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# Vision-Based Relative Altitude Estimation of Small Unmanned Aerial Vehicles in Target Localization

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*Abstract*— **This paper presents a UAV vision-based relative altitude estimation method using a given size (length) of the ground vehicle. In conjunction with a well known target localization technique, the proposed method relaxes the flat ground assumption and provides an independent approach of altitude correction. It offers possible improvement or expansion in small UAV applications in target localization, tracking, and terrain exploration missions, such as ground terrain mapping. In addition, the proposed kinematic method is reliable and computationally efficient. Its feasibility and effectiveness is further demonstrated by extensive experiments, including both stationary and moving target estimations.**

## I. INTRODUCTION

Small unmanned aerial vehicles (UAVs) provide much potential for autonomous surveillance, reconnaissance, and search and rescue [1][2]. Target localization and tracking using small, low cost UAVs have received much attention in recent years. Often, these UAVs employ a single monocular camera for target tracking due to payload size limitations. Traditionally, two approaches are used to estimate the inertial location of the target - single-frame estimation and multi-frame estimation. Single-frame localization methods are able to estimate the inertial location of the target using a single camera frame as well as data from other sensors on board (altimeter, GPS, inertial measurement unit (IMU), etc.). Since the target's position is estimated at each time interval, velocity of the target can also be estimated. Multi-frame localization uses multiple views of the target to triangulate its position using a method similar to stereo cameras. This method becomes more challenging if the target is moving, and restrictions often need to be placed on the target's movement (such as constant velocity) for the estimation to work.

In addition, vision-based target localization estimates the position of a ground target in the earth-fixed frame using the local position of the UAV as well as the position of the target relative to the UAV captured by the monocular camera onboard. However, the depth of the target, or the elevation of ground surface where the ground target is located, is not directly observable given a single image from a monocular camera. To compensate for this problem, the altitude of the UAV needs to be independently determined from pressure altimeters onboard the UAV or from GPS readings, although GPS is rarely ever used for altitude due to its update latency and its inaccuracies in vertical position estimation. Pressure altimeters are also vulnerable to changing weather conditions and are not sensitive to changing ground elevation. As a result, a "flat ground " assumption is often made in many localization methods [3]. To attempt to relax this assumption, Dobrokhodov used elevation maps to improve the accuracy of localization on uneven terrain [4].

However, to the authors' knowledge, the UAV's absolute altitude is not necessary for localization. Instead, the relative altitude between the UAV and the target is what is required. On the other hand, in many real world applications involving target tracking and localization, it is observed that the size of the target is often known. Examples include the length of a specific vehicle or the width of a road or pipeline. With this additional piece of information, it is possible to obtain the relative altitude measurement based on the vision system alone. In this paper, we propose a UAV vision-based relative altitude estimation method using a given size (length) information of the ground vehicle in conjunction with a well known target localization technique. The novelty of this paper, through proposed novel algorithm in relative altitude estimation, opens doors for possible improvement or expansion in small UAV applications in target localization, tracking, and terrain exploration missions. For example, the vision-based relative altitude estimate between the ground vehicle and UAV, if compensated by an independently obtained UAV altitude, enables estimation of the ground surface elevation where the ground vehicle is located. It may lead to a three-dimensional terrain map development by coordination of UAV and UGV vision systems. Further, it relaxes flat ground assumption without any prior knowledge of elevation maps. Even on a flat surface, the relative altitude may help to correct UAV altitude readings due to errors originated from other sensors or measurements. In addition, the proposed method is derived by the system's kinematics. It is accurate and computationally efficient. In this paper, the feasibility and effectiveness of the proposed algorithm is further demonstrated by extensive experiments, including both stationary and moving target estimations. An artist's view is illustrated in Figure 1 to demonstrate the main concept.

