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An Asymmetric Small-Gain Technique to Construct Lyapunov-Krasovskii Functionals for Nonlinear Time-Delay Systems with Static Components

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Abstract— **The standard ISS small-gain theorem and the recently-developed iISS small-gain theorem assume that system components are characterized in a symmetric way with respect to the equilibrium. Dissipative properties of a nonlinear system is usually asymmetric. Formulating them into symmetric properties sometimes causes crucial conservativeness. The purpose of this paper is to develop a technique to take the asymmetry into account in stability analysis of time-delay systems. The result is based on decomposition of a system into integral input-to-state stable dynamic components and static components characterized in an asymmetric way with respect to the equilibrium. A Lyapunov-Krasovskii functional establishing robustness with respect to disturbances in the presence of time-delays is constructed, and its effectiveness is illustrated by a network flow control example. The proposed iISS methodology covers a broader class of systems than existing approaches based on operator norms or the ISS gain.**

I. Introduction

As recent technology and society have tended to focus on systems of larger scale, coping with time-delays and nonlinearities has become more important. One of major approaches to stability and robustness of such systems is the small-gain technique which makes use of gain-type properties belonging to the concept of dissipation. Nonlinearities suggest the use of gain functions which are not linear. The input-to-state stability (ISS) provides us with such a frameworks [22]. The integral input-to-state stability (iISS) extends the idea of ISS to a broader class of nonlinearities [1]. Both ISS small-gain and iISS small-gain theorems are available in the literature [14], [25], [15], [21], [7], [10]. The small-gain analysis is based on decomposition of a system into subsystems, and dissipation properties of the individual subsystems are aggregated to establish the stability and robustness of the overall system. Naturally, dissipation properties of nonlinear systems are nonlinear, which appears as asymmetry of the dissipativity with respect to equilibria. The standard formulation of small-gain theorems relies on ISS and iISS properties of subsystems characterized in a symmetric way. Therefore, the loop gain computed is a function which is symmetric with respect to the equilibrium. Application of symmetric dissipation and loop gains sometimes results in an useless answer to the problem of stability and robustness analysis.

This paper aims at proposing a small-gain methodology to make use of asymmetry in verifying stability and robustness of nonlinear time-delay systems consisting of iISS dynamic components (functional differential equations) and static components (functional algebraic equations). Since components which are not necessarily ISS hamper existing trajectory-based treatments, this paper develops a new way to construct Lyapunov-Krasovskii functional. As an application of the developed methodology, a network flow control system proposed in [16] is studied in this paper, and it is shown that the flow control system is robust with respect to disturbances in the presence of arbitrarily large time-delays in communication and processing if utility and penalty functions satisfy a certain criterion.

Notation: The logical sum and the logical product are denoted by $\vee$ and $\wedge$, respectively. For a real vector $x \in \mathbb{R}^n$, the symbol $|x|$ stands for the Euclidean norm. A function $\omega : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is said to be positive definite if it is continuous and satisfies $\omega(0) = 0$ and $\omega(s) > 0$ for all $s > 0$, and written as $\omega \in \mathcal{P}$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A continuous function $\omega : \mathbb{R} \to \mathbb{R}$ is said to be of both-sided class $\mathcal{K}$ and written as $\omega \in \mathcal{BK}$ if both $\omega(s)$ and $-\omega(-s)$ are class $\mathcal{K}$ functions for $s \in \mathbb{R}_+$. The class $\mathcal{BK}_\infty$ and class $\mathcal{BP}$ are defined in a similar manner.

II. Time-Delay Systems: Interconnection through Asymmetric Dissipation Inequalities

We consider the system described by

$$\begin{aligned} \dot{x}_0(t) &= f_0(x_{0,t}, x_{1,t}, ..., x_{K,t}, r_0(t)), \quad t \geq 0 \\ x_k(t) &= f_k(x_{0,t}, r_k(t)), \quad k = 1, 2, ..., K \\ x_{0,0} &= \xi_{0,0}, \quad x_{k,0} = \xi_{k,0}, \end{aligned} \tag{1}$$

where $x_i(t)$, $i = 0, 1, ..., K$, are real vector-valued functions of time and written as $x_i(t) \in \mathbb{R}^{n_i}$ with positive integers n_i. The signals $r_i(t) \in \mathbb{R}^{m_i}$, $i = 0, 1, ..., K$, are external inputs (measurable, locally essentially bounded). For $t \in \mathbb{R}_+$, $x_{i,t} : [-\Delta, 0] \to \mathbb{R}^{n_i}$ denotes the function given by $x_{i,t}(\tau) = x_i(t + \tau)$, where $\Delta > 0$ is the maximum involved delay. Let $\mathcal{C}_{n_i}$ denote the space of continuous functions mapping the interval $[-\Delta, 0]$ into $\mathbb{R}^{n_i}$. For $\phi_i \in \mathcal{C}_{n_i}$, we use $\|\phi_i\|_\infty = \sup_{-\Delta \leq \theta \leq 0} |\phi_i(\theta)|$. Suppose that $\xi_{0,0} \in \mathcal{C}_{n_0}$, and that $f_0 : \mathcal{C}_{n_0} \times \mathcal{C}_{n_1} \times ... \times \mathcal{C}_{n_K} \times \mathbb{R}^{m_0} \to \mathbb{R}^{n_0}$ and $f_k : \mathcal{C}_{n_0} \times \mathbb{R}^{m_k} \to \mathbb{R}^{n_k}$, $k = 1, 2, ..., K$, are functionals which are Lipschitz on any bounded set. It is also assumed that $f_0(0, 0, ..., 0, 0) = 0$ and $f_1(0, 0) = 0$ thus ensuring that $x_0(t) = 0$ is the solution corresponding to zero input and zero initial conditions. The state vector of the overall

