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Zero Duality Gap for Classical OPF Problem Convexifies Fundamental Nonlinear Power Problems

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Abstract— **Most of the fundamental optimization problems for power systems are highly non-convex and NP-hard (in the worst case), partially due to the nonlinearity of certain physical quantities, e.g. active power, reactive power and magnitude of voltage. The classical optimal power flow (OPF) problem is one of such problems, which has been studied for half a century. Recently, we obtained a condition under which the duality gap is zero for the classical OPF problem and hence a globally optimal solution to this problem can be found efficiently by solving a semidefinite program. This zero-duality-gap condition is satisfied for IEEE benchmark systems and holds widely in practice due to the physical properties of transmission lines. The present paper studies the case when there are other common sources of non-convexity, such as variable shunt elements, variable transformer ratios and contingency constraints. It is shown that zero duality gap for the classical OPF problem implies zero duality gap for a general OPF-based problem with these extra sources of non-convexity. This result makes it possible to find globally optimal solutions to several fundamental power problems in polynomial time.**

I. INTRODUCTION

The classical optimal power flow (OPF) problem aims to find a steady-state operating point of a power system that minimizes a desirable cost function, e.g. power loss or generation cost, and satisfies network and physical constraints on loads, powers, voltages and line flows [1]. The OPF problem is not only non-convex but also NP-hard, because of its possible reduction in a special case to the $(0, 1)$-quadratic optimization. Started by the work [2] in 1962, many of the existing optimization techniques have been adapted to solve the OPF problem, leading to algorithms based on linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point method, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithm, evolutionary programming and particle swarm optimization [3], [4]. Due to the non-convexity of the OPF problem, these algorithms are not robust, lack performance guarantees, and may not be able to find a global optimum.

In an effort to convexify the OPF problem, it is shown in [5] that the load flow problem for a radial distribution system can be modeled as a convex optimization problem in the form of a conic program. Nonetheless, this result fails to hold for a general network, due to the presence of arctangent equality constraints [6]. By exploiting the physical properties of transmission lines, we have proven in our recent papers [7] and [8] that the classical OPF problem corresponding to a practical power system can be convexified naturally and then solved efficiently. More precisely, we considered some equivalent form of the OPF problem whose dual can be cast as a semidefinite program [9], [10]. Although this dual problem is solvable in polynomial time, its solution may not help solve the OPF problem, in light of the duality gap being possibly nonzero. We derived a zero-duality-gap condition for the OPF problem in [7] and [8] under which a globally optimal solution to the OPF problem can be recovered from a solution to its dual. This condition is satisfied for all IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, and is expected to hold for every practical power system (for more details, see the studies provided in [7], [8]).

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Many of the fundamental optimization problems arising in power systems are based on a single or a set of coupled classical OPF problems with more constraints and more variables. A question arises as whether these problems can also be convexified. The present paper aims to address this question. The objective is to show that zero duality gap for the classical OPF problem implies zero duality gap for the following important problems (and any combinations of them) as well:

- The OPF problem with extra variables associated with the unknown shunt elements [1].
- The OPF problem with extra variables associated with the unknown transformer ratios [1].
- The security-constraint OPF problem (known also as contingency-constrained OPF), which corresponds to a set of coupled OPF problems [11].

The technique developed in this paper can be used to generalize the above-mentioned zero-duality-gap result to several other OPF-based problems such as the dynamic OPF problem or the power system planning with renewable resources [12].

The rest of the paper is organized as follows. A summary of our previous results is provided in Section II and the problem is formulated accordingly. The main results are given in Section III, which are applied to the IEEE test systems with 14 and 30 buses in Section IV. Finally, some concluding remarks are drawn in Section V.

Notations: The following notations will be used throughout the paper:

- i : The imaginary unit.
- $\mathbf{R}$: The set of real numbers.
- Re$\{\cdot\}$ and Im$\{\cdot\}$: The operators returning the real and imaginary parts of a complex matrix.
- T : The transpose operator.
- $*$: The conjugate transpose operator.
- $\succeq$: The matrix inequality sign in the positive semidef-

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inite sense [9].