The rest of this paper is presented as follows. The relevant work to our proposed method is briefly described in Section II. The relative altitude estimation problem is then formulated in Section III. The main result of this paper, in the format of a proposed estimation algorithm, is presented in Section IV. Section V gives details of the experimental set up, while Section VI presents the experimental results,

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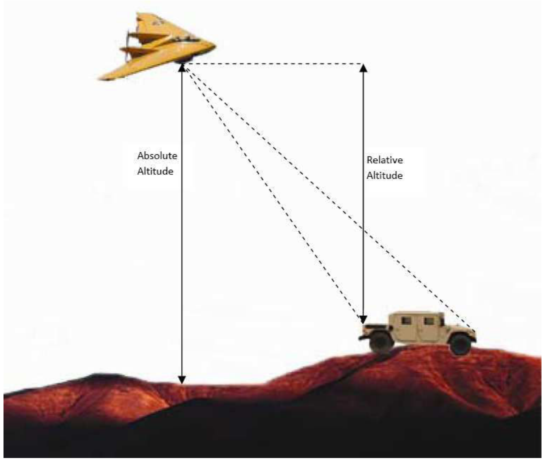


Fig. 1. An artist's view of the proposed scenario

followed by some conclusive remarks in Section VII.

## II. Relevant Work

The proposed relative altitude method will be used in conjunction with a well known target localization technique. Target localization involves identifying the target and estimating its position and/or velocity in the inertial frame. Many techniques are used to identify and track the target using a single camera vision system onboard the UAV. The target is first captured by the camera and identified through image processing. Several set ups have been implemented to attempt to maintain visual tracking of the target. Redding uses a gimbal-mounted camera with 2 degrees of freedom (DOF), in which the gimbal is controlled to try to keep the target in the center of the image plane. To minimize the localization error, the authors used a recursive least squares filter to allow the UAV to track a single stationary target to an accuracy of 10 m [3][5][6]. A major component of localization is the transformation between the camera reference frame and the inertial frame. The process is described in detail in [3][5] and involves a series of transformations from the camera frame (c) to the gimbal frame (g), the gimbal frame to the body frame (b), the body frame to the vehicle frame (v), and finally the vehicle frame to the inertial frame (I). This formulation is presented here because it takes into account the offset of the camera from the UAV center of mass and the distance between the gimbal center of rotation and the center of the camera. This is accomplished through the use of homogenous transformation matrices (HTMs) $T_i^j$, which transforms from the $i$ frame to the $j$ frame. The inertial coordinates of the target is given in [3] by:

$$p^I_{obj} = \lambda [C T^c_g T^g_b T^b_v T^v_I]^{-1} q \tag{1}$$

where $C$ is the camera calibration matrix, $q$ is the coordinates of the target in the image plane and $\lambda$ is the depth of the target, or relative altitude between the (ground) target and the UAV.

There are two predominant techniques for estimating the target depth in literature. In the method proposed by Redding, the altitude of the UAV is assumed to be known and the ground is assumed to be flat. By using the known altitude of the camera center (cc) and the focal length of the camera, the depth $\lambda$ can be solved.

Another technique for solving the target location involves using multiple frames and triangulation methods. This is presented in [7] and [4]. However, restrictions on the target velocity often have to be in place to ensure proper localization.

Han and DeSouza presented a method of target localization that relaxes many of the constraints that others posed. They claimed to be able to track multiple, slow moving targets without assuming a flat ground [8]. Their method uses optical flow techniques to track the moving target, and the SIFT algorithm to track features surrounding the target. The background features are triangulated between subsequent frames to provide an improved estimate for the altitude of the UAV, which allows the depth of the target to be calculated. The main assumption in this case is that the features detected near the target are at the same altitude as the target itself. Furthermore, the camera used in this model is fixed with respect to the UAV, allowing it to track multiple targets simultaneously [8].

In this paper, the proposed relative altitude estimation algorithm will be developed in conjunction with Redding's target localization method, i.e., once the relative altitude $\lambda$ is obtained, the inertial position of the target can then be calculated using Equation 2,

$$p^I_{obj} = p^I_{cc} + \lambda(\overline{p}^I_{obj} - p^I_{cc}) \tag{2}$$

## III. Problem Formulation

This section will outline the set up of the problem and the mathematical derivations of the altitude estimation formula.