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system is $x_t = x_{0,t}$. Let the entire disturbance be in the vector form of $r(t) = [r_0(t)^T, r_1(t)^T, ..., r_K(t)^T]^T \in \mathbb{R}^m$, $m = m_0 + m_1 + ... + m_K$. The components described by f_k, $k = 1, 2, ..., K$, are static in the sense that $x_k(t)$ is uniquely determined for $t \in \mathbb{R}_+$ by $x_{0,t}$ and $r_k(t)$, and no initial function of x_k is involved in the $x_k(t)$-equation. Note that the $x_0(t)$-equation requires $x_k(t)$ in the interval of $[-\Delta, 0)$. We suppose that $\xi_{k,0} \in \mathcal{C}_{n_k}$ satisfies the matching condition $\xi_{k,0}(0) = f_k(\xi_{0,0}, r_k(0))$. The coupled equations of (1) are not troubled with any algebraic loop since the x_0-subsystem has no direct-feed-through term and the x_k-subsystems which are static have no self-feedback terms.

We use the following three groups of functions:

Assumption 1: *(i)* The functional $M_{a,0} : \mathcal{C}_{n_0} \to \mathbb{R}_+$ and the functions $\underline{\gamma}_{a,0}, \overline{\gamma}_{a,0} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy

$$\underline{\gamma}_{a,0} \in \mathcal{K}_\infty, \quad \overline{\gamma}_{a,0} \in \mathcal{K}_\infty \tag{2}$$

$$\underline{\gamma}_{a,0}(|\phi_0(0)|) \leq M_{a,0}(\phi_0) \leq \overline{\gamma}_{a,0}(\|\phi_0\|_\infty), \ \forall \phi_0 \in C_{n_0}. \tag{3}$$

(ii) The functionals $M_{0,k} : \mathcal{C}_{n_0} \to \mathbb{R}$ and the functions $\underline{\gamma}_{0,k} : \mathbb{R}^{n_0} \to \mathbb{R}$, $k = 1, 2, ..., K$, are continuous and satisfy

$$\underline{\gamma}_{0,k}(0) = 0 \tag{4}$$

$$\underline{\gamma}_{0,k}(\phi_0(0))(M_{0,k}(\phi_0) - \underline{\gamma}_{0,k}(\phi_0(0))) \geq 0, \ \forall \phi_0 \in \mathcal{C}_{n_0}. \tag{5}$$

(iii) The functionals $M_k : \mathcal{C}_{n_k} \to \mathbb{R}$ and the functions $\underline{\gamma}_k : \mathbb{R}^{n_k} \to \mathbb{R}$, $k = 1, 2, ..., K$, are continuous and satisfy

$$\underline{\gamma}_k(0) = 0 \tag{6}$$

$$\underline{\gamma}_k(\phi_k(0))(M_k(\phi_k) - \underline{\gamma}_k(\phi_k(0))) \geq 0, \ \forall \phi_k \in \mathcal{C}_{n_k}. \tag{7}$$

These functions have yet to be determined. Note that the range of the pair ($M_{0,k}$,$\underline{\gamma}_{0,k}$) and the pair (M_k,$\underline{\gamma}_k$), $k = 1, 2, ..., K$, is two-sided, while the range of ($M_{a,0}$,$\underline{\gamma}_{a,0}$,$\overline{\gamma}_{a,0}$) is one-sided. We assume that the x_0-subsystem is iISS with respect to input $(x_1, ..., x_K, r_0)$ and state x_0 as follows:

Assumption 2: There exist a locally Lipschitz functional $V_0 : \mathcal{C}_{n_0} \to \mathbb{R}_+$, a continuous functional $\alpha_0 : \mathcal{C}_{n_0} \to \mathbb{R}_+$, $\hat{\alpha}_0 \in \mathcal{P}$, $\sigma_{0,k,j} \in \mathcal{BK}$, $\sigma_{r,0} \in \mathcal{K} \cup \{0\}$, $\overline{\alpha}_0, \underline{\alpha}_0 \in \mathcal{K}_\infty$, a continuous functional $M_{a,0} : \mathcal{C}_{n_0} \to \mathbb{R}_+$, continuous functions $\underline{\gamma}_{a,0}, \overline{\gamma}_{a,0} : \mathbb{R}_+ \to \mathbb{R}_+$, continuous functionals $M_k : \mathcal{C}_{n_k} \to \mathbb{R}$, continuous functions $\underline{\gamma}_k : \mathbb{R}^{n_k} \to \mathbb{R}$ and $S_{0,k,j} \in \{0, 1\}$ for $k = 1, 2, ..., K$ and $j = 0, 1, \ldots, h + h_d$ such that