II. PRELIMINARIES AND PROBLEM FORMULATION

Given two natural numbers m and n such that $m \leq n$, consider a power network with n buses, labeled as $1, 2, ..., n$, and m generators connected to buses $1, 2, ..., m$. Assume that each bus $k \in \{1, 2, ..., n\}$ is connected to a load with the given apparent power $P_{D_k} + Q_{D_k}\mathrm{i}$ (this number is zero whenever a bus is not connected to any load). For every $l \in \{1, 2, ..., m\}$, let P_{G_l} and Q_{G_l} denote the unknown active and reactive powers supplied by generator l, respectively, and $f_l(P_{G_l}) = c_{l2}P_{G_l}^2 + c_{l1}P_{G_l} + c_{l0}$ denote a cost function associated with this generator, for some nonnegative numbers c_{l0}, c_{l1}, c_{l2}. Define V_k as the unknown complex voltage at bus $k \in \{1, 2, ..., n\}$ and $\mathbf{V}$ as the vector of all bus voltages. The classical OPF problem aims to minimize $\sum_{l=1}^{m} f_l(P_{G_l})$ over the unknown parameters $\mathbf{V}, P_{G_1}, ..., P_{G_m}, \; Q_{G_1}, ..., Q_{G_m}$ subject to the constraints that every bus $k \in \{1, 2, ..., n\}$ must be able to deliver the power $P_{D_k} + Q_{D_k}\mathrm{i}$ to its load and that

$$
\begin{aligned}
&P_{k,\min} \leq P_{G_k} \leq P_{k,\max}, && k \in \{1, ..., m\} \\
&Q_{k,\min} \leq Q_{G_k} \leq Q_{k,\max}, && k \in \{1, ..., m\} \\
&V_{k,\min} \leq |V_k| \leq V_{k,\max}, && k \in \{1, ..., n\} \\
&|S_{kl}| \leq S_{kl,\max}, && k, l \in \{1, ..., n\}
\end{aligned}
$$

for some given limits $P_{k,\min}, P_{k,\max}, Q_{k,\min}, Q_{k,\max}, V_{k,\min}, V_{k,\max}, S_{kl,\max}$, where S_{kl} denotes the apparent power transferred from bus k to the rest of the network through the line (k, l) (note that S_{kl} is zero if the line (k, l) does not exist).

In order to mathematically formulate the problem, the first step is to find an equivalent circuit model of the network with only three types of lumped elements: resistors, capacitors and inductors. This model can be obtained by replacing every transmission line and transformer with their equivalent Π models [1]. In the derived equivalent circuit, let y_{kl} denote the mutual admittance between buses k and l, and y_{kk} denote the admittance-to-ground at bus k, for every $k, l \in \{1, 2, ..., n\}$. Define the admittance matrix Y of the network as an $n \times n$ complex-valued matrix whose (k, l) entry is equal to $-y_{kl}$ if $k \neq l$ and $y_{kk} + \sum_{r \in \mathcal{N}(k)} y_{kr}$ otherwise, where $\mathcal{N}(k)$ is the set of those buses that are connected to bus k. It is worth mentioning that Y plays the role of a complex-valued (generalized) Laplacian matrix for a weighted graph associated with the power system. Define the current vector $\mathbf{I}$ as $Y\mathbf{V}$ and represent its k^{th} element with I_k, for every $k \in \{1, 2, ..., n\}$. Note that I_k is indeed the net current injected to bus k.

Let $e_1, e_2, ..., e_n$ denote the standard basis vectors in $\mathbf{R}^n$. For every $k, l \in \{1, 2, ..., n\}$, define

$$
\begin{aligned}
Y_k &:= e_k e_k^T Y \\
Y_{kl} &:= \frac{1}{2} e_k (b_{kl}\mathrm{i}) e_k^T + e_k y_{kl} e_k^T - e_k y_{kl} e_l^T \\
\mathbf{Y}_k &:= \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Y}_{kl} &:= \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{Y_{kl} + Y_{kl}^T\} & \mathrm{Im}\{Y_{kl}^T - Y_{kl}\} \\ \mathrm{Im}\{Y_{kl} - Y_{kl}^T\} & \mathrm{Re}\{Y_{kl} + Y_{kl}^T\} \end{bmatrix} \\
\bar{\mathbf{Y}}_k &:= \frac{-1}{2}\begin{bmatrix} \mathrm{Im}\{Y_k + Y_k^T\} & \mathrm{Re}\{Y_k - Y_k^T\} \\ \mathrm{Re}\{Y_k^T - Y_k\} & \mathrm{Im}\{Y_k + Y_k^T\} \end{bmatrix} \\
\bar{\mathbf{Y}}_{kl} &:= \frac{-1}{2}\begin{bmatrix} \mathrm{Im}\{Y_{kl} + Y_{kl}^T\} & \mathrm{Re}\{Y_{kl} - Y_{kl}^T\} \\ \mathrm{Re}\{Y_{kl}^T - Y_{kl}\} & \mathrm{Im}\{Y_{kl} + Y_{kl}^T\} \end{bmatrix} \\
\mathbf{X} &:= \begin{bmatrix} \mathrm{Re}\{\mathbf{V}\}^T & \mathrm{Im}\{\mathbf{V}\}^T \end{bmatrix}^T
\end{aligned}
$$

