}^2$, it follows that $e^{\theta[u]_\times} \in \mathsf{SO}(3)$ and

$$\mathcal{R}(\theta, u) := e^{\theta[u]_\times} = I + \sin\theta\,[u]_\times + (1 - \cos\theta)\,[u]_\times^2, \quad (1)$$

where for some $A \in \mathbb{R}^{n\times n}$, e^A denotes the matrix exponential of A. This is commonly known as the angle-axis parametrization of $\mathsf{SO}(3)$, or the Rodrigues formula. One can recover the angle and axis (non-uniquely) as

$$u \sin\theta = \psi(\mathcal{R}(\theta, u)) \qquad 2\cos\theta = \operatorname{trace}(\mathcal{R}(\theta, u)) - 1.$$

We denote the canonical basis for $\mathbb{R}^n$ as $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, we define

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} \qquad \operatorname{vec} A = \begin{bmatrix} A\mathbf{e}_1 \\ \vdots \\ A\mathbf{e}_n \end{bmatrix},$$

as the Kronecker product of A and B and the vectorization of A, respectively. Given vectors $x, y \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{m\times n}$, their inner product is defined as $\langle x, y\rangle := x^\top y$ and $\langle A, B\rangle := \operatorname{trace}(A^\top B) = (\operatorname{vec} A)^\top \operatorname{vec} B$, respectively. The 2-norm of a vector $y \in \mathbb{R}^n$ is $|y| = \sqrt{\langle y, y\rangle}$ and the Frobenius norm of a matrix $A \in \mathbb{R}^{n\times m}$ is $\|A\|_F = \sqrt{\langle A, A\rangle}$.

For differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}$, we define their Jacobian matrix and gradient vector as

$$[\mathfrak{D}\, f(x)]_{ij} = \frac{\partial f_i(x)}{\partial x_j} \qquad \nabla g(x) := (\mathfrak{D}\, g(x))^\top,$$

respectively. For a differentiable function $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$, we use the matrix calculus proposed in [18], where

$$\mathfrak{D}\, F(X) = \frac{\partial \operatorname{vec} F(X)}{\partial \operatorname{vec} X}.$$

This definition induces the following chain and product rules. Suppose that $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$, $G : \mathbb{R}^{p\times q} \to \mathbb{R}^{r\times s}$, $H = G \circ F : \mathbb{R}^{m\times n} \to \mathbb{R}^{r\times s}$, $K : \mathbb{R}^{m\times n} \to \mathbb{R}^{q\times r}$ are differentiable functions. Then, we have

$$\mathfrak{D}\, H(X) = \mathfrak{D}\, G(F(X))\, \mathfrak{D}\, F(X) \quad (2)$$

$$\begin{aligned} \mathfrak{D}(F(X)K(X)) = {} & (K(X)^\top \otimes I_m)\, \mathfrak{D}\, F(X) \\ & + (I_p \otimes F(X))\, \mathfrak{D}\, K(X). \end{aligned} \quad (3)$$

As special cases of $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$, when either the domain or range of F is $\mathbb{R}$, we retain some useful notation that groups the partial derivatives together in a matrix, rather than applying vec to them. In this direction, for differentiable functions $G : \mathbb{R} \to \mathbb{R}^{m\times n}$ and $H : \mathbb{R}^{m\times n} \to \mathbb{R}$, we define

$$\left[\frac{dG(t)}{dt}\right]_{ij} = \frac{dG_{ij}(t)}{dt} \qquad [\nabla H(X)]_{ij} = \frac{\partial H(X)}{\partial X_{ij}}.$$

Also, we denote $\dot{G}(t) := dG(t)/dt$. Then, we have

$$\mathfrak{D}\, G(t) = \frac{d \operatorname{vec} G(t)}{dt} = \operatorname{vec} \frac{dG(t)}{dt}$$

$$\mathfrak{D}\, H(X) = \frac{\partial H(X)}{\partial \operatorname{vec} X} = (\operatorname{vec} \nabla H(X))^\top.$$

Let $V = H \circ G$. Applying the chain rule (2), we have

$$\dot{V}(t) = (\operatorname{vec} \nabla H(G(t)))^\top \operatorname{vec} \dot{G}(t) = \left\langle \nabla H(G(t)), \dot{G}(t) \right\rangle.$$

This agrees with the usual notation for differentiable functions $\gamma : \mathbb{R} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}$, $v = \sigma \circ \gamma$, where

$$\dot{v}(t) = \nabla\sigma(\gamma(t))^\top \dot{\gamma}(t) = \langle \nabla\sigma(\gamma(t)), \dot{\gamma}(t)\rangle.$$