The relative altitude problem involves a single UAV equipped with an onboard monocular camera, inertial measurement sensors, and a GPS receiver. The UAV will be flying above a target of given length. The target is assumed to lie on a plane parallel to the ground. It is assumed that the UAV will be able to maintain tracking on the target using its onboard camera while the target is in view. The camera is assumed to be calibrated so that its intrinsic parameters and most importantly, its focal length, is known. The set up is described in Figure 2.

Table I describes each point in Figure 2 in detail. The camera model and the coordinate frames are defined in the same way as [3] for consistency. The transformations are simplified in this investigation to assume that all the rotations from the camera frame to the inertial frame are about the point $G$. In order to conduct flight tests using UAVs, the offsets between the center of rotation for each transformation should be taken into account. This can be achieved by using homogeneous transformation matrices. Table II defines the parameters that will be used in the derivation in the next section.

The estimation problem is then formulated as *to find the kinematic solution to calculate the altitude*: $\underrightarrow{GH}$.

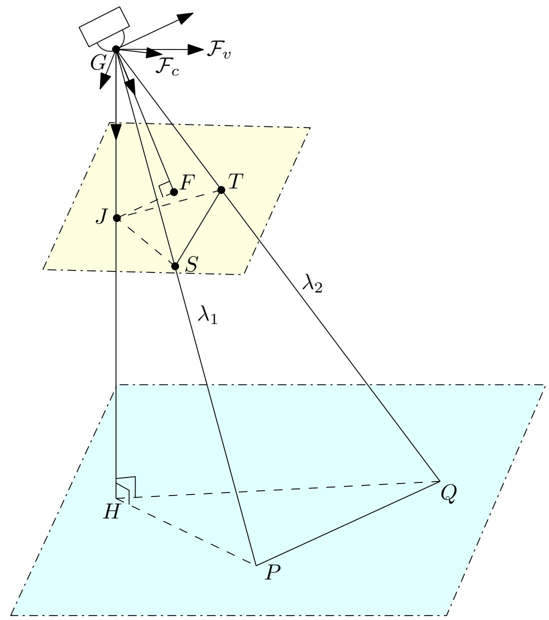


Fig. 2. Geometry of the Problem

TABLE I
DESCRIPTION OF GEOMETRY POINTS

| Point | Description |
|---|---|
| $G$ | Center of the camera aboard the UAV |
| $F$ | Center of the image plane. $\|\underset{\rightarrow}{GF}\|$ is equal to the focal length |
| $P$ | First target point. Lies on the ground plane |
| $Q$ | Second target point. Also lies on the ground plane |
| $H$ | Point on the ground plane vertically below the camera |
| $J$ | Intersection of $\underset{\rightarrow}{GF}$ with the image plane |
| $S$ | First target point as it appears on the image plane |
| $T$ | Second target point as it appears on the image plane |
| $\lambda_1$ | Depth of the first target point. Measured in meters |
| $\lambda_2$ | Depth of the second target point. Measured in meters |

## IV. ALTITUDE ESTIMATION METHOD

The altitude $h$ can be determined from Equation (3) or Equation 4 if the depth of either of the target points, $\lambda_1$ or $\lambda_2$ is known as well as the angles $\alpha$ and $\beta$.

$$h = \lambda_1 \cos\alpha \tag{3}$$

$$h = \lambda_2 \cos\beta \tag{4}$$

The depths $\lambda_1$ and $\lambda_2$ are also related to the length of the target, $l$, through the cosine law.