where b_{kl} denotes the capacitance of the transmission line (k, l) (note that Y_{kl}, $\mathbf{Y}_{kl}$, $\bar{\mathbf{Y}}_{kl}$ are all zero if $k = l$ or the line (k, l) does not exist). For every $k \in \{1, 2, ..., n\}$, denote the net active and reactive powers injected to bus k as $P_{k,\text{inj}}$ and $Q_{k,\text{inj}}$, respectively. Given $l \in \{1, ..., m\}$, $l' \in \{m+1, ..., n\}$ and $k, k' \in \{1, ..., n\}$, it can be shown that (see [7])

$$
\begin{aligned}
P_{l,\text{inj}} &= P_{G_l} - P_{D_l} \\
Q_{l,\text{inj}} &= Q_{G_l} - Q_{D_l} \\
P_{l',\text{inj}} &= -P_{D_l'} \\
Q_{l',\text{inj}} &= -Q_{D_l'} \\
|V_k|^2 &= \text{trace}\left\{M_k \mathbf{X}\mathbf{X}^T\right\} \\
P_{k,\text{inj}} &= \text{trace}\left\{\mathbf{Y}_k \mathbf{X}\mathbf{X}^T\right\}, \quad Q_{k,\text{inj}} = \text{trace}\left\{\bar{\mathbf{Y}}_k \mathbf{X}\mathbf{X}^T\right\} \\
|S_{kk'}|^2 &= \left(\text{trace}\left\{\mathbf{Y}_{kk'} \mathbf{X}\mathbf{X}^T\right\}\right)^2 + \left(\text{trace}\left\{\bar{\mathbf{Y}}_{kk'} \mathbf{X}\mathbf{X}^T\right\}\right)^2
\end{aligned}
$$

where $M_k \in \mathbf{R}^{2n \times 2n}$ is a diagonal matrix whose entries are all equal to zero, except for its (k, k) and $(n+k, n+k)$ entries that are equal to 1. To simplify the presentation, assume that $f_l(P_{G_l}) = P_{G_l}$ for every $l \in \{1, 2, ..., m\}$, implying that the cost to be minimized is simply the total power generation (the results being developed here are valid for the general case as well). Hence, the classical OPF problem corresponds to the minimization of

$$
\sum_{l=1}^{m} \left(\text{trace}\left\{\mathbf{Y}_l \mathbf{X}\mathbf{X}^T\right\} + P_{D_l}\right) \tag{1}
$$

over the variable $\mathbf{X} \in \mathbf{R}^{2n}$ subject to the constraints

$$
\begin{aligned}
P_{k,\min} - P_{D_k} \leq \text{trace}\left\{\mathbf{Y}_k \mathbf{X}\mathbf{X}^T\right\} \leq P_{k,\max} - P_{D_k} && \text{(2a)} \\
Q_{k,\min} - Q_{D_k} \leq \text{trace}\left\{\bar{\mathbf{Y}}_k \mathbf{X}\mathbf{X}^T\right\} \leq Q_{k,\max} - Q_{D_k} && \text{(2b)} \\
(V_{k,\min})^2 \leq \text{trace}\left\{M_k \mathbf{X}\mathbf{X}^T\right\} \leq (V_{k,\max})^2 && \text{(2c)} \\
\text{trace}\left\{\mathbf{Y}_{kl} \mathbf{X}\mathbf{X}^T\right\}^2 + \text{trace}\left\{\bar{\mathbf{Y}}_{kl} \mathbf{X}\mathbf{X}^T\right\}^2 \leq (S_{kl,\max})^2 && \text{(2d)}
\end{aligned}
$$

for all $k, l \in \{1, 2, ..., n\}$, where $P_{k,\min}$, $P_{k,\max}$, $Q_{k,\min}$ and $Q_{k,\max}$ are considered as zero (by convention) if $k > m$. In order to avoid triviality, assume that $\mathbf{X} = 0$ (or equivalently $\mathbf{V} = 0$) is not a solution to the OPF problem. We introduce four optimization problems in the sequel whose interrelation and relation to the OPF problem are illustrated in the diagram given in Figure 1.