III. Potential Functions and Critical Points

In this section, we describe the family of potential functions on $\mathsf{SO}(3)$. Given a continuously differentiable function $V : \mathsf{SO}(3) \to \mathbb{R}$, there will exist at least four critical points[1] where the shape of the state space interacts with the gradient of V to eliminate infinitesimal change of V with respect to $R \in \mathsf{SO}(3)$. Clearly, we have

$$\dot{V}(R(t)) = \left\langle \nabla V(R), R\,[\omega]_\times \right\rangle = 2\left\langle \omega, \psi(R^\top \nabla V(R)) \right\rangle,$$

where we have used the property that $\operatorname{trace}(A\,[y]_\times) = 2y^\top \psi(A^\top)$ for any $A \in \mathbb{R}^{3\times 3}$ and $y \in \mathbb{R}^3$. Thus, no matter the value of ω, when $R^\top \nabla V(R) = \nabla V(R)^\top R$, there is no infinitesimal change in V. Thus the critical points of V are

$$\operatorname{Crit} V = \{R \in \mathsf{SO}(3) : \psi(R^\top \nabla V(R)) = 0\}.$$

Definition 1. A continuously differentiable function $V : \mathsf{SO}(3) \to \mathbb{R}_{\geq 0}$ is a *potential function on* $\mathsf{SO}(3)$ *(with respect to I)* if $V(R) > 0$ for all $R \in \mathsf{SO}(3) \setminus \{I\}$ and $V(I) = 0$. The class of potential functions on $\mathsf{SO}(3)$ is denoted $\mathscr{P}$.

[1]The fact that there are at least four critical points of any continuously differentiable function on $\mathsf{SO}(3)$ follows from the calculation of its Lusternik-Schnirelmann category, defined as the minimum number of contractible sets needed to cover $\mathsf{SO}(3)$. We refer the reader to [19] for the original work by Lusternik and Schnirelmann, and to [20] for a more modern and thorough treatment of the ideas in [19]. We note that the Lusternik-Schnirelmann category of $\mathsf{SO}(3)$ is listed in [21], [22].

We note that since $\mathsf{SO}(3)$ is a Lie group, one can use the group properties to translate potential functions. That is, if V is a potential function on $\mathsf{SO}(3)$ with respect to I, then $V_d(R) = V(R_d^\top R)$ is a potential function on $\mathsf{SO}(3)$ with respect to $R_d \in \mathsf{SO}(3)$.

A commonly used potential function on $\mathsf{SO}(3)$ is the so-called "modified trace function," defined as

$$P_A(X) = \operatorname{trace}(A(I - X)) = \tfrac{1}{2}\langle I - X, A(I - X)\rangle, \quad (4)$$

which, taking inspiration from [8], was first introduced for attitude control in [5]. It is helpful to see P_A in the angle-axis representation through (1). Given $\theta \in \mathbb{R}$ and $u \in \mathbb{S}^2$,

$$P_A(\mathcal{R}(\theta, u)) = (1 - \cos\theta)(\operatorname{trace}(A) - u^\top A u). \quad (5)$$

As we now show, the critical points of P_A are the identity element and rotations of $180°$ about the (real) eigenvectors of A. In this direction, for each $A \in \mathbb{R}^{n\times n}$, we define

$$\mathscr{E}(A) = \{(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n : Av = \lambda v,\ |v| = 1\}$$

and its projections onto $\mathbb{C}$ and $\mathbb{C}^n$ as

$$\mathscr{E}_\lambda(A) = \{\lambda \in \mathbb{C} : \exists(\lambda, v) \in \mathscr{E}(A)\}$$
$$\mathscr{E}_v(A) = \{v \in \mathbb{C}^n : \exists(\lambda, v) \in \mathscr{E}(A)\}.$$

Recall that when A is symmetric, $\mathscr{E}_\lambda(A) \subset \mathbb{R}$. In this case, it is helpful to constrain the sets of unit eigenvectors to be purely real as well. In this direction, we let

$$\mathscr{E}_v^{\mathbb{R}}(A) = \{v \in \mathbb{R}^n : \exists(\lambda, v) \in \mathscr{E}(A)\}.$$

Lemma 2. *Let $A \in \mathbb{R}^{3\times 3}$ be symmetric and positive definite. Then, the function P_A satisfies*

$$\operatorname{Crit} P_A = \{I\} \cup \mathcal{R}(\pi, \mathscr{E}_v^{\mathbb{R}}(A)). \quad (6)$$

When A has distinct eigenvalues, P_A obtains the minimal number of critical points possible.

