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A Generalized Zames-Falb Multiplier

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Abstract— **Lur'e systems represent feedback interconnections of a Linear Time-Invariant system with a nonlinear operator. The Zames-Falb criterion is a powerful tool to determine the stability of a Lur'e system when the nonlinear operator is defined by a monotonic, odd and single-valued function. The article provides a generalization of the Zames-Falb criterion for the analysis of stability of a Lur'e system for nonlinear static operators that are "approximately" monotonic and odd. This result extends the standard Zames-Falb criterion with an additional notion of "robustness".**

I. Introduction

In this work, we derive a generalized form of the Zames-Falb multiplier for the analysis of stability of Lur'e systems [1]. Lur'e systems are given by the feedback interconnection of a linear time-invariant system $\mathcal{G}$ with a nonlinear block Δ (see Figure 1). A large number of real systems have this structure. Recent examples are provided by Atomic Force Microscopes [2],[3] and other Microelectromechanical systems [4], [5], [6]. Many studies have targeted the problem of "absolute stability", where the global asymptotic stability of the origin is sought with respect to a class of nonlinear operators, and thus provide a robust notion of stability. Classical results include the Popov (see [7]) and the circle criteria (see [8] and [9]) that provide sufficient conditions for global asymptotic stability when the nonlinearity is restricted to be time-invariant and time varying respectively. A relatively new approach involving only input-output maps is given by the Integral Quadratic Constraint (IQC) methods pioneered in [10]. In this approach, a quadratic constraint is used to characterize the nonlinearity playing the role of the sector condition used in the other classical absolute stability criteria. Since the sector relations can be derived via special IQC conditions, such an approach provides a unifying theoretical framework. Furthermore, it provides the powerful capability of seamlessly integrating different characterizations of the nonlinearity, as well. The main limitation to the general applicability of this technique is given by the necessity of identifying an IQC that properly characterizes the nonlinearity of interest. For example, the adoption of "standard" IQC's such as those definining the circle or the Popov criteria can lead to conservative results. Indeed, the sector conditions to which they are equivalent could include a class of nonlinearities that is too broad. On the other hand, an IQC such as the Zames-Falb one, is satisfied by nonlinearities with very strict characteristics (odd symmetry and monotonicity), but it does not provide any notion of "robustness" in the case of uncertainties in the knowledge of the nonlinearity. In this paper we introduce a natural generalization of the Zames-Falb multiplier that bridges the gap between the circle criterion condition and the standard Zames-Falb one. The paper is structured as follows: in Section II we derive an IQC that is satisfied by a class of nonlinearities that are approximatively odd and monotonic; in Section III we use such a result in order to provide a stability criterion; finally Section IV shows the utility and limits of our techniques through numerical examples.

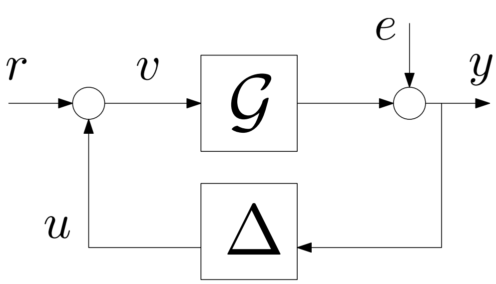


Fig. 1. A Lur'e system

Notation

$\mathbb{R}$: set of real numbers
C : set of complex numbers
L_2 : set of square integrable functions on $\mathbb{R}$
L_{2e} : set of functions on $\mathbb{R}$ that are square integrable on any compact set.

II. Generalized Zames-Falb Multiplier

We introduce the class of quasi-monotonic-and-odd functions that will be used to formulate a stability criterion.