Optimization 1: This optimization is obtained from the OPF problem formulated in (1) and (2) by replacing its constraint (2d) with the equivalent condition of the positive semi-definiteness of the matrix

$$\begin{bmatrix} (S_{kl,\max})^2 & \text{trace}\left\{\mathbf{Y}_{kl}\mathbf{X}\mathbf{X}^T\right\} & \text{trace}\left\{\bar{\mathbf{Y}}_{kl}\mathbf{X}\mathbf{X}^T\right\} \\ \text{trace}\left\{\mathbf{Y}_{kl}\mathbf{X}\mathbf{X}^T\right\} & 1 & 0 \\ \text{trace}\left\{\bar{\mathbf{Y}}_{kl}\mathbf{X}\mathbf{X}^T\right\} & 0 & 1 \end{bmatrix}$$

Optimization 2: This optimization is defined as the dual of Optimization 1, which indeed minimizes

$$\sum_{k=1}^{n}\Bigg\{\boldsymbol{\lambda}_k P_{D_k} + \bar{\boldsymbol{\lambda}}_k Q_{D_k} + \lambda_{k,\min}P_{k,\min} - \lambda_{k,\max}P_{k,\max} + \bar{\lambda}_{k,\min}Q_{k,\min} - \bar{\lambda}_{k,\max}Q_{k,\max} + \mu_{k,\min}\left(V_{k,\min}\right)^2 - \mu_{k,\max}\left(V_{k,\max}\right)^2 - \sum_{l=1}^{n}\left(\left(S_{kl,\max}\right)^2 h_{kl}^{11} + h_{kl}^{22} + h_{kl}^{33}\right)\Bigg\}$$

over the nonnegative scalar variables $\lambda_{k,\min}, \lambda_{k,\max}, \bar{\lambda}_{k,\min}, \bar{\lambda}_{k,\max}, \mu_{k,\min}, \mu_{k,\max}$, and the positive semidefinite matrices $H_{kl} \in \mathbf{R}^{3\times 3}$, $\forall k, l \in \{1, 2, ..., n\}$, subject to

$$A(\boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}, \boldsymbol{\mu}, \mathbf{H}) := \sum_{k=1}^{n}\Bigg\{\boldsymbol{\lambda}_k \mathbf{Y}_k + \bar{\boldsymbol{\lambda}}_k \bar{\mathbf{Y}}_k + \boldsymbol{\mu}_k M_k + 2\sum_{l=1}^{n}\left(h_{kl}^{12}\mathbf{Y}_{kl} + h_{kl}^{13}\bar{\mathbf{Y}}_{kl}\right)\Bigg\} \succeq 0$$

where h_{kl}^{ij} denotes the (i,j) entry of H_{kl} for every $i, j \in \{1, 2, 3\}$, and

$$\boldsymbol{\lambda}_k := \begin{cases} -\lambda_{k,\min} + \lambda_{k,\max} + 1 & \text{if } k = 1, ..., m \\ -\lambda_{k,\min} + \lambda_{k,\max} & \text{otherwise} \end{cases},$$

$$\bar{\boldsymbol{\lambda}}_k := -\bar{\lambda}_{k,\min} + \bar{\lambda}_{k,\max}, \quad \boldsymbol{\mu}_k := -\mu_{k,\min} + \mu_{k,\max},$$

$$\boldsymbol{\lambda} := \{\lambda_{k,\min}, \lambda_{k,\max}\}_{k=1}^{n}, \quad \bar{\boldsymbol{\lambda}} := \left\{\bar{\lambda}_{k,\min}, \bar{\lambda}_{k,\max}\right\}_{k=1}^{n},$$

$$\boldsymbol{\mu} := \{\mu_{k,\min}, \mu_{k,\max}\}_{k=1}^{n}, \quad \mathbf{H} = \{H_{kl}\}_{k,l=1}^{n}$$

(note that H_{kl} can be taken as zero if the line (k, l) does not exist in the power system).

Optimization 3: This optimization is obtained from Optimization 1 by first replacing every term $\mathbf{X}\mathbf{X}^T$ with a symmetric matrix variable $W \in \mathbf{R}^{2n\times 2n}$ and then adding the constraint $W \succeq 0$ (thus, the variable has changed from $\mathbf{X}$ to W).

Optimization 4: This optimization is obtained from Optimization 3 by including the additional constraint $\text{rank}\{W\} = 1$.

A. Previous Results

As illustrated in Figure 1 and proven in our recent papers [7], [8], Optimization 1 is naturally equivalent to the OPF problem, Optimization 2 is the dual of Optimization 1, Optimization 3 is the dual of Optimization 2 (strongly duality holds), Optimization 4 is different from Optimization 3 by an extra rank constraint, and finally Optimization 1 is equivalent to Optimization 4 via the change of variable $W = \mathbf{X}\mathbf{X}^T$. Due to the natural equivalence between Optimization 1 and the OPF problem, the names *OPF problem*, *dual of the OPF problem* and *dual of the dual of the OPF problem* will be used interchangeably for Optimizations 1, 2 and 3, respectively. The dual of the OPF problem is always feasible, but its optimal objective value can be: (i) infinite or (ii) finite. In Case (i), the OPF problem must be infeasible. In Case (ii), since the OPF problem is nonconvex, the optimal objective values of the OPF problem and its dual might not be identical. Whenever Case (i) happens (which detects the infeasibility of the OPF problem) or the optimal objective values of the OPF problem and its dual are the same, it is said that *the duality gap is zero for the OPF problem*. We proved the following important result in [7].