Proof. By an obvious calculation, we have that

$$\nabla P_A = -A$$
$$\operatorname{Crit} P_A = \{R \in \mathsf{SO}(3) : R^\top A = AR\}.$$

Then, since $R^{-1} = R^\top$, it follows that

$$RA^2 = A^2 R.$$

Then, since $RR^\top = R^\top R = I$ for every $R \in \mathsf{SO}(3)$, it follows that R is normal (that is, $R^\top R = RR^\top$) and so it is diagonalizable [23, Ch. 7.5]. Clearly, both A^2 and A are also normal (and diagonalizable). Then, by [24, Theorem 1.3.19], since A^2 and R commute, they are simultaneously diagonalizable and hence, have the same set of eigenvectors. Moreover, since A commutes with A^2, they, as normal matrices, must also have the same set of eigenvectors. Hence, A and R must have the same set of eigenvectors.

In this direction, let $R = \mathcal{R}(\theta, v)$ for some $\theta \in \mathbb{R}$ and $v \in \mathbb{S}^2$. Then, $\mathscr{E}_\lambda(R) = \{1, e^{i\theta}, e^{-i\theta}\}$. Let $(\phi, u) \in \mathscr{E}(R)$ and let $(\lambda, u) \in \mathscr{E}(A)$. It follows that

$$ARR^\top Au = \lambda^2\phi^2 u = A^2 u = \lambda^2 u.$$

Since A is positive definite, it follows that $\lambda > 0$ and so, $\phi^2 = 1$. This implies that $\theta \in \{0, \pi\}$. Finally, since $v \in \mathscr{E}_v(A)$, it follows that R is either the identity, or a $180°$ rotation about an eigenvector of A. This proves (6).

When A has distinct eigenvalues, for any $(\lambda, v) \in \mathscr{E}(A)$, the dimension of the null space of $A - \lambda I$ is one (see [23, Ch. 7]). Since $\mathcal{R}(\pi, -v) = \mathcal{R}(-\pi, v) = \mathcal{R}(\pi, v)$, it follows that $\mathcal{R}(\pi, \mathscr{E}_v^{\mathbb{R}}(A))$ has only 3 points. Hence P_A has four critical points, the minimal number possible [19]. ☐

IV. Synergistic Potential Functions on $\mathsf{SO}(3)$

Definition 3. Let $Q \subset \mathbb{Z}$ be a finite set of N elements and define $\mu : \mathscr{P}^N \to \mathbb{R}_{\geq 0}$ such that for each family of potential functions $\mathscr{V} = \{V_q\}_{q\in Q}$,

$$\mu(\mathscr{V}) = \min_{\substack{q\in Q \\ x \in \operatorname{Crit} V_q \setminus \{I\}}} \max_{p \in Q} V_q(x) - V_p(x). \quad (7)$$

The family $\mathscr{V}$ is *synergistic* if there exists $\delta > 0$ such that

$$\mu(\mathscr{V}) > \delta, \quad (8)$$

where we say that $\mathscr{V}$ is synergistic *with gap exceeding δ*.

Since one can use modified trace functions, defined in (4), to produce any number of potential functions on $\mathsf{SO}(3)$ having only I as a common critical point, one might wonder whether one can find a synergistic family of potential functions of the form $\mathscr{V} = \{P_{A_q}\}_{q\in Q}$. It would seem possible, especially since it is easy to separate the critical sets between functions by choosing different A's with different eigenvectors; however, the following result shows that this is not the case.

Theorem 4. *Any finite family of modified trace functions is not synergistic.*

Proof. Let $\mathscr{V} = \{P_{A_q}\}_{q\in Q}$ be a family of modified trace functions as defined in (4). Recalling Lemma 2, we have $\operatorname{Crit} P_{A_q} \setminus \{I\} = \mathcal{R}(\pi, \mathscr{E}_v^{\mathbb{R}}(A_q))$. It follows from (5) that for any $p, q \in Q$ and any $(\lambda, u) \in \mathscr{E}(A_q)$ we have

$$P_{A_q}(\mathcal{R}(\pi, u)) - P_{A_p}(\mathcal{R}(\pi, u)) =$$
$$2(\operatorname{trace}(A_q) - \lambda - \operatorname{trace}(A_p) + u^\top A_p u).$$

By definition, the family $\mathscr{V}$ is synergistic when for every $q \in Q$ and $(\lambda, u) \in \mathscr{E}(A_q)$, there exists p such that

$$\operatorname{trace}(A_q) - \lambda - \operatorname{trace}(A_p) + u^\top A_p u > 0. \quad (9)$$