$$\underline{\alpha}_0(M_{a,0}(\phi_0)) \leq V_0(\phi_0) \leq \overline{\alpha}_0(M_{a,0}(\phi_0)), \quad \forall \phi_0 \in \mathcal{C}_{n_0} \tag{8}$$

$$D^+V_0(\phi_0, \phi_1, ..., \phi_K, r_0) \leq \rho_0(\phi_0, \phi_1, ..., \phi_K, r_0), \quad \forall \phi_i \in \mathcal{C}_{n_i}, i = 0, 1, ..., K, \ \forall \ r_0 \in \mathbb{R}^{m_0} \tag{9}$$

and (2), (3), (6), (7) hold, where $\rho_0 : \mathcal{C}_{n_0} \times \mathcal{C}_{n_1} \times ... \times \mathcal{C}_{n_K} \times \mathbb{R}^{m_0} \to \mathbb{R}$ is defined by

$$\begin{aligned}\rho_0(\phi_0, \phi_1, ..., \phi_K, r_0) &= -\alpha_0(\phi_0) + \sigma_{r,0}(|r_0|) \\ &+ \sum_{k=1}^{K} \Bigg\{ S_{0,k,0} \left| \sigma_{0,k,0}(M_k(\phi_k)) \right| \\ &+ \sum_{j=1}^{h} S_{0,k,j} \left| \sigma_{0,k,j} \left(\underline{\gamma}_k(\phi_k(-\Delta_j)) \right) \right| \\ &+ \sum_{j=h+1}^{h+h_d} S_{0,k,j} \int_{-\Delta_j}^{0} \left| \sigma_{0,k,j} \left(\underline{\gamma}_k(\phi_k(\tau)) \right) \right| d\tau \Bigg\}\end{aligned} \tag{10}$$

$$\hat{\alpha}_0(M_{a,0}(\phi_0)) \leq \alpha_0(\phi_0) \tag{11}$$

for some $\Delta_j \in (0, \Delta]$, $j = 1, 2, \ldots, h+h_d$ with non-negative integers h and h_d. By $h = 0$ (resp., $h_d = 0$), we mean that the first (resp., second) sum in terms of j in (10) vanishes.

In Assumption 2, the derivative $D^+V_0(\phi_0, \phi_1, ..., \phi_K, r_0)$ is defined by

$$D^+V_0(\phi_0, \phi_1, ..., \phi_K, r_0) = \limsup_{h \to 0^+} \frac{V_0(\phi_0^h) - V_0(\phi_0)}{h}$$

$$\phi_0^h(s) = \begin{cases} \phi_0(s+h), & s \in [-\Delta, -h), \\ \phi_0(0) + (s+h) f_0(\phi_0, ..., \phi_K, r_0), & s \in [-h, 0] \end{cases}.$$

This derivative plays a central role in the Lyapunov-Krasovskii methodology [2]. The functional V_0 is chosen locally Lipschitz according to [18], [19]. We next characterize the x_k-subsystems for $k = 1, 2, .., K$ as follows:

Assumption 3: There exist $\alpha_k, \sigma_{k,j} \in \mathcal{BK}$, continuous functions $\sigma_{r,k} : \mathbb{R}^{m_k} \to \mathbb{R}$, continuous functionals $M_{0,k} : \mathcal{C}_{n_0} \to \mathbb{R}$, continuous functions $\underline{\gamma}_{0,k} : \mathbb{R}^{n_0} \to \mathbb{R}$ and $S_{k,j} \in \{0, 1\}$ for $k = 1, 2, ..., K$ and $j = 0, 1, ..., h$ such that

$$\sigma_{r,k}(0) = 0 \tag{12}$$

$$M_k(\phi_k)\rho_k(\phi_k, \phi_0, r_k) \geq 0, \ \forall \phi_0 \in \mathcal{C}_{n_0}, \ r_k \in \mathbb{R}^{m_k} \tag{13}$$

and (4)-(5) hold, where $\rho_k : \mathcal{C}_{n_k} \times \mathcal{C}_{n_0} \times \mathbb{R}^{m_k} \to \mathbb{R}$ is

$$\begin{aligned}\rho_k(\phi_k, \phi_0, r_k) &= -\alpha_k(M_k(\phi_k)) + S_{k,0}\sigma_{k,0}(M_{0,k}(\phi_0)) \\ &+ \sum_{j=1}^{h} S_{k,j}\sigma_{k,j} \left(\underline{\gamma}_{0,k}(\phi_0(-\Delta_j)) \right) \\ &+ \sigma_{r,k}(r_k),\end{aligned} \tag{14}$$

and $\Delta_j \in (0, \Delta]$, $j = 1, 2, \ldots, h$, $M_k : \mathcal{C}_{n_k} \to \mathbb{R}$ and $\underline{\gamma}_k : \mathbb{R}^{n_k} \to \mathbb{R}$ are defined in Assumption 2. By $h = 0$, we mean that the first sum in terms of j in (14) vanishes.

Furthermore, we assume the following.

Assumption 4: If $\sum_{j=1}^{h+h_d} S_{0,k,j} > 0$ holds for an integer $k = 1, 2, ..., K$, it holds that $x_k(t) = f_k(x_0(t))$.