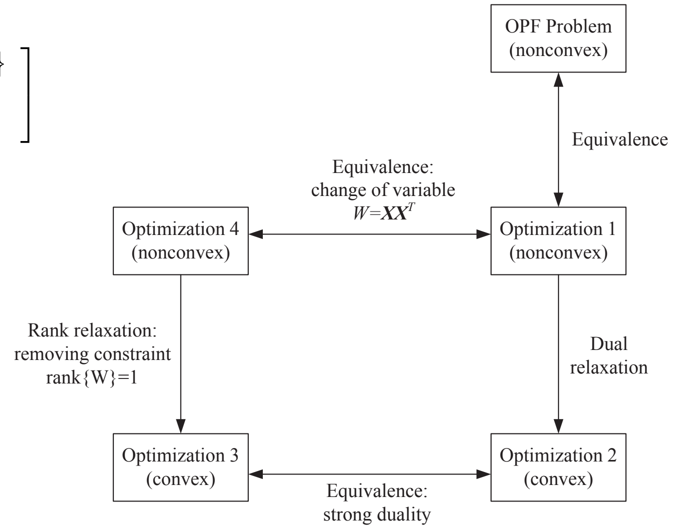

Fig. 1. This diagram demonstrates how Optimizations 1-4 are interrelated and also related to the OPF problem.

Theorem 1: The duality gap is zero for the OPF problem if its dual has a solution $(\boldsymbol{\lambda}_{\text{opt}}, \bar{\boldsymbol{\lambda}}_{\text{opt}}, \boldsymbol{\mu}_{\text{opt}}, \mathbf{H}_{\text{opt}})$ such that the matrix $A(\boldsymbol{\lambda}_{\text{opt}}, \bar{\boldsymbol{\lambda}}_{\text{opt}}, \boldsymbol{\mu}_{\text{opt}}, \mathbf{H}_{\text{opt}})$ has rank at least $2n - 2$. In this case, two properties hold:

- The dual of the dual of the OPF problem has a rank-one solution W_{opt}.
- Given any nonzero vector $\begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ in the null space of $A(\boldsymbol{\lambda}_{\text{opt}}, \bar{\boldsymbol{\lambda}}_{\text{opt}}, \boldsymbol{\mu}_{\text{opt}}, \mathbf{H}_{\text{opt}})$, there exist two scalars ζ_1 and ζ_2 such that $\mathbf{V}_{\text{opt}} = (\zeta_1 + \zeta_2 \mathrm{i})(U_1 + U_2 \mathrm{i})$ is a solution to the OPF problem.

Consider a special case of the OPF problem formulated in (1) and (2) where Y is a real-valued matrix, the constraints given in (2a), (2b) and (2d) are removed (by setting the corresponding lower and upper bounds as $-\infty$ and $+\infty$), all reactive loads are zero, and finally $V_{k,\min} = V_{k,\max}$ for every $k \in \{1, 2, ..., n\}$. In this case, the feasibility region for the real-valued vector $(V_1, ..., V_n)$ consists of 2^n points in the form of $(\pm V_{1,\min}, ..., \pm V_{n,\min})$. This substantiates that the OPF problem may have a complicated feasibility region, which can make it NP-complete for an arbitrary Y and hence create a nonzero duality gap [7]. However, the admittance matrix Y corresponding to a power system is structured in light of the physical properties of transmission lines. Using this fact, we showed in [7] and [8] that the zero-duality-gap condition stated in Theorem 1 is satisfied for all IEEE test systems with 14, 30, 57, 118 and 300 buses and, moreover, this condition is likely to hold for every practical power system.

B. Problem Statement

Define $\mathcal{D}$ as the set of every admittance matrix Y whose associated OPF problem has no duality gap for all possible values of the limits $P_{k,\min}, P_{k,\max}, Q_{k,\min}, Q_{k,\max}, V_{k,\min}, V_{k,\max}, S_{kl,\max}$, $k, l \in \{1, 2, ..., n\}$. The set $\mathcal{D}$ characterizes every network topology (including those corresponding to practical power systems) for which a globally optimal solution of the OPF problem can be found efficiently (by solving its dual). Recall that the classical OPF problem was non-convex partially due to the nonlinearity of *active power*, *reactive power* and *magnitude of voltage* with respect to the state variable $\mathbf{V}$. This source of non-convexity appears in almost all fundamental optimization-based power problems. These problems are often based on a single or a set of coupled classical OPF problems with more sources of non-convexity, e.g. variable transformer ratios, variable shunt elements, stability constraints and security constraints. The objective of this paper is to prove the following statement: *zero duality gap for the classical OPF problem implies zero duality gap for harder power problems with more sources of non-convexity*. In other words, it is intended to show that if Y belongs to $\mathcal{D}$, fundamental power problems (based on OPF) can be convexified naturally via the duality theory.