Definition 1: Let $n : \mathbb{R} \to \mathbb{R}$ be a single-valued function. We say that n is a quasi-monotonic-and-odd function with *spread* $D < 1$ and *skeleton* $\overline{n} : \mathbb{R} \to \mathbb{R}$ if there exists a function $\delta : \mathbb{R} \to \mathbb{R}$ such that

- $n(y) = \overline{n}(y)[1 + \delta(y)]$
- $\overline{n}(y)$ is monotonic non-decreasing and odd
- $|\delta(y)| \le D < 1$ for every $y \in \mathbb{R}$.

A graphical representation of a quasi-monotonic-and-odd function is given in Figure 2.

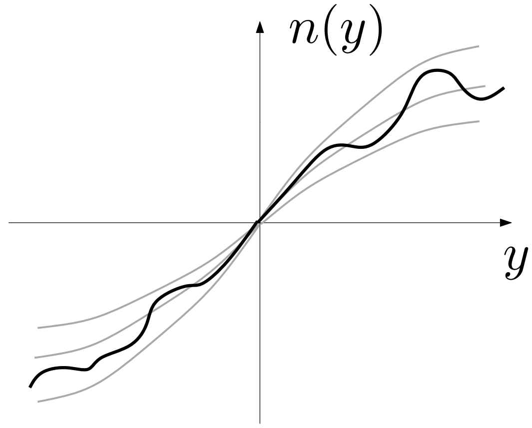


Fig. 2. A quasi-monotonic-and-odd function $n(y)$ is contained in an "envelope" of a monotonic and odd function $\overline{n}(y)$.

We introduce the main technical result of the paper.

Theorem 2.1 (Generalized Zames-Falb multiplier): Let $n:\mathbb{R}\to\mathbb{R}$ be a quasi-monotonic-and-odd function with spread $D<1$ and skeleton $\overline{n}$. Assume that

- $yn(y)\geq 0$ for every y
- $y(t)\in L_2$ implies $n(y(t))\in L_2$
- $h(t)$ is a absolutely integrable function such that

$$\|h(t)\|_1 \leq \left(\frac{1-D}{1+D}\right)^2. \tag{1}$$

Then, for every $y(t)\in L_2$, we have that

$$\int_{-\infty}^{+\infty} n(y(t))\left[y(t)-\int_{-\infty}^{+\infty} h(\tau)y(t+\tau)d\tau\right]dt \geq 0 \tag{2}$$

Proof: The derivation follows the line of [11] for exactly monotonic and odd nonlinearities, but the additional degree of freedom given by the presence of the uncertainty D introduces technical complications that need to be taken into account.

Define the potential function

$$P(y) := \int_0^y n(y')dy'. \tag{3}$$

Observe that

$$\int_a^b n(y')dy' = \tag{4}$$

$$= P(b)-P(a) \leq \begin{cases} (b-a)\sup_{a\leq y'\leq b} n(y') & \text{if } a\leq b \\ (a-b)\sup_{b\leq y'\leq a} -n(y') & \text{if } a>b. \end{cases} \tag{5}$$

Observe that, since $n(y)y\geq 0$, it holds that

$$\overline{n}(y)(1-D\mathrm{sgn}(y)) \leq n(y) \leq \overline{n}(y)(1+D\mathrm{sgn}(y)). \tag{6}$$

Since $\overline{n}(y)$ is monotonic non-decreasing and $0\leq D<1$, we have that

$$\sup_{a\leq y'\leq b} n(y') \leq \overline{n}(b)[1+D\mathrm{sgn}(b)] \qquad \text{if } a\leq b \tag{7}$$

$$\sup_{b\leq y'\leq a} -n(y') \leq -\overline{n}(b)[1-D\mathrm{sgn}(b)] \qquad \text{if } a>b. \tag{8}$$

Since $\overline{n}$ is odd , we have that

$$\sup_{a\leq y'\leq b} n(y') \leq -\overline{n}(-b)[1+D\mathrm{sgn}(b)] \qquad \text{if } a\leq b \tag{9}$$

$$\sup_{b\leq y'\leq a} -n(y') \leq \overline{n}(-b)[1-D\mathrm{sgn}(b)] \qquad \text{if } a>b. \tag{10}$$