Assumption 4 guarantees that the Lyapunov-Krasovskii functional V_{cl} to be constructed for the entire system (1) is a functional of ϕ_0 and independent of the disturbance r_k. In contrast to the symmetric formulation in [12], [11], this paper employs $\sigma_{0,k,j} \in \mathcal{BK}$ and $\alpha_0 : \mathcal{C}_{n_0} \to \mathbb{R}_+$ so that the functional ρ_0 is allowed to be asymmetric with respect to the origin. In a similar manner, the functions $\alpha_k, \sigma_{k,j} \in \mathcal{BK}$ allow the functional ρ_k to be asymmetric for $k = 1, 2, ..., K$.

III. Asymmetric Small-Gain Condition

For $k = 1, 2, ..., K$, define

$$\sigma_{0,k}(s) = \sum_{j=0}^{h} S_{0,k,j}\sigma_{0,k,j}(s) + \sum_{j=h+1}^{h+h_d} S_{0,k,j}\Delta_j\sigma_{0,k,j}(s) \quad (15)$$

$$v_k = \sum_{j=0}^{h} S_{k,j} \quad (16)$$

which are class $\mathcal{BK}$ functions. We also employ the functions

$$\eta_k(s) = \begin{cases} \lim_{s\to-\infty} \sigma_{0,k}(s), & s \in (-\infty, \alpha_k(-\infty)] \\ \sigma_{0,k} \circ \alpha_k^{-1}(s), & s \in (\alpha_k(-\infty), \alpha_k(+\infty)) \\ \lim_{s\to+\infty} \sigma_{0,k}(s), & s \in [\alpha_k(+\infty), \infty) \end{cases} \quad (17)$$

for $k = 1, 2, ..., K$. Let

$$F_{i,j}(\tau) = \frac{-\tau}{\Delta_j} + (1+\epsilon_i)\frac{\tau+\Delta_j}{\Delta_j}, \ i = 0, 1, ..., K, \quad (18)$$

where the positive real numbers ϵ_i have yet to be determined. The following is the main result which establishes iISS and ISS properties of the system (1) and constructs a Lyapunov-Krasovskii functional.

Theorem 1: Assume that

$$\lim_{s\to+\infty} \alpha_k(s) = +\infty \ \vee \ \lim_{s\to+\infty} \sigma_{0,k}(s) < +\infty \quad (19)$$

$$\lim_{s\to-\infty} \alpha_k(s) = -\infty \ \vee \ \lim_{s\to-\infty} \sigma_{0,k}(s) > -\infty \quad (20)$$

for $k = 1, 2, ..., K$. Suppose that there exist $\tilde{\alpha}_{0,k} \in \mathcal{BK}$, $c_{0,k} > 1$ and $c_k > 1$, $k = 1, 2, ..., K$, such that

$$\sum_{k=1}^{K} \left|\tilde{\alpha}_{0,k}(M_{0,k}(\phi_0))\right| \le \alpha_0(\phi_0), \quad \forall\phi_0 \in \mathcal{C}_{n_0} \quad (21)$$

$$\sum_{j=0}^{h} \left|c_{0,k}\sigma_{0,k} \circ \alpha_k^{-1} \circ c_k v_k S_{k,j}\sigma_{k,j}(s)\right| \le \left|\tilde{\alpha}_{0,k}(s)\right|, \quad \forall s \in \mathbb{R}. \quad (22)$$

Then the system (1) is iISS with respect to input r and state x. In addition, an iISS Lyapunov-Krasovskii functional is

$$\begin{aligned} V_{cl}(\phi_0) &= V_0(\phi_0) \\ &+ \sum_{k=1}^{K}\Bigg\{\sum_{j=1}^{h} S_{0,k,j}\int_{-\Delta_j}^{0} F_{0,j}(\tau)\left|\sigma_{0,k,j}\left(\underline{\gamma}_k(\phi_k(\tau))\right)\right|d\tau \\ &+ \sum_{j=h+1}^{h+h_d} S_{0,k,j}\int_{-\Delta_j}^{0} F_{0,j}(\tau)\int_{\tau}^{0}\left|\sigma_{0,k,j}\left(\underline{\gamma}_k(\phi_k(\theta))\right)\right|d\theta d\tau \\ &+ (1+\epsilon_0)\sum_{j=1}^{h} S_{k,j}\int_{-\Delta_j}^{0} F_{k,j}(\tau)\big|\eta_k \\ &\qquad \circ (1+\zeta_k)v\sigma_{k,j}(\underline{\gamma}_{0,k}(\phi_0(\tau)))\big|d\tau\Bigg\}, \end{aligned} \quad (23)$$

where

$$0 < \zeta_k \le c_k - 1, \quad k = 1, 2, ..., K \quad (24)$$

$$0 < \epsilon_0, \ 0 < \epsilon_k, \ 0 < (1+\epsilon_0)(1+\epsilon_k) < c_{0,k}. \quad (25)$$

Furthermore, the system (1) is ISS with respect to input r and state x, and the function (23) is an ISS Lyapunov-Krasovskii functional in the case of $\hat{\alpha}_0 \in \mathcal{K}_\infty$.