In fact, this is impossible to satisfy. Let

$$q^* = \operatorname*{argmin}_{q\in Q} \left(\operatorname{trace}(A_q) - \max \mathscr{E}_\lambda(A_q)\right).$$

Now, let $\lambda^* = \max \mathscr{E}_\lambda(A_{q^*})$ and let u^* be a unit eigenvector corresponding to λ^* for A_{q^*}. Then, it follows that

$$\operatorname{trace}(A_{q^*}) - \lambda^* - \operatorname{trace}(A_p) + u^{*\top} A_p u^* \leq$$
$$\operatorname{trace}(A_{q^*}) - \lambda^* - \operatorname{trace}(A_p) + \max \mathscr{E}_\lambda(A_p) \leq 0$$

for every $p \in Q$. That is, there exists $q \in Q$ and $(\lambda, u) \in \mathscr{E}(A_q)$ such that (9) is not satisfied for every $p \in Q$. This completes the proof. ☐

While Theorem 4 may seem counter-intuitive, the following lemma sheds some light on why this is the case. Recall that for any function $V : \mathsf{SO}(3) \to \mathbb{R}$, that $\dot{V}(R) = 2\omega^\top \psi(R^\top \nabla V(R))$. Then, to flow down the gradient of V, one can set $\omega = -2\psi(R^\top \nabla V(R))$.

Lemma 5. *Let $A \in \mathbb{R}^{3\times 3}$ be symmetric and positive definite. Suppose the kinematic feedback $\omega = -2\psi(R^\top \nabla P_A(R)) = 2\psi(R^\top A)$ is applied to the system $\dot{R} = R\,[\omega]_\times$. Then, $\mathcal{R}(\pi, \mathbb{S}^2)$ is invariant for the closed-loop system*

$$\dot{R} = R\left[2\,\mathrm{vec}_\times\,\mathrm{skew}\,R^\top A\right]_\times = R - RAR.$$

Proof. Suppose that $R = R^\top$. Then, $\dot{R}^\top = (R - RAR)^\top = R - RAR = \dot{R}$. Since $\dot{R}$ is symmetric when R is symmetric, the set of symmetric rotations, $\{I\} \times \mathcal{R}(\pi, \mathbb{S}^2)$ is invariant to the flow $\dot{R} = R - RAR$. Note that $\{I\}$ and $\mathcal{R}(\pi, \mathbb{S}^2)$ are disjoint and in fact, there exists an open neighborhood $U \ni I$ such that $U \cap \mathcal{R}(\pi, \mathbb{S}^2) = \emptyset$. This implies that $\mathcal{R}(\pi, \mathbb{S}^2)$ is invariant. □

These results raise some important questions. Does there exist a synergistic family of potentials? If so, what is the minimum number N such that there exists a synergistic family of N potential functions? In the following section, we give the answers: yes and two, respectively, by construction.

V. Synergistic Potentials via Angular Warping

Let $\mathscr{C}^1(\mathsf{SO}(3))$ denote the set of continuously differentiable real-valued functions on $\mathsf{SO}(3)$ and let

$$\mathscr{C}^1_I(\mathsf{SO}(3)) = \{P \in \mathscr{C}^1(\mathsf{SO}(3)) : P(I) = 0\}.$$

Then, define the function $\mathcal{T} : \mathsf{SO}(3) \to \mathsf{SO}(3)$ as

$$\mathcal{T}(R, k, P, u) = e^{kP(R)[u]_\times} R, \tag{10}$$

where $k \in \mathbb{R}$, $P \in \mathscr{C}^1_I(\mathbb{S}^2)$, and $u \in \mathbb{S}^2$ are fixed parameters. In context, given a gain $k \in \mathbb{R}$, a function $P \in \mathscr{C}^1_I(\mathbb{S}^2)$, and a rotation axis $u \in \mathbb{S}^2$, $\mathcal{T}$ applies a rotation in the amount of $kP(R)$ to R about the vector u.