$$l^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\delta \tag{5}$$

If the angles $\alpha$, $\beta$, and $\delta$ are known, Equations 3, 4, and 5 can be combined to solve for $h$. $\delta$ can be determined from the image frame. From geometry, it is clear that $\delta$ is the angle between the vectors $\underset{\rightarrow}{s}$ and $\underset{\rightarrow}{t}$. The coordinates of $\underset{\rightarrow}{s}$ and $\underset{\rightarrow}{t}$ can be easily obtained from the camera frame, where $\underset{\rightarrow}{s} = \mathscr{F}_c^T[s_x\ s_y\ f]^T$ and $\underset{\rightarrow}{t} = \mathscr{F}_c^T[t_x\ t_y\ f]^T$. Thus, $\delta$ is given by Equation 6.

$$\cos\delta = \frac{\underset{\rightarrow}{s}\cdot\underset{\rightarrow}{t}}{\left\|\underset{\rightarrow}{s}\right\|\left\|\underset{\rightarrow}{t}\right\|} \tag{6}$$

TABLE II
PARAMETERS USED IN THE DERIVATION

| Parameter | Definition |
|---|---|
| $h$ | $\|\underset{\rightarrow}{GH}\|$ altitude in meters |
| $l$ | $\|\underset{\rightarrow}{PQ}\|$ length of target in meters |
| $f$ | $\|\underset{\rightarrow}{GF}\|$ focal length in pixels |
| $\delta$ | $\angle PGQ \equiv \angle SGT$ |
| $\alpha$ | $\angle PGH \equiv \angle SGJ$ |
| $\beta$ | $\angle QGH \equiv \angle TGJ$ |
| $\underset{\rightarrow}{f}$ | $\underset{\rightarrow}{GF}$ |
| $\underset{\rightarrow}{s}$ | $\underset{\rightarrow}{GS}$ |
| $\underset{\rightarrow}{t}$ | $\underset{\rightarrow}{GT}$ |
| $\underset{\rightarrow}{j}$ | $\underset{\rightarrow}{GJ}$ |
| $\phi$ | roll angle |
| $\theta$ | pitch angle |
| $\psi$ | yaw angle |
| $el$ | gimbal elevation angle |
| $az$ | gimbal azimuth angle |
| $C_{ab}$ | rotation matrix from reference frame $b$ to $a$ |

The angles $\alpha$ and $\beta$ can be determined in a similar manner, shown in Equations 7 and 8, but the vector $\underset{\rightarrow}{j}$ is required in the camera frame.

$$\cos\alpha = \frac{\underset{\rightarrow}{s}\cdot\underset{\rightarrow}{j}}{\left\|\underset{\rightarrow}{s}\right\|\left\|\underset{\rightarrow}{j}\right\|} \tag{7}$$

$$\cos\beta = \frac{\underset{\rightarrow}{t}\cdot\underset{\rightarrow}{j}}{\left\|\underset{\rightarrow}{t}\right\|\left\|\underset{\rightarrow}{j}\right\|} \tag{8}$$

Since the vector $j$ is pointing down vertically (i.e. along the z axis of the vehicle frame $\mathscr{F}_v$), its coordinates in the vehicle frame is give by Equation 9.

$$\underset{\rightarrow}{j} = \mathscr{F}_v^T\mathbf{j_v} = \mathscr{F}_v^T\begin{bmatrix}0\\0\\j_{v_z}\end{bmatrix} \tag{9}$$

Several transformations are required to express $\underset{\rightarrow}{j}$ in the camera frame. First, the vector is transformed from the vehicle frame to the UAV body frame through a rotation $C_{bv}$. Next, it is transformed from the body frame to the gimbal frame through $C_{gb}$. Finally, it is transformed from the gimbal frame to the camera frame by the rotation matrix $C_{cg}$. This is shown in Equation 10.

$$\mathbf{j_c} = C_{cg}C_{gb}C_{bv}\mathbf{j_v} = C_{cv}\mathbf{j_v} \tag{10}$$

The rotation matrices are shown in Equations 11, 12 and 13.