The pair of (21) and (22) in Theorem 1 forms an asymmetric small-gain condition. The functions $\sigma_{0,k}$, α_k, $\sigma_{k,j}$, $\tilde{\alpha}_{0,k}$ and the functional $M_{0,k}$ are bidirectional, so that the loop gain does not have to be symmetric with respect to the origin. It can be seen in the proof of Theorem 1 that $c_k = 1$ and $\zeta_k = 0$ can be used in the case of $\sigma_{r,k} = 0$.

Remark 1: The existence of $c_0, c_k > 1$ fulfilling (22) does not require

$$\lim_{s\to\infty} \alpha_k(s) = \infty \ \vee \ \lim_{s\to\infty} \alpha_k(s) > v_k S_{k,j} \lim_{s\to\infty} \sigma_{k,j}(s) \quad (26)$$

since (19) and (20) are assumed. The properties (19) and (20) ensure that the inverse operation of α_k is not necessarily well-defined for the entire $\mathbb{R}$ in (22).

Remark 2: The formulation this paper employs has redundancy in dividing a time-delay system into an x_0-subsystem and x_k-subsystems. Although different decompositions may result in Lyapunov-Krasovskii functionals and small-gain conditions essentially in the same form, the conservativeness stemming from the non-unique decomposition varies. The search for a decomposition minimizing the conservativeness is an interesting subject of further study.

Remark 3: The definition of the functions M_k, $M_{0,k}$ and $\underline{\gamma}_k$, $\underline{\gamma}_{0,k}$ in (7) and (5) allows the dissipation properties with (10) and (14) to be asymmetric. The supply rates (10) and (14) reduce to the symmetric ones in [11] if we restrict $\underline{\gamma}_k(\phi_k(\tau))$ and $\underline{\gamma}_{0,k}(\phi_0(\tau))$ satisfying (7) and (5) to $\underline{\gamma}_k(|\phi_k(\tau)|)$, $\underline{\gamma}_{0,k}(|\phi_0(\tau)|)$ satisfying

$$M_{0,k}(\phi_0) \ge \underline{\gamma}_{0,k}(|\phi_0(0)|), \ \forall\phi_0 \in \mathcal{C}_{n_0}$$

$$M_k(\phi_k) \le \underline{\gamma}_k(|\phi_k(0)|), \ \forall\phi_k \in \mathcal{C}_{n_k}$$

with $\underline{\gamma}_k$ and $\underline{\gamma}_{0,k} \in \mathcal{K}_\infty$. In such a case, the small-gain condition consisting of (21) and (22) reduces to the symmetric one in [11].

IV. Proof of Theorem 1

The properties (19) and (20) imply $\eta_k \in \mathcal{BP}$ for each $k = 1, 2, ..., K$, and they are non-decreasing. The function η_k is of class $\mathcal{BK}$ if $\alpha_k \in \mathcal{BK}_\infty$. From (5) and (7), we have

$$D^+V_{cl}(\phi_0, \phi_1, ..., \phi_K, r_0) \le \bar{\rho}_0 + \beta, \quad (27)$$

where

$$G_j(\phi_0,\phi_1,...,\phi_K) = \sum_{k=1}^{K}\Bigg\{S_{0,k,j}\frac{\epsilon_0}{\Delta_j}\int_{-\Delta_j}^{0}\left|\sigma_{0,k,j}\left(\underline{\gamma}_k(\phi_k(\tau))\right)\right|d\tau + S_{k,j}\frac{(1+\epsilon_0)\epsilon_k}{\Delta_j}\int_{-\Delta_j}^{0}\bigg|\eta_k\circ (1+\zeta_k)v_k\sigma_{k,j}\left(\underline{\gamma}_{0,k}(\phi_0(\tau))\right)\bigg|d\tau\Bigg\}$$

$$H_j(\phi_1,...,\phi_K) = \sum_{k=1}^{K}S_{0,k,j}\frac{\epsilon_0}{\Delta_j}\int_{-\Delta_j}^{0}\int_{\tau}^{0}\left|\sigma_{0,k,j}\left(\underline{\gamma}_k(\phi_k(\theta))\right)\right|d\theta d\tau$$

$$\bar{\rho}_0(\phi_0,\phi_1,...,\phi_K,r_0) = -\alpha_0(\phi_0)+\sigma_{r,0}(|r_0|) + (1+\epsilon_0)\sum_{k=1}^{K}|\sigma_{0,k}(M_k(\phi_k))| - \sum_{j=1}^{h}G_j(\phi_0,\phi_1,...,\phi_K) - \sum_{j=h+1}^{h+h_d}H_j(\phi_1,...,\phi_K)$$

$$\beta(\phi_0) = (1+\epsilon_0)\sum_{k=1}^{K}\Bigg\{\sum_{j=1}^{h}(1+\epsilon_k)S_{k,j}\left|\eta_k\circ(1+\zeta_k)v_k\sigma_{k,j}(M_{0,k}(\phi_0))\right| - S_{k,j}\left|\eta_k\circ(1+\zeta_k)v_k\sigma_{k,j}\left(\underline{\gamma}_{0,k}(\phi_0(-\Delta_j))\right)\right|\Bigg\}.$$