Using the last four relations, the fact that $\overline{n}(b)b\geq 0$ and $0\leq D<1$, we get the following bounds in terms of $n(b)$ and $n(-b)$

$$\sup_{a\leq y'\leq b} n(y') \leq n(b)\frac{1+D\mathrm{sgn}(b)}{1-D\mathrm{sgn}(b)} \tag{11}$$

$$\sup_{b\leq y'\leq a} -n(y') \leq -n(b)\frac{1-D\mathrm{sgn}(b)}{1+D\mathrm{sgn}(b)} \tag{12}$$

$$\sup_{a\leq y'\leq b} n(y') \leq -n(-b)\frac{1+D\mathrm{sgn}(b)}{1-D\mathrm{sgn}(b)} \tag{13}$$

$$\sup_{b\leq y'\leq a} -n(y') \leq n(-b)\frac{1-D\mathrm{sgn}(b)}{1+D\mathrm{sgn}(b)}. \tag{14}$$

Plugging the two relations (11) and (12) in (4) we get

$$P(b)-P(a) \leq n(b)(b-a)\frac{1+D\mathrm{sgn}(b)}{1-D\mathrm{sgn}(b)} \qquad \text{if } a\leq b \tag{15}$$

$$P(b)-P(a) \leq n(b)(b-a)\frac{1-D\mathrm{sgn}(b)}{1+D\mathrm{sgn}(b)} \qquad \text{if } a>b, \tag{16}$$

or, more succintly,

$$P(b)-P(a) \leq n(b)(b-a)\frac{1+D\mathrm{sgn}[b(b-a)]}{1-D\mathrm{sgn}[b(b-a)]}. \tag{17}$$

Using the other two relations (13) and (14) we find

$$P(b)-P(a) \leq -n(-b)(b-a)\frac{1+D\mathrm{sgn}[b(b-a)]}{1-D\mathrm{sgn}[b(b-a)]}. \tag{18}$$

For every $y(t)\in L_2(-\infty,+\infty)$, $P(y(t))$ is Lebesgue-integrable since, from (17) with $b=y(t)$ and $a=0$, we have

$$0\leq P(y(t)) \leq y(t)n(y(t))\frac{1+D}{1-D}. \tag{19}$$

Analogously, from (18) with $b=-y(t)$ and $a=0$, we find that $P(-y(t))$ is Lebesgue-integrable

$$0\leq P(-y(t)) \leq y(t)n(y(t))\frac{1+D}{1-D}. \tag{20}$$

Fix $y(t)\in L_2(-\infty,+\infty)$. Now, let $b=y(t)$ and $a=y(t+\tau)$. We obtain

$$P(y(t))-P(y(t+\tau)) \leq n(y(t))[y(t)-y(t+\tau)]\frac{1}{q(t)} \tag{21}$$

where

$$\frac{1-D}{1+D} \leq q(t) := \frac{1-D\mathrm{sgn}[y(t)(y(t)-y(t+\tau))]}{1+D\mathrm{sgn}[y(t)(y(t)-y(t+\tau))]} \leq \frac{1+D}{1-D}$$

is a bounded signal that can assume only positive values. From (19) and the fact that $q(t)>0$, we have

$$q(t)[P(y(t))-P(y(t+\tau))] \leq n(y(t))[y(t)-y(t+\tau)].$$

Integrating both sides of the above inequality, we obtain

$$\int_{-\infty}^{\infty} q(t)P(y(t))dt - \int_{-\infty}^{\infty} q(t)P(y(t+\tau))dt \leq$$
$$\int_{-\infty}^{\infty} n(t)[y(t)-y(t+\tau)]dt.$$