III. Main Results

Different generalizations to the classical OPF problem will be studied in the sequel.

A. Security-Constrained Optimal Power Flow

As far as the steady-state operation of a power system is concerned, there are two types of parameters: (i) a state vector $\mathbf{X}$ containing the real and imaginary parts of the bus voltages, (ii) a control vector $\mathbf{U}$ containing the controllable parameters of the power system. Note that every power system has certain controllable parameters (depending on its control strategy) such as active powers and voltage magnitudes at generator buses, sizes of capacitor banks, and transformer tap ratios. A general OPF-based problem can be formulated as:

$$\min_{\mathbf{X},\mathbf{U}} f(\mathbf{X},\mathbf{U}) \tag{4a}$$

$$\text{s.t.} \quad g(\mathbf{X},\mathbf{U}) = 0 \tag{4b}$$

$$h(\mathbf{X},\mathbf{U}) \geq 0, \tag{4c}$$

where

- $f(\mathbf{X},\mathbf{U})$ is an appropriate cost to be minimized (such as power loss or total generation cost).
- The relation (4b) describes the set of all equality constraints resulting from the power flow equations.
- The relation (4c) describes the set of all inequality constraints resulting from the physical limits imposed on the parameters of the system.

Assume that the power system is subject to c different contingencies, where each contingency corresponds to a new configuration in which certain transmission lines and generators are disconnected. The security-constrained optimal power flow (SCOPF) problem aims to optimize the performance of the power system under the normal condition such that the load and physical constraints are still satisfied after every pre-specified contingency. This problem can be formulated as:

$$\min_{\mathbf{X}^{(0)},...,\mathbf{X}^{(c)},\mathbf{U}^{(0)},...,\mathbf{U}^{(c)}} f\left(\mathbf{X}^{(0)},\mathbf{U}^{(0)}\right) \tag{5a}$$

$$\text{s.t.} \quad g_t\left(\mathbf{X}^{(t)},\mathbf{U}^{(t)}\right) = 0, \quad t = 0,...,c \tag{5b}$$

$$h_t\left(\mathbf{X}^{(t)},\mathbf{U}^{(t)}\right) \geq 0, \quad t = 0,...,c \tag{5c}$$

$$\left|\mathbf{U}^{(r)} - \mathbf{U}^{(0)}\right| \leq \Delta\mathbf{U}^{(r)}_{\max}, \quad r = 1,...,c \tag{5d}$$

where

- $\mathbf{X}^{(t)}$ and $\mathbf{U}^{(t)}$ denote the state and control vectors for the t^{th} configuration ($t = 0$ is the normal configuration and $t > 0$ is a contingency case).
- The equality and inequality constraints for the t^{th} configuration are given by (5b) and (5c).
- Given the constant vector $\Delta\mathbf{U}^{(r)}_{\max}$, the constraint (5d) accounts for the fact that the controllable parameters of a power system may not be able to change arbitrarily fast after a reconfiguration (this is partially due to physical ramp-up and ramp-down constraints).

It is worth mentioning that if $\Delta\mathbf{U}^{(r)}_{\max}$ is zero, the corresponding control strategy is said to be *preventive* in light of taking no control action after a contingency; otherwise, it is said to be *corrective*. Note that some of the entries of $\Delta\mathbf{U}^{(r)}_{\max}$ can be infinity, implying that the corresponding controllable parameters can change arbitrarily after a reconfiguration.

The objective of this part is to prove the following statement: *zero duality gap for the OPF problem implies zero duality gap for the SCOPF problem*. To this end, for the sake of simplifying the presentation, assume that every controllable parameter can only be an active power, a reactive power or a voltage magnitude. Using the techniques being developed in the next subsections, the results can be generalized to incorporate loads, shunt elements and transformer ratios into the control vector $\mathbf{U}$. Moreover, with no loss of generality, suppose that the cost function $f_0(\mathbf{X}^{(0)},\mathbf{U}^{(0)})$ is the total power generation. As before, we use the superscript (t) for every parameter of the power system in the t^{th} configuration, $t = 0, 1, ..., c$. For instance, $Y^{(0)}$ is equal to Y, and $Y^{(r)}$, $r = 1, 2, ..., c$, is the admittance matrix of the power system under the r^{th} contingency. The SCOPF problem can be expressed as the minimization of