From (13), the non-decreasing property of η_k and $\eta_k(0)=0$ it follows that

$$\bar{\rho}_0+\beta \le -\alpha_0(\phi_0)+\sigma_{r,0}(|r_0|) - \sum_{j=1}^{h}G_j - \sum_{j=h+1}^{h+h_d}H_j + \beta + (1+\epsilon_0)\sum_{k=1}^{K}\Bigg|\eta_k\circ\Bigg(\sigma_{r,k}(r_k)+S_{k,0}\sigma_{k,0}(M_{0,k}(\phi_0)) + \sum_{j=1}^{h}S_{k,j}\sigma_{k,j}\left(\underline{\gamma}_{0,k}(\phi_0(-\Delta_j))\right)\Bigg)\Bigg|. \tag{28}$$

Using $c_k-1\ge\zeta_k>0$, (22), the non-decreasing property of η_k defined in (17), we can verify that there exists $\mu\in(0,1)$ such that

$$\begin{aligned}
&\beta + (1+\epsilon_0)\sum_{k=1}^{K}\Bigg|\eta_k\circ\Bigg(\sigma_{r,k}(r_k)+S_{k,0}\sigma_{k,0}(M_{0,k}(\phi_0)) + \sum_{j=1}^{h}S_{k,j}\sigma_{k,j}\left(\underline{\gamma}_{0,k}(\phi_0(-\Delta_j))\right)\Bigg)\Bigg| \\
&\le \beta + (1+\epsilon_0)\sum_{k=1}^{K}\Bigg\{S_{k,0}\Big|\eta_k\circ(1+\zeta_k)v_k\sigma_{k,0}(M_{0,k}(\phi_0))\Big| + \sum_{j=1}^{h}S_{k,j}\Big|\eta_k\circ(1+\zeta_k)v_k\sigma_{k,j}\left(\underline{\gamma}_{0,k}(\phi_0(-\Delta_j))\right)\Big| + \Big|\eta_k\circ(1+\tfrac{1}{\zeta_k})\sigma_{r,k}(r_k)\Big|\Bigg\} \\
&\le (1+\epsilon_0)\sum_{k=1}^{K}\Bigg\{\sum_{j=0}^{h}(1+\epsilon_k)\Big|\sigma_{0,k}\circ\alpha_k^{-1}\circ(1+\zeta_k)v_k \cdot S_{k,j}\sigma_{k,j}(M_{0,k}(\phi_0))\Big| + \Big|\eta_k\circ(1+\tfrac{1}{\zeta_k})\sigma_{r,k}(r_k)\Big|\Bigg\} \\
&\le (1+\epsilon_0)\sum_{k=1}^{K}\Bigg\{\frac{(1+\epsilon_k)}{c_{0,k}}|\tilde{\alpha}_{0,k}(M_{0,k}(\phi_0))| + \Big|\eta_k\circ(1+\tfrac{1}{\zeta_k})\sigma_{r,k}(r_k)\Big|\Bigg\} \\
&\le \mu\alpha_0(\phi_0) + (1+\epsilon_0)\sum_{k=1}^{K}\Big|\eta_k\circ(1+\tfrac{1}{\zeta_k})\sigma_{r,k}(r_k)\Big|.
\end{aligned} \tag{29}$$

Here, the property (21) is used in the last inequality. The inequality (27) and substitution of (29) into (28) yield

$$D^+V_{cl} \le -(1-\mu)\alpha_0(\phi_0) - \sum_{j=1}^{h}G_j - \sum_{j=h+1}^{h+h_d}H_j + \sigma_{r,0}(|r_0|) + (1+\epsilon_0)\sum_{k=1}^{K}\Big|\eta_k\circ(1+\tfrac{1}{\zeta_k})\sigma_{r,k}(r_k)\Big|. \tag{30}$$

Thus, by virtue of the Lyapunov-type characterizations presented in [20], [12], the property (30) together with (12), (11), (4), (6), Assumption 4 and $1\le F_{i,j}(\tau)\le 1+\epsilon_i, \forall\tau\in[-\Delta_j,0]$ implies that V_{cl} given in (23) is an iISS Lyapunov-Krasovskii functional for (1). Furthermore, the functional V_{cl} is an ISS Lyapunov-Krasovskii functional if $\hat{\alpha}_0\in\mathcal{K}_\infty$.

V. An Example: Network Flow Control

This section investigates robustness of a flow control algorithm for communication networks proposed in [16]. The algorithm is based on a static optimization problem and a dynamic stabilization which provides an update law to drive the flow into the desired operating point determined by the optimization problem. The aim of the flow control is to allocate the available bandwidth to N competing users within the limits of link capacities. Let R denote a routing matrix describing the user-link connection as

$$R_{ji} = \begin{cases} 1 & \text{If the } i\text{-th user uses the } j\text{-th link} \\ 0 & \text{otherwise} \end{cases}.$$