$$C_{cg} = \begin{bmatrix}0 & 0 & -1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} \tag{11}$$

$$C_{gb} = \begin{bmatrix}c_{el}c_{az} & c_{el}s_{az} & s_{el}\\-s_{az} & c_{az} & 0\\-s_{el}c_{az} & -s_{el}s_{az} & c_{el}\end{bmatrix} \tag{12}$$

$$C_{bv} = \begin{bmatrix}c_\theta c_\psi & c_\theta s_\psi & -s_\theta\\s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta\\c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta\end{bmatrix} \tag{13}$$

where $c_* = \cos(*)$ and $s_* = \sin(*)$. Since the only non-zero component of $\mathbf{j_v}$ is the z-component, $j_{v_z}$, its value can be determined from Equation 14[1].

$$j_{v_z} = \frac{2f}{\mathrm{tr}C_{cv} - 1} \tag{14}$$

Once $\mathbf{j_c}$ is found, the angles $\alpha$ and $\beta$ can be determined through Equation 7 and 8. The final equation solving for $h$ is given in Equation 15.

$$h = \lambda_1 \cos\alpha = \cos\alpha \sqrt{\frac{l^2}{1 + \left(\frac{\cos\alpha}{\cos\beta}\right)^2 - 2\frac{\cos\alpha}{\cos\beta}\cos\delta}} \tag{15}$$

## V. Experimental Setup

A small scale experiment is conducted to verify the fidelity of the algorithm presented. A Logitech webcam is set up to track a target in an indoor environment while both the camera and the target are tracked simultaneously by the OptiTrack™ system from NaturalPoint® Inc. The camera attitude is initially calibrated such that it is pointed straight downward. The video stream from the camera and the output from OptiTrack™ are fed into a program which uses the $x$ and $y$ coordinates of the camera as well as the roll, pitch and yaw angles to calculate the relative altitude between the camera and the target as well as the target $x$ and $y$ coordinates. The OptiTrack™ measurement of the target position is used as the ground truth.

### A. Testing Environment

The experiment is performed in the Flight Systems and Control Lab at the University of Toronto Institute for Aerospace Studies (UTIAS). The experiment uses 3 OptiTrack™ infrared cameras covering a volume of approximately 3 $m^3$. Both the camera and the target are tagged with 4 reflective markers in a unique arrangement distinguishable to the OptiTrack™ software. Figure 3 shows the webcam, the target and an OptiTrack™ camera.

Fig. 3. Test Area with Camera and Target


Fig. 4. Screenshot of the experiment program

### B. Vision-based Target Tracking

Visual tracking of the target is performed using the OpenCV library. A Pyramidal Lucas-Kanade Optical Flow algorithm is implemented as well as Shi and Tomasi's corner detection algorithm [9], implemented in OpenCV as the GoodFeaturesToTrack function. The user first selects a point on the image with the mouse. OpenCV then looks for trackable corners nearby. If a corner is found, it is tracked by the optical flow algorithm. Once two points are selected, the program begins to calculate the height and position of the target. A screenshot of the program window is shown on Figure 4. The green dots on Figure 4 show the 2 points being tracked by the optical flow algorithm. In this experiment, the distance between the 2 points selected is 125 mm. The user must set this distance for localization to work.

In a situation where the UAV is expected to recognize and track the target autonomously, another vision algorithm must be implemented. In most of these cases, the UAV would most likely need to know what the target looks like beforehand. If an image of the target exists and is stored in memory, the UAV can attempt to match the stored image with the frames from the video stream. Many algorithms exist to serve this purpose including Template Matching, CAMShift, and SIFT/SURF. Choosing which algorithm to use depends largely on the computational resources available.

## VI. Experimental Results

Two experiments are performed in this investigation. In the first experiment the webcam is used to track a stationary target while following a random trajectory. In the second experiment, a moving target is tracked. Since localization is performed at each timestep, each estimate is independent. This means that the trackable velocity of the target is only limited by the frame rate of the camera and the speed of the

[1] Proof is given in the Appendix.