Theorem 6. *Let $k \in \mathbb{R}$, $P \in \mathscr{C}^1_I(\mathsf{SO}(3))$, $u \in \mathbb{S}^2$, $V \in \mathscr{P}$, $U = V \circ \mathcal{T}$, and define $\Theta : \mathsf{SO}(3) \to \mathbb{R}^{3\times 3}$ as*

$$\Theta(R) = I + 2kR^\top u \psi(\nabla P(R) R^\top)^\top R.$$

Then, the transformation $\mathcal{T} : \mathsf{SO}(3) \to \mathsf{SO}(3)$ satisfies

$$\mathcal{T}(I) = I \tag{11}$$

$$\begin{aligned}\mathfrak{D}\mathcal{T}(R) = \left(I \otimes \exp(kP(R)\,[u]_\times)\right)(I+\\ k\,\mathrm{vec}([u]_\times R)(\mathrm{vec}\,\nabla P(R))^\top)\end{aligned} \tag{12}$$

$$\det \mathfrak{D}\mathcal{T}(R) = 1 + 2k\left\langle u, \psi(\nabla P(R) R^\top)\right\rangle \tag{13}$$

$$\psi(R^\top \nabla U(R)) = \Theta(R)^\top \psi\left(\mathcal{T}(R)^\top \nabla V(\mathcal{T}(R))\right) \tag{14}$$

$$\dot{\mathcal{T}}(R) = \mathcal{T}(R)\left[\Theta(R)\omega\right]_\times. \tag{15}$$

Moreover, when $\det \mathfrak{D}\mathcal{T}(x) \neq 0$,

$$\Theta(R)^{-1} = I - \frac{2kR^\top u \psi(\nabla P(R) R^\top)^\top R}{\det \mathfrak{D}\mathcal{T}(R)} \tag{16}$$

$$\mathrm{Crit}\, U = \mathcal{T}^{-1}\left(\mathrm{Crit}\, V\right). \tag{17}$$

for all $R \in \mathsf{SO}(3)$.

Proof. See Appendix. □

Definition 7. A map $h : X \to Y$ is a *diffeomorphism* if it is bijective, differentiable, and has a differentiable inverse. The map h is a *local diffeomorphism* if for each $x \in X$, there exists an open set $U \subset X$ containing x such that $h : U \to h(U)$ is a diffeomorphism.

When $\det \mathfrak{D}\mathcal{T}(R) \neq 0$, $\mathcal{T}$ is a local diffeomorphism of $\mathsf{SO}(3)$ by the inverse function theorem. Additionally, some special structure of $\mathcal{T}$ implies that $\mathcal{T}$ is also a *global* diffeomorphism for k sufficiently small in magnitude.

Theorem 8. *Given $k \in \mathbb{R}$, $P \in \mathscr{C}^1_I(\mathsf{SO}(3))$, $u \in \mathbb{S}^2$, if*

$$\sqrt{2}|k| \max \|\nabla P(\mathsf{SO}(3))\|_F < 1, \tag{18}$$

then $\mathcal{T} : \mathsf{SO}(3) \to \mathsf{SO}(3)$ is a diffeomorphism.

Proof. See Appendix. □

A simple consequence of Theorem 6 is that if V is a potential function, then so is $V \circ \mathcal{T}$.

Corollary 9. *Let $V \in \mathscr{P}$, $P \in \mathscr{C}^1_I(\mathsf{SO}(3))$, and $u \in \mathbb{S}^2$. If k satisfies* (18) *then, $V \circ \mathcal{T} \in \mathscr{P}$.*

Applying the diffeomorphism $\mathcal{T}$ to an existing potential function produces a new potential function, with new critical points that are calculated by applying the inverse of $\mathcal{T}$ to the critical points of the original potential function, as shown by (17) of Theorem 6.

We now provide an existence result by example and show that it is possible to generate a synergistic family of potential functions using the transformation $\mathcal{T}$ and a particular potential function of the type P_A, defined in (4). In fact, this example shows that one can generate such a family of potential functions from a single potential function.

Example 10. Let $w = [11\ \ 12\ \ 13]^\top$, $\Delta = 3\,\mathrm{diag}(w)/\sum_{i=1}^3 w_i$, $u = w/\|w\|_2$. Then, define $V_1^k(R) = P_\Delta(\mathcal{T}(R, k, u, P_\Delta))$, $V_2^k(R) = P_\Delta(\mathcal{T}(R, -k, u, P_\Delta))$. By Theorem 8, $\mathcal{T}$ is a diffeomorphism (and $V_i^k \in \mathscr{P}$ for $i \in \{1, 2\}$) when k satisfies (18), or

$$|k| < 1/(\sqrt{2}\|\Delta\|_F) \approx 0.4073.$$

Define the family $\mathscr{V}^k = \{V_1^k, V_2^k\}$. It is not difficult to compute $\mu(\mathscr{V}^k)$ for every k satisfying this bound, for which we provide a plot in Fig. 1.