It is assumed that each user uses fixed links at any given time, i.e., R_{ji}'s are constant. Let x_i, $i=1,2,...,N$, denote the sending rate of users. Then the link rate y_i, $i=1,2,...,L$, satisfies

$$y = Rx \tag{31}$$
$$x=[x_1,x_2,...,x_N]^T, \quad y=[y_1,y_2,...,y_L]^T.$$

Consider

$$p_j = h_j(y_j), \quad p=[p_1,p_1,...,p_L]^T, \tag{32}$$

where $h_i(y_j)$ is the penalty function that enforces the capacity constraint of the j-th link. The link price p_i is sent back to the users by

$$q = R^Tp, \quad q=[q_1,q_1,...,q_N]^T. \tag{33}$$

The primal algorithm proposed in [16] updates the sending rate x_i based on the price feedback q_i as follows:

$$\dot{x}_i = \lambda_i \left(\frac{dU_i(x_i)}{dx_i} - q_i \right)^+_{x_i}, \ i = 1, 2, ..., N \quad (34)$$

Here, $\lambda_i > 0$, $i = 1, 2, ..., N$, are design parameters The function $U_i(x_i)$ is called the utility function describe how happy each user is. The capacity of the links prevent the users from maximize $U_i(x_i)$ without limit, which is approximately described by the penalty functions [16]. The projection

$$(f_i(x_i))^+_{x_i} = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } f_i(x_i) < 0 \\ f_i(x_i) & \text{otherwise} \end{cases}$$

guarantees the non-negativity of x_i's, i.e., each $x_i(t)$ remains non-negative for all $t \geq 0$ for arbitrary $x_i(0) \geq 0$, $i = 1, 2, ..., N$, which holds naturally. We assume that each $U_i(x_i)$ is twice continuously differentiable and satisfies

$$\frac{d^2U_i(x_i)}{dx_i^2} < 0, \ \forall x_i \in \mathbb{R}_+$$

$$\lim_{x_i \to 0} U_i(x_i) = -\infty, \quad \lim_{x_i \to \infty} \frac{dU_i(x_i)}{dx_i} \leq 0,$$

while each penalty function $h_i(y_j)$ is a non-decreasing continuous function on $\mathbb{R}_+$ and satisfies

$$h_j(0) \geq 0, \quad \lim_{y_j \to \infty} h_j(y_j) > 0.$$

These assumptions guarantee the existence of an equilibrium x_*. In the presence of time-delays in the communication between users and links, the model (34) becomes

$$\dot{x}_i(t) = \lambda_i \left(\frac{dU_i(x_i(t))}{dx_i} - q_i(t - \Delta_i) + d_i(i) \right)^+_{x_i}, \ i = 1, 2, ..., N. \quad (35)$$

The signals $d_i(t) \in \mathbb{R}$, $i = 1, 2, ..., N$, are disturbances. Notice that the delays in the forward and backward paths can be combined into the round trip delays Δ_i without loss of generality since the x_i-equations (34) by themselves are decoupled from each other. The lumped notation only shifts the initial conditions of x componentwisely in time. Define

$$W_i(x_i - x_i^*) = \frac{1}{\lambda_i} \int_{x_i^*}^{x_i} \frac{dU_i}{dx_i}(x_i^*) - \frac{dU_i}{dx_i}(s) ds.$$

Using Young's inequality, we obtain

$$\dot{W}_i \leq -\left(\frac{1}{2} - \delta\right) \left(\frac{dU_i}{dx_i}(x_i^*) - \frac{dU_i}{dx_i}(x_i(t)) \right)^2 + \frac{1}{2}(q_i(t-\Delta_i) - q_i^*)^2 + \frac{1}{4\delta} d_i(t)^2 \quad (36)$$

along the solution $x_i(t)$ of (35), for $0 < \delta < 1/2$, where $q_i^* > 0$ corresponds to the equilibrium x_i^*. We use $\tilde{x}_i = x_i - x_i^*$ and $\tilde{q}_i = q_i - q_i^*$. The inequality (36) indicates that the x_i-subsystem (35) is iISS with respect to input $(\tilde{q}_i, d_i)$[1], [20]. The x_i-subsystem is not ISS if dU_i/dp_i is bounded [23], [20]. For the network flow control system, the dynamic part corresponding to the x_0-subsystem in Sections II and III is (35) where $[\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N]^T$ and $[\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_N]^T$ are its state vector and feedback input vector, respectively. The static part corresponding to the x_k-subsystems in Sections II and III is obtained from (31), (32) and (33) as

$$\tilde{q}_i = \chi_i(x), \quad i = 1, 2, ..., N, \quad (37)$$

where $\chi_i : \mathbb{R}_+^N \to \mathbb{R}$ is

$$\chi_i(x) = [R_{1i}, R_{2i}, ..., R_{Li}] \begin{bmatrix} h_1([Rx]_1) - h_1([Rx^*]_1) \\ h_2([Rx]_2) - h_j([Rx^*]_2) \\ \vdots \\ h_L([Rx]_L) - h_L([Rx^*]_L) \end{bmatrix}.$$