$$\sum_{l=1}^{m} P^{(0)}_{G_l} \tag{6}$$

subject to

$$P_{k,\min}^{(t)} \leq P_{G_k}^{(t)} \leq P_{k,\max}^{(t)}, \qquad k \in \{1,...,m\} \tag{7a}$$
$$Q_{k,\min}^{(t)} \leq Q_{G_k}^{(t)} \leq Q_{k,\max}^{(t)}, \qquad k \in \{1,...,m\} \tag{7b}$$
$$V_{k,\min}^{(t)} \leq \left|V_k^{(t)}\right| \leq V_{k,\max}^{(t)}, \qquad k \in \{1,...,n\} \tag{7c}$$
$$\left|S_{kl}^{(t)}\right| \leq S_{kl,\max}^{(t)}, \qquad k,l \in \{1,...,n\} \tag{7d}$$
$$\left|P_{G_k}^{(r)} - P_{G_k}^{(0)}\right| \leq \Delta P_{k,\max}^{(r)}, \qquad k \in \{1,...,m\} \tag{7e}$$
$$\left|Q_{G_k}^{(r)} - Q_{G_k}^{(0)}\right| \leq \Delta Q_{k,\max}^{(r)}, \qquad k \in \{1,...,m\} \tag{7f}$$
$$\left|\left|V_k^{(r)}\right|^2 - \left|V_k^{(0)}\right|^2\right| \leq \left(\Delta V_{k,\max}^{(r)}\right)^2, \quad k \in \{1,...,n\} \tag{7g}$$

for every $t \in \{0,...,c\}$ and $r \in \{1,...,c\}$, where $\Delta P_{k,\max}^{(r)}$, $\Delta Q_{k,\max}^{(r)}$ and $\Delta V_{k,\max}^{(r)}$ are some given nonnegative numbers. Note that

- If $\Delta P_{k,\max}^{(r)}$ is zero, it implies that no corrective action is taken for the controllable parameter P_{G_k}. Furthermore, if $\Delta P_{k,\max}^{(r)}$ is infinity (so that the corresponding inequality can be removed from the SCOPF problem), it implies that P_{G_k} is either a non-controllable parameter or a controllable parameter with no ramp constraint. A similar remark can be made about Q_{G_k} and V_k.
- The formulation given in (6) and (7) is capable of modeling faults in both the transmission network and the generators. For instance, $Y^{(1)} \neq Y$ implies that some of the transmission lines are disconnected under the first contingency, while $Y^{(1)} = Y$ and $P_{1,\min}^{(1)} = P_{1,\max}^{(1)} = 0$ imply that the generator of bus 1 is removed under the first contingency.

The dual of the SCOPF problem can be derived based on the method presented in [7], which turns out to be a semidefinite program similar to Optimization 2. A question arises as whether a globally optimal solution of the non-convex SCOPF problem can be found by solving this semidefinite program. This question is answered in the next theorem.

Theorem 2: Assume that the duality gap is zero for every classical OPF problem associated with each of the configurations $Y^{(0)}, Y^{(1)}, ..., Y^{(c)}$ (i.e. $Y^{(0)}, ..., Y^{(c)} \in \mathcal{D}$). Then, the duality gap is zero for the SCOPF problem as well so that a globally optimal solution of this problem can be recovered from an optimal solution of its convex dual problem.

Proof: Consider the optimization problem of minimizing

$$\sum_{l=1}^{m} \left(\text{trace}\left\{\mathbf{Y}_l^{(0)} W^{(0)}\right\} + P_{D_l}\right) \tag{8}$$

over the positive semidefinite matrices $W^{(0)}, W^{(1)}, ..., W^{(c)}$ subject to

$$P_{k,\min}^{(t)} - P_{D_k} \leq \text{trace}\left\{\mathbf{Y}_k^{(t)} W^{(t)}\right\} \leq P_{k,\min}^{(t)} - P_{D_k} \tag{9a}$$
$$Q_{k,\min}^{(t)} - Q_{D_k} \leq \text{trace}\left\{\bar{\mathbf{Y}}_k^{(t)} W^{(t)}\right\} \leq Q_{k,\min}^{(t)} - Q_{D_k} \tag{9b}$$
$$\text{trace}\left\{\mathbf{Y}_{kl}^{(t)} W^{(t)}\right\}^2 + \text{trace}\left\{\bar{\mathbf{Y}}_{kl}^{(t)} W^{(t)}\right\}^2 \leq \left(S_{kl,\max}^{(t)}\right)^2 \tag{9c}$$
$$\left(V_{k,\min}^{(t)}\right)^2 \leq \text{trace}\left\{M_k W^{(t)}\right\} \leq \left(V_{k,\max}^{(t)}\right)^2 \tag{9d}$$
$$\left|\text{trace}\left\{\mathbf{Y}_k^{(r)} W^{(r)}\right\} - \text{trace}\left\{\mathbf{Y}_k^{(0)} W^{(0)}\right\}\right| \leq \Delta P_{k,\max}^{(r)} \tag{9e}$$
$$\left|\text{trace}\left\{\bar{\mathbf{Y}}_k^{(r)} W^{(r)}\right\} - \text{trace}\left\{\bar{\mathbf{Y}}_k^{(0)} W^{(0)}\right\}\right| \leq \Delta Q_{k,\max}^{(r)} \tag{9f}$$
$$\left|\text{trace}\left\{M_k W^{(r)}\right\} - \text{trace}\left\{M_k W^{(0)}\right\}\right| \leq \left(\Delta V_{k,\max}^{(r)}\right)^2 \tag{9g}$$