VI. Conclusion

The task of global asymptotic stabilization of rigid body attitude is impossible with smooth feedback; however, this obstacle can be overcome if one is willing to relax the control laws to allow for a hybrid feedback that coordinates more than one potential-based feedback in a hysteretic fashion. As we have shown in the companion paper [14], to achieve global asymptotic stability, such a hybrid controller requires a "synergism" property of the potential functions, which is not attributable to any family of "modified trace functions."

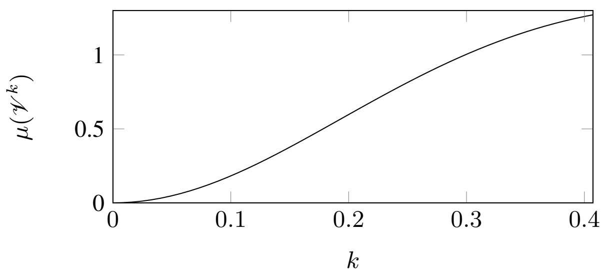

Fig. 1. Plot of $\mu(\mathscr{V}^k)$ for the family of potential functions in Example 10. A choice of $0 < \delta < \mu(\mathscr{V}^k)$ provides a synergistic family of potential functions on $\mathsf{SO}(3)$ with respect to I.

To generate new potential functions capable of forming a synergistic family, we proposed a parametrized diffeomorphism that stretches and compresses $\mathsf{SO}(3)$ while leaving the identity element a fixed point. When composed with an existing potential function, this diffeomorphism is capable of relocating critical points. Applying this diffeomorphism with different parameters to the same potential function allows one to construct a synergistic family, paving the way for global asymptotic stabilization by hybrid feedback.

Appendix

In what follows, we employ many identities relating to vectorization of matrices, Kronecker products, and the vector cross product. Most helpful is the operator $\Gamma : \mathbb{R}^{3\times 3} \to \mathbb{R}^{9\times 3}$ defined as

$$\Gamma(A) = -\begin{bmatrix} [A\mathbf{e}_1]_\times & [A\mathbf{e}_2]_\times & [A\mathbf{e}_3]_\times \end{bmatrix}^\top .$$

With this definition, we use the identities

$$\operatorname{vec} ABC = (C^\top \otimes A) \operatorname{vec} B \tag{19}$$
$$(\operatorname{vec} A)^\top \operatorname{vec} B = \operatorname{trace}(A^\top B) \tag{20}$$
$$\operatorname{trace}(A\,[z]_\times) = 2z^\top \psi(A^\top) \tag{21}$$
$$|\psi(A)|^2 = (1/2)\|\operatorname{skew} A\|_F^2 \tag{22}$$
$$\psi(AB^\top) = (1/2)\sum_{i=1}^{3} [B\mathbf{e}_i]_\times A\mathbf{e}_i \tag{23}$$
$$\psi(AB^\top) = (1/2)\Gamma(A)^\top \operatorname{vec} B \tag{24}$$
$$(I \otimes R)\Gamma(A)R^\top = \Gamma(RA) \tag{25}$$
$$\operatorname{vec}\,[z]_\times A = (I \otimes [z]_\times) \operatorname{vec} A = -\Gamma(A)z \tag{26}$$
$$\operatorname{vec} R\,[z]_\times = \operatorname{vec}\,[Rz]_\times R = -\Gamma(R)Rz, \tag{27}$$

where A, B, C are matrices of appropriate sizes, $z \in \mathbb{R}^3$, and $R \in \mathsf{SO}(3)$. We note that (25) follows from the identity $R\,[u]_\times R^\top = [Ru]_\times$ for all $R \in \mathsf{SO}(3)$ and $u \in \mathbb{R}^3$.

Proof of Theorem 6. Property (11) is obvious since $P(I) = 0$ by assumption, so $\exp 0 = I$ and $\mathcal{T}(I) = I$. To prove (12), we appeal to the product and chain rules for matrix differentiation, (3) and (2), respectively. We have

$$\mathfrak{D}\mathcal{T}(R) = \left(R^\top \otimes I\right)\mathfrak{D}\exp(kP(R)\,[u]_\times) + (I \otimes R)\,\mathfrak{D}R.$$

Clearly, $\mathfrak{D}R = I$. Next, we recall that $d/dt\exp(At) = A\exp(At) = \exp(At)A$ and use the chain rule to calculate

$$\begin{aligned}
&\mathfrak{D}\exp(kP(R)\,[u]_\times) \\
&\qquad = \left.\frac{d\operatorname{vec}\exp(t\,[u]_\times)}{dt}\right|_{t=kP(R)} \frac{\partial kP(R)}{\partial \operatorname{vec} R} \\
&\qquad = \operatorname{vec}\left(k\exp(kP(R)\,[u]_\times)\right)[u]_\times (\operatorname{vec}\nabla P(R))^\top .
\end{aligned}$$

Recalling property (19), we have

$$\begin{aligned}
&\mathfrak{D}\mathcal{T}(R) = \\
&\left(I \otimes \exp(kP(R)\,[u]_\times)\right)\left(I + k\operatorname{vec}\,[u]_\times R(\operatorname{vec}\nabla P(R))^\top\right).
\end{aligned}$$

This proves (12).