To apply Theorem 1 to the flow control system, take

$$K = h = N, \ h_d = 0, \ V_0(\tilde{x}) = \sum_{i=1}^N W_i(\tilde{x}_i)$$
$$\phi_0 = [\phi_{0,1}, \phi_{0,1}, ..., \phi_{0,N}]^T$$
$$\alpha_0(\phi_0) = \sum_{i=1}^N \left(\frac{1}{2} - \delta\right) \left(\frac{dU_i}{dx_i}(x_i^*) - \frac{dU_i}{dx_i}(\phi_{0,i}(0) + x_i^*) \right)^2$$
$$\tilde{\alpha}_{0,k}(s) = \frac{c}{2}s^2, \quad c > 1$$
$$M_{0,k}(\phi_0) = \chi_k(\phi_0(0) + x^*), \ M_k(\phi_k) = \phi_k(0)$$
$$v_k = 1, \ \alpha_k(s) = s, , \ \sigma_{r,k} = 0$$
$$S_{k,0} = 1, \ \sigma_{k,0}(s) = s, \ S_{k,j} = 0, j = 1, 2, ..., h$$
$$S_{0,k,0} = 0, \ S_{0,k,j} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases},$$
$$\sigma_{0,k,j}(s) = \begin{cases} \frac{1}{2}s^2 \text{sgn}(s) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$
$$\sigma_{0,k}(s) = \frac{1}{2}s^2 \text{sgn}(s), \quad \sigma_{r,0} = \frac{s^2}{4\delta},$$

where ϕ_0 and ϕ_k corresponds to $\tilde{x}(t + \tau)$ and $\tilde{q}_k(t + \tau)$, respectively, where $\tau \in [-\Delta, 0]$ and $k = 1, 2, ..., N$. Notice that the choices $\alpha_0(\phi_0)$ and $M_{0,k}(\phi_0)$ are asymmetric with respect to the equilibrium. Requiring the utility and penalty functions to be almost symmetric is unrealistic for meaningful flow optimization, which implies that an symmetric small-gain analysis becomes purely local. By virtue of the development in the previous section, we obtain the following theorem allowing for the asymmetry.

Theorem 2: If the utility functions $U_i(x_i)$, $i = 1, 2, ..., N$, and the penalty functions $h_j(y_j)$, $j = 1, 2, ..., L$, satisfy

$$\sum_{i=1}^N \left(\frac{dU_i}{dx_i}(x_i^*) - \frac{dU_i}{dx_i}(x_i) \right)^2 \geq c \sum_{i=1}^N \chi_i(x)^2, \ x \in \mathbb{R}_+^N \quad (38)$$

for some $c > 1$, then the primal flow control system is iISS with respect to input d and state $\tilde{x}$ for arbitrary time-delays. In addition, it is ISS with respect to input d and state $\tilde{x}$ if

$$\lim_{x_i \to \infty} \frac{dU_i}{dx_i}(x_i) = -\infty. \quad (39)$$

The unboundedness property (39) is not met by typical choices of the utility functions such as $U_i(x_i) = -a_i/x_i$, $U_i(x_i) = a_i \log x_i$, $U_i(x_i) = a_i \tan^{-1}(x_i/a_i)$, $a_i > 0$,

[24], [17]. Therefore, ISS approaches naturally fail to verify robustness properties with respect to disturbances in the presence of large time-delays(see e.g. [5]). In contrast, the iISS approach proposed in this paper demonstrates that the flow control system implementing these utility functions can possess robustness in the sense of iISS regardless of time-delays. It is stressed that the iISS property implies global asymptotic stability of the equilibrium.

In comparison with preceding studies [26], [5], [4], [6], the iISS framework developed in this paper not only allows us to make use of nonlinearity efficiently, but also address arbitrarily large time-delays and disturbances simultaneously. The basic idea of the approach formulated rigorously in this paper has been used for stability analysis of a power control algorithm for CDMA wireless communication [13]. Deriving a Lyapunov-Krasovskii functional instead of using trajectory-based approaches is the key to the stability analysis without imposing the ISS requirement on the utility functions.

VI. Concluding Remarks

This paper has presented an asymmetric small-gain condition for verifying the iISS of the nonlinear time-delay system consisting of an iISS dynamical subsystem and a static subsystem. The time-delays are allowed to be both discrete and distributed. An iISS Lyapunov-Krasovskii functional has been derived. The iISS framework enables us to deal with a much broader class of nonlinearities than existing approaches based on operator norms or the ISS gain. An example of network flow control has illustrated how the asymmetric iISS characterization reduces potential conservativeness in global stability and robustness analysis of a time-delay system subject to disturbances. Extension to time-varying delays along the line of [11] is straightforward.

For interconnections consisting only of dynamical subsystems, the utilization of asymmetric small-gain characterization remains unsolved. It is known that, for the interconnection of two iISS dynamical subsystems, a Lyapunov function can be constructed as a sum of nonlinearly transformed Lyapunov functions V_i of the individual subsystems, i.e., $V = F_1(V_1) + F_2(V_2)$ (see e.g. [7], [10]). Incorporation of the asymmetry into such a construction is not at all straightforward unless we restrict the transformations F_1 and F_2 to linear functions. For nonlinear systems, limiting the transformations to linear functions in constructing Lyapunov functions often results in far too conservative stability criteria [3]. In [8], it is shown that, for a special class of interconnected systems, the utilization of the asymmetry can be still combined with nonlinear F_1 and F_2. Extending this idea to time-delay systems is a topic of further study.

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