Observe that

$$\frac{1-D}{1+D}\int_{-\infty}^{\infty}P(y(t))dt \le \int_{-\infty}^{\infty}q(t)P(y(t))dt \tag{22}$$

$$-\frac{1+D}{1-D}\int_{-\infty}^{\infty}P(y(t))dt \le -\int_{-\infty}^{\infty}q(t)P(y(t+\tau))dt. \tag{23}$$

Then, we can write

$$-\frac{4D}{(1-D)^2}\int_{-\infty}^{\infty}n(y(t))y(t)dt = \tag{24}$$

$$-\left(\frac{1+D}{1-D}-\frac{1-D}{1+D}\right)\int_{-\infty}^{\infty}\left(\frac{1+D}{1-D}\right)n(y(t))y(t)dt \le \tag{25}$$

$$-\left(\frac{1+D}{1-D}-\frac{1-D}{1+D}\right)\int_{-\infty}^{\infty}P(y(t))dt \le \tag{26}$$

$$\le \frac{1-D}{1+D}\int_{-\infty}^{\infty}P(y(t))dt - \frac{1+D}{1-D}\int_{-\infty}^{\infty}P(y(t+\tau))dt \le \tag{27}$$

$$\le \int_{-\infty}^{\infty}q(t)[P(y(t))-P(y(t+\tau))]dt \le \tag{28}$$

$$\le \int_{-\infty}^{\infty}n(t)[y(t)-y(t+\tau)]dt. \tag{29}$$

Thus, we get

$$\int_{-\infty}^{\infty}n(y(t))y(t+\tau)dt \le \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty}n(y(t))y(t)dt. \tag{30}$$

Proceeding in a similar way, using (20), we would find

$$-\int_{-\infty}^{\infty}n(y(t))y(t+\tau)dt \le \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty}n(y(t))y(t)dt. \tag{31}$$

Thus, we have an inequality involving the absolute value

$$\left|\int_{-\infty}^{\infty}n(y(t))y(t+\tau)dt\right| \le \frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty}n(y(t))y(t)dt. \tag{32}$$

Let us evaluate

$$\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}d\tau h(\tau)y(t+\tau)n(y(t)) = \tag{33}$$

$$\int_{-\infty}^{\infty}d\tau h(\tau)\int_{-\infty}^{\infty}dt y(t+\tau)n(y(t)) \le \tag{34}$$

$$\int_{-\infty}^{\infty}d\tau|h(\tau)|\frac{(1+D)^2}{(1-D)^2}\int_{-\infty}^{\infty}dt y(t)n(y(t)) \le \tag{35}$$

$$\int_{-\infty}^{\infty}n(y(t))y(t)dt. \tag{36}$$

This proves the assertion. ∎

III. Stability Criterion via IQC's

We provide the definition of well-posedness and (input-output) stability of a Lur'e interconnection.

Definition 2: Consider a Lur'e system where the linear part $\mathcal{G}$ is defined by a transfer function $G(s)$ and the nonlinear part is given by a nonlinear operator $\Delta : L_{2e} \to L_{2e}$. The interconnection, as represented in Figure 1 is described by the relations

$$\begin{cases} e = y - \mathcal{G}(u) \\ r = u - \Delta(y) \end{cases} \tag{37}$$

which define a map $\mathcal{M} : L_{2e} \times L_{2e} \to L_{2e} \times L_{2e}$ from the signals (u, y) to the L_{2e} signals (e, r). We say that the Lur'e interconnection is *well-posed* if $\mathcal{M}$ is invertible. We say that the Lur'e interconnection is *stable* if the restriction of $\mathcal{M}$ on L_2 is bounded.

The IQC framework for the stability analysis of Lur'e systems allows the expression of many absolute stability criteria in terms of a single unifying theory [10].
The fundamental result provided in [10] is reported for the sake of completeness.