Recalling that for $A, B \in \mathbb{R}^{n\times n}$, we have $\det AB = \det A \det B$, $\det(\exp(A)) = \exp(\operatorname{trace}(A))$ [23], and for any skew-symmetric matrix S, $\operatorname{trace}(S) = 0$, it follows that

$$\det \mathfrak{D}\mathcal{T}(R) = \det\left(I + k\operatorname{vec}\,[u]_\times R(\operatorname{vec}\nabla P(R))^\top\right).$$

To show (13) we calculate this determinant from the formula for "rank-one updates": for any $y, z \in \mathbb{R}^n$, $\det(I + yz^\top) = 1 + z^\top y$ [23]. Recalling (20) and (21), we have

$$\det \mathfrak{D}\mathcal{T}(R) = 1 + 2k\left\langle u, \psi(\nabla P(R)R^\top)\right\rangle .$$

Continuing, we now show (15). First, we show that

$$\mathfrak{D}\mathcal{T}(R)\Gamma(R) = \Gamma(\mathcal{T}(R))\exp(kP(R)\,[u]_\times)\Lambda(R),$$

where

$$\Lambda(R) = (I + ku\psi(\nabla P(R)R^\top)) = R\Theta(R)R^\top .$$

In the following development, we let $\Phi = \exp(kP(R)\,[u]_\times)$. Applying (26) we have

$$\mathfrak{D}\mathcal{T}(R)\Gamma(R) = (I \otimes \Phi)\Gamma(R)\left(I - ku(\operatorname{vec}\nabla P(R))^\top \Gamma(R)\right)$$

Then, (24), (25), and (26), and $[u]_\times^\top = -[u]_\times$, imply

$$\mathfrak{D}\mathcal{T}(R)\Gamma(R) = \Gamma(\mathcal{T}(R))\exp(kP(R)\,[u]_\times\Lambda(R).$$

Since $\operatorname{vec}\dot{R} = -\Gamma(R)R\omega$, it follows that

$$\operatorname{vec}\dot{\mathcal{T}}(R) = \mathfrak{D}\mathcal{T}(R(t))\mathfrak{D}R(t) = \operatorname{vec}\mathcal{T}(R)\,[\Theta(R)\omega]_\times .$$

This proves (15). Then, noting that

$$\begin{aligned}
\frac{dV(\mathcal{T}(R(t)))}{dt} &= 2\omega^\top \psi(R^\top \nabla(V \circ \mathcal{T})(R)) \\
&= \left\langle \nabla V(\mathcal{T}(R)), \mathcal{T}(R)\,[\Theta(R)\omega]_\times \right\rangle \\
&= 2\omega^\top \Theta(R)^\top \psi\left(\mathcal{T}(R)^\top \nabla V(\mathcal{T}(R))\right),
\end{aligned}$$

it follows that

$$\psi(R^\top \nabla(V \circ \mathcal{T})(R)) = \Theta(R)^\top \psi\left(\mathcal{T}(R)^\top \nabla V(\mathcal{T}(R))\right).$$

This proves (14).

Finally, (16) follows from the follows from the Sherman-Morrison formula (see [23, Ch. 3.8]) when $\det \mathfrak{D}\mathcal{T}(R) \neq 0$, and (17) follows from the fact that $\Theta(R)$ has full rank when $\det \mathfrak{D}\mathcal{T}(R) \neq 0$. $\square$

Proof of Theorem 8. It follows from (13) that if for all $R \in \mathsf{SO}(3)$, $\det \mathfrak{D}\mathcal{T}(R) = 1 + 2k \left\langle u, \psi(\nabla P(R)R^\top)\right\rangle \neq 0$, then $\mathcal{T}$ is a local diffeomorphism. This is guaranteed when $2|k|| \left\langle u, \psi(\nabla P(R)R^\top)\right\rangle|_2 < 1$, for all $R \in \mathsf{SO}(3)$. It follows from the general Cauchy-Bunyakovskii-Schwarz inequality ([23, Ch. 5.3]) and $|u|_2 = 1$ that

$$| \left\langle u, \psi(\nabla P(R)R^\top)\right\rangle | = |\psi(\nabla P(R)R^\top)|_2.$$