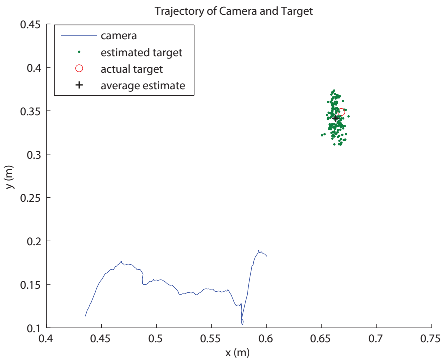


Fig. 5. Top View of Results

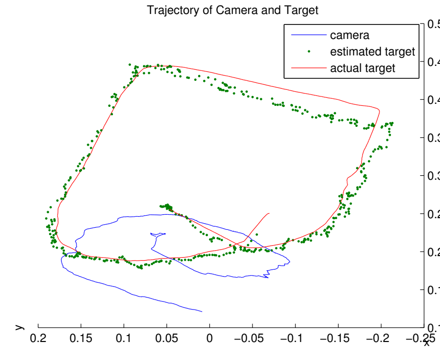


Fig. 7. Top View of Moving Target Localization

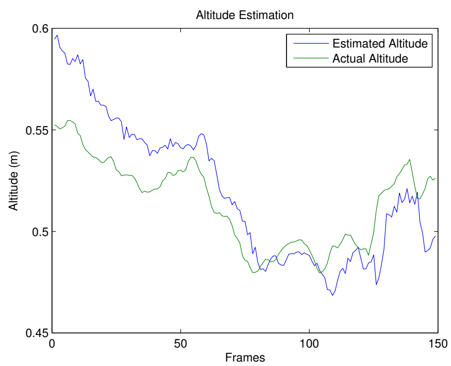


Fig. 6. Stationary Target Altitude Estimation

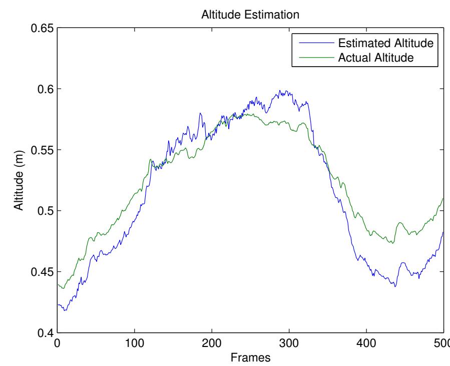


Fig. 8. Moving Target Altitude Estimates

algorithm. The experimental results are collected and plotted. The next 2 sections show the results of the 2 experiments performed.

### A. Stationary Target Estimation

Figure 5 shows a top view of the camera trajectory, target estimates and the real target location.

Note that the spread of the estimates do not exceed 2 cm in the $x$ direction and 5 cm in the $y$ direction. This shows that the algorithm is robust to changes in the camera roll, pitch and yaw angles. The results of the individual estimates are averaged over the entire set of data. The average estimated position is also indicated on Figure 5. The error of the average estimate is under 7 mm in both $x$ and $y$ directions without any filtering.

The altitude estimation results are plotted against the number of camera frames, shown on Figure 6. Each frame represents a single position estimate. In this experiment, the frame rate averaged 10 fps. The maximum error in the altitude estimation in this case is $\pm$ 8 percent. Section VI-C discusses possible sources of this error.

### B. Moving Target Estimation

In the moving target experiment, the target is slowly moved manually in a rough circle within the testing area. The camera is moved in such a way so that the target is always in view. The trajectories of the camera and the target are shown in Figure 7. Both of the $x$ and $y$ estimates are within 3 cm of the actual target.

The altitude estimates are also plotted for this experiment and shown on Figure 8. We see that the altitude errors are within 8% of the actual altitude, which is consistent with the stationary results.