Theorem 3.1 (Megretski-Rantzer Theorem): Consider a linear system $\mathcal{G}$ defined by a stable transfer function $G(s)$ and an operator $\Delta : L_{2e} \to L_{2e}$ bounded on its restriction on L_2. Let $\Pi(i\omega) : i\mathbb{R} \to C^{2\times 2}$ be a measurable and hermitian function. Let $v := \Delta(y)$ for any $y \in L_2$ and let $\hat{y}(i\omega)$ and $\hat{v}(i\omega)$ be the Fourier transform of $y(t)$ and $v(t)$ respectively. Assume that

- for every $\tau \in [0, 1]$ the feedback interconnection of $\mathcal{G}$ and $\tau\Delta$ is well-posed
- for every $\tau \in [0, 1]$ and for every $y \in L_2$

$$\int \begin{pmatrix} \hat{y}(i\omega) \\ \tau\hat{v}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{y}(i\omega) \\ \tau\hat{v}(i\omega) \end{pmatrix} d\omega \ge 0 \tag{38}$$

- there exists $\epsilon > 0$ such that

$$\begin{pmatrix} G(i\omega) \\ 1 \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} G(i\omega) \\ 1 \end{pmatrix} \le -\epsilon G(i\omega)^* G(i\omega). \tag{39}$$

Then, the feedback interconnection of $\mathcal{G}$ and Δ is stable.

We formulate the following criterion

Theorem 3.2 (Generalized Zames-Falb): Consider a Lur'e system given by the the interconnection of a LTI system $\mathcal{G}$, described by a stable transfer function $G(s)$, and a nonlinear operator $\Delta : L_{2e} \to L_{2e}$ defined as $\Delta(y) = v$, where $v(t) := n(y(t))$ for every $t \in \mathbb{R}$ and n is quasi-monotonic-and-odd function. Assume that

- the feedback inteconnection of $\mathcal{G}$ and $\tau\Delta$ is well-posed for every $\tau \in [0, 1]$
- $yn(y) \ge 0$
- $y(t) \in L_2$ implies $n(y(t)) \in L_2$
- Δ is bounded in its restriction on L_2
- there exists $\epsilon > 0$ and a summable function $h(t)$ such that

$$\|h\|_1 \le \left(\frac{1-D}{1+D}\right)^2 \tag{40}$$

$$\mathrm{Real}\{G(i\omega)[1 + H(i\omega)]\} \le -\epsilon G^*(i\omega)G(i\omega) \tag{41}$$

where $H(i\omega)$ is the Fourier transform of $h(t)$.

Then, the Lur'e system is stable.

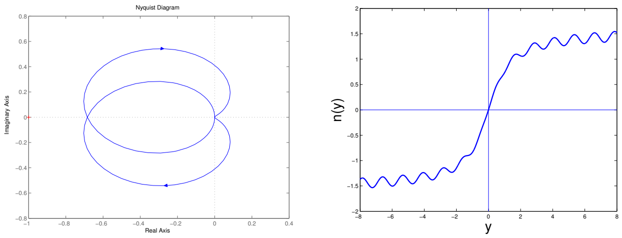
Proof: Denote the Fourier transform of $y(t)$ and $v(t)$ as $\hat{y}(i\omega)$ and $\hat{v}(i\omega)$ respectively. Define

$$\Pi(i\omega) := \frac{1}{2}\begin{pmatrix} 0 & 1 - H(i\omega) \\ 1 - H(i\omega)^* & 0 \end{pmatrix}. \tag{42}$$

Then, the condition defined in (38) is equivalent to

$$\frac{\tau}{2}\int_{-\infty}^{+\infty}[\hat{y}(i\omega)^*(1 - H(i\omega))\hat{v}(i\omega) + \tag{43}$$

$$+\, \hat{y}(i\omega)(1 - H(i\omega)^*)\hat{v}(i\omega)^*]d\omega \ge 0 \tag{44}$$



(a) (b)