Applying (22) and noting that $\|\operatorname{skew} A\|_F \leq \|A\|_F$ and $\|AU\|_F = \|A\|_F$ for any $A \in \mathbb{R}^{m\times n}$ and any orthogonal matrix U, we have

$$|\psi(\nabla P(R)R^\top)|_2 \leq \tfrac{1}{\sqrt{2}}\|\nabla P(R)\|_F$$

So, if $2|k|\|\nabla P(R)\|_F/\sqrt{2} = \sqrt{2}|k|\|\nabla P(R)\|_F < 1$ for all $R \in \mathsf{SO}(3)$, it follows that $\det \mathfrak{D}\mathcal{T}(R) \neq 0$ for all $R \in \mathsf{SO}(3)$. This is clearly satisfied if

$$\sqrt{2}|k| \max \|\nabla P(\mathsf{SO}(3))\|_F < 1,$$

which is (18). Moreover, when $\det \mathfrak{D}\mathcal{T}(R) \neq 0$ for all $R \in \mathsf{SO}(3)$, $\Theta(R)^{-1}$ exists for all $R \in \mathsf{SO}(3)$. At this point, the inverse function theorem guarantees that $\mathcal{T}(R)$ is everywhere a local diffeomorphism. We now prove that $\mathcal{T}$ is a diffeomorphism when k satisfies (18).

Since $\mathsf{SO}(3)$ is compact, $\mathcal{T}$ is a proper map, that is, the inverse image of any compact set is compact. This implies that $\mathcal{T}$ is surjective [25]. We now verify that $\mathcal{T}$ is injective.

Suppose that $P \in \mathscr{P}$ and $u \in \mathbb{S}^2$ are fixed parameters and let $\mathcal{T}_k$ denote a particular instance of $\mathcal{T}$, with the specified value of k. Suppose that k^* satisfies (18). Define

$$\tau = \sup\{k' \in [0, |k^*|] : \mathcal{T}_k \text{ is a diffeomorphism } \forall k \in [-k', k']\}. \quad (28)$$

Now, if $\mathcal{T}_\tau$ is a diffeomorphism, there exists $\epsilon > 0$ such that $\mathcal{T}_k$ is a diffeomorphism for all $k \in (\tau - \epsilon, \tau + \epsilon)$ [1, Ch. 1, "Stability Theorem"]. But this contradicts the definition of τ in (28). We now assume that $\mathcal{T}_\tau$ is not injective.

Define the function $\mathcal{W} : \mathsf{SO}(3) \times \mathbb{R} \to \mathsf{SO}(3) \times \mathbb{R}$ as

$$\mathcal{W}(R, k) = (\mathcal{T}_k(R), k).$$

Using (12) of Theorem 6 and recalling again that $d/dt \exp(At) = A \exp(At) = \exp(At)A$, we calculate the partial derivatives of $\mathcal{W}$ as

$$\mathfrak{D}\mathcal{W}(R, k) = \begin{bmatrix} \mathfrak{D}\mathcal{T}_k(R) & \operatorname{vec} P(R)\,[u]_\times\, \mathcal{T}_k(R) \\ 0 & 1 \end{bmatrix}.$$

Since $\mathfrak{D}\mathcal{T}_k(R)$ is nonsingular whenever (18) is satisfied, $\mathfrak{D}W(R, k)$ is also nonsingular under the same condition, so the inverse function theorem implies that $\mathcal{W}$ is also a local diffeomorphism.

Since $\mathcal{T}_\tau$ is not injective, let $R_1, R_2, R^* \in \mathsf{SO}(3)$ satisfy $\mathcal{T}_\tau(R_1) = \mathcal{T}_\tau(R_2) = R^*$. But now, since $\mathcal{W}$ is a local diffeomorphism, there exist disjoint open sets $U_i \subset \mathsf{SO}(3)$, and $\epsilon > 0$ such that $R_i \in U_i$, and that $\mathcal{W}$ restricted to $U_i \times (\tau - \epsilon, \tau + \epsilon)$ is a diffeomorphism. But then, there exist $R_1' \neq R_2'$ and $k' < \tau$ such that $\mathcal{T}_{k'}(R_1') = \mathcal{T}_{k'}(R_2') = R^*$. This again contradicts the definition of τ in (28). Hence, for all k satisfying (18), $\mathcal{T}_k$ must be a diffeomorphism. □

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