### C. Velocity Estimation

In the moving target experiment, the elapsed time between each video frame is also recorded. This is used to calculate of the velocity of the target. The x-velocity of the target in the previous experiment is shown in Figure 9. The real velocity of the target is computed using its real positions as measured by OptiTrack$^{\text{TM}}$and shown for comparison. It is clear that although the estimates do follow the trends of the real velocity, they are very noisy. A state estimation filter may be used in this case to smooth out the results. This will be an area of future work. Since the errors in these results are on the order of the velocities themselves, a definitive conclusion cannot be reached as to the effectiveness of this method. The limitations on the target velocity depend largely on the robustness and speed of the vision algorithm. In this experiment, the vision algorithm ran at approximately 10 frames per second. In this case, the target speed is limited to the tracking capabilities of the optical flow algorithm between each frame.

The primary source of error in these experiments is the vision tracking algorithm. Even though it works reasonably

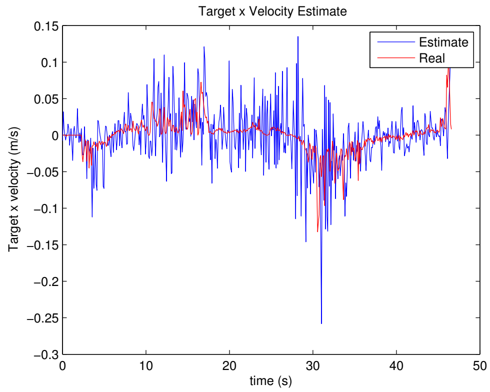


Fig. 9. Target x velocity

well, the tracked feature points may drift when the camera or the target is moving. This drift, to some degree, represents a loss of tracking and affects both the perceived size of the target as well as the centroid of the target, causing errors in the altitude estimation and localization. Other visual tracking algorithms are readily available but are not within the scope of this research.

The mounting of the camera to the reflective markers of the OptiTrack$^{\text{TM}}$system may cause attitude measurement errors due to misalignment. This was taken into account by measuring the misalignment angles. The mounting position of the camera also causes an altitude displacement of the camera center, which was also taken into account.

In the moving target experiment, changes in the target's attitude may have induced some error in the altitude estimation. By assumption, the target must be parallel to the ground plane. Errors may also occur even when the target is parallel to the ground if the camera's initial attitude was not calibrated correctly. If the target attitude is pitched a known angle, the altitude estimation algorithm can be readily modified to accommodate this change.

## VII. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a UAV vision-based relative altitude estimation method using a given size (length) information of ground vehicle in conjunction with a well known target localization technique. The novelty of this paper, through proposed novel algorithm in relative altitude estimation, opens doors for possible improvement or expansion in small UAV applications in target localization, tracking, and terrain exploration missions. It has been shown that using a given target size to estimate UAV altitude yields reliable results and can be applied to existing methods of target localization and tracking. Through small scale experiments, it was found that even an 8% error in the altitude estimation yielded good localization results without filtering. Furthermore, this method may be applied to target velocity estimation. The experimental results can be improved by implementing a state estimation algorithm such as the Extended Kalman Filter. Other vision tracking algorithms can be investigated to maintain robust tracking on the target. Further verification of the algorithm's applicability can be obtained through flight tests.

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## APPENDIX

From Equation 9, it is clear that

$$\left\| \underset{\rightarrow}{j} \right\| = j_{v_z}$$

$$f = j_{v_z} \cos \sigma$$

where

$\sigma =$ angle between the camera frame and the vehicle frame

Using Euler parameters, or quaternions, we have the definition

$$\eta = \cos \frac{1}{2}\sigma$$

It can also be shown that [10]

$$\eta = \pm\frac{1}{2}(1 + \mathrm{tr}C_{cv})^{\frac{1}{2}}$$

This may be manipulated into

$$2\eta^2 - 1 = \frac{\mathrm{tr}C_{cv} - 1}{2}$$

Therefore,

$$j_{v_z} = \frac{f}{\cos\sigma} = \frac{f}{2\cos^2(\frac{1}{2}\sigma) - 1} = \frac{f}{2\eta^2 - 1}$$

and it follows that

$$j_{v_z} = \frac{2f}{\mathrm{tr}C_{cv} - 1}$$

$\square$