Fig. 3. (a) Nyquist plot of the transfer function $G(s) = -\frac{10(s+0.25)s}{s^3+2s^2+2s+1}$; (b) representation of the feedback nonlinearity

Using the Parseval theorem, Equation (43) becomes

$$\tau \int_{-\infty}^{+\infty} n(y(t)) \left[y(t) - \int_{-\infty}^{+\infty} h(\sigma) y(t+\sigma) d\sigma \right] dt \geq 0 \qquad (45)$$

that is satisfied because of Theorem 2.1. Finally, Equation (39) is met by hypothesis. Thus, we can apply Theorem 3.1 and conclude the stability of the interconnection. ■

When $D = 0$, we find as a special case the standard Zames-Falb multiplier. Also, observe how the circle criterion (but in a formulation limited to time-invariant nonlinearities) can be obtained by using $H(i\omega) = 0$.

IV. Numerical Examples

Condition (41) can be checked by solving a linear matrix inequality as proved in [12], however, we make use of an analytically tractable example, just for the purpose of showing the use of the generalized Zames-Falb stability criterion.

Consider a Lur'e system defined as the *positive* feedback interconnection of a linear time-invariant operator described by a transfer function (Nyquist plot in Figure 3a)

$$G(s) = -\frac{10(s+0.25)s}{s^3 + 2s^2 + 2s + 1} \qquad (46)$$

and the nonlinear function represented in (3)

$$n(y) := K \arctan(y)[1 + D \sin(5y + \phi)] \qquad (47)$$

with $K \geq 0$, $D \in [0, 0.05]$ and $\phi \in [-\pi, \pi]$. Nonlinearities of this kind are common in many applications, such as when interference fringes disturb the measurement obtained using photodiodes ([13] and [14]). Consider $H(s) = 0.75/(s+1)$ and observe that its impulse response $h(t)$ satisfies $\|h(t)\|_1 < 0.75$. Observe that

$$\begin{aligned} &\mathrm{Real}\{[1 - H^*(i\omega)]G(i\omega)\} = \\ &= -|1 - H^*(i\omega)|^2 \mathrm{Real}\left\{\frac{i\omega}{-\omega^2 + i\omega + 1}\right\} = \\ &= -\frac{\omega^2 |1 - H^*(i\omega)|^2}{\omega^4 - \omega^2 + 1} \leq -\epsilon \frac{\omega^2[0.25^2 + \omega^2]}{\omega^6 + 1} = -\epsilon |G(i\omega)|^2, \end{aligned}$$

that is satisfied for some $\epsilon > 0$ sufficiently small. The generalized Zames-Falb condition holds for

$$0.75 < \left(\frac{1-D}{1+D}\right)^2 \Rightarrow D \leq \frac{1 - \sqrt{0.75}}{1 + \sqrt{0.75}} \simeq 0.0718, \qquad (48)$$

that defines for $G(s)$ the maximum deviation D of $n(y)$ from the monotonic function $\overline{n}(y) = K \arctan(y)$ that can be tolerated keeping the interconnection stable. Thus, the generalized Zames-Falb criterion guarantees stability for any $K > 0$ with a relative tolerance on the nonlinearity $\overline{n}$ of about 7%. The quasi-monotonic-and-odd nonlinearity has a spread of 5%, thus the stability is ensured.

The circle criterion guarantees stability for $K < \overline{K}_c \simeq 2.858$, while the Popov criterion guarantees stability for $K < \overline{K}_p \simeq 2.97$. The standard Zames-Falb criterion guarantees stability for any $K > 0$, but it can not be applied for $D \neq 0$.

V. Conclusions

In this paper, we have derived a generalization of the Zames-Falb multiplier for the stability analysis of Lur'e systems. The standard Zames-Falb multiplier allows for a formulation of a stability criterion when the feedback nonlinearity is strictly odd and monotonic. The new formulation takes into account possible deviations from the odd and monotonic behavior introducing a notion of robustness in the criterion.

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