for every $k,l \in \{1,2,...,n\}$, $r \in \{1,2,...,c\}$ and $t \in \{0,1,...,c\}$ (as before, $P_{k,\min}^{(t)}$, $P_{k,\max}^{(t)}$, $Q_{k,\min}^{(t)}$ and $Q_{k,\max}^{(t)}$ are set to zero by convention if $k > m$). A diagram similar to the one depicted in Figure 1 for the OPF problem can be derived to deduce the following two properties:

i) The convex optimization given in (8) and (9) is the dual of the dual of the SCOPF problem specified in (6) and (7).
ii) The optimization given in (8) and (9) under the additional non-convex constraints rank$\{W^{(0)}\} = \cdots =$ rank$\{W^{(c)}\} = 1$ can be equivalently converted to the SCOPF problem via the change of variables $W^{(t)} = \mathbf{X}^{(t)} {\mathbf{X}^{(t)}}^T$, $t = 0,1,...,c$.

It can be concluded from Property (ii) that the SCOPF problem is infeasible if the optimization problem (8) and (9) is infeasible. Therefore, assume that the latter optimization problem is feasible. Using the above properties and in line with the argument made in [7], one can infer that the duality gap between the SCOPF problem and its dual is zero, provided the optimization problem given in (8) and (9) has a minimizer $(W_{\text{opt}}^{(0)}, W_{\text{opt}}^{(1)} ..., W_{\text{opt}}^{(c)})$ such that

$$\text{rank}\left\{W_{\text{opt}}^{(0)}\right\} = \text{rank}\left\{W_{\text{opt}}^{(1)}\right\} = \cdots = \text{rank}\left\{W_{\text{opt}}^{(c)}\right\} = 1.$$

To prove the existence of such a solution to the dual of the dual of the SCOPF problem, let $(W_{\text{opt}}^{(0)}, W_{\text{opt}}^{(1)} ..., W_{\text{opt}}^{(c)})$ be an arbitrary minimizer of this optimization problem. Consider a feasibility problem with the variable $W^{(c)}$ and the constraints $(\forall k,l \in \{1,...,n\})$

$$\text{trace}\left\{\mathbf{Y}_k^{(c)} W^{(c)}\right\} = \text{trace}\left\{\mathbf{Y}_k^{(c)} W_{\text{opt}}^{(c)}\right\} \tag{10a}$$
$$\text{trace}\left\{\bar{\mathbf{Y}}_k^{(c)} W^{(c)}\right\} = \text{trace}\left\{\bar{\mathbf{Y}}_k^{(c)} W_{\text{opt}}^{(c)}\right\} \tag{10b}$$
$$\text{trace}\left\{\mathbf{Y}_{kl}^{(c)} W^{(c)}\right\}^2 + \text{trace}\left\{\bar{\mathbf{Y}}_{kl}^{(c)} W^{(c)}\right\}^2 \leq \left(S_{kl,\max}^{(c)}\right)^2 \tag{10c}$$
$$\text{trace}\left\{M_k W^{(c)}\right\} = \text{trace}\left\{M_k W_{\text{opt}}^{(c)}\right\} \tag{10d}$$

Obviously, $W^{(c)} = W_{\text{opt}}^{(c)}$ is a solution to this feasibility problem (i.e. it satisfies the above constraints). In addition, it can be shown that $(W_{\text{opt}}^{(0)}, ..., W_{\text{opt}}^{(c-1)}, W_f)$ is a solution to the optimization given in (8) and (9) for some matrix W_f if $W^{(c)} = W_f$ is a solution to the feasibility problem (10). Now, the goal is to show that this feasibility problem has a rank-one solution W_f. To this end, convert this feasibility problem into an optimization problem by minimizing $\sum_{l=1}^{m}(\text{trace}\{\mathbf{Y}_l^{(c)}W^{(c)}\} + P_{D_l})$. The diagram given in Figure 1 yields that this optimization problem is the dual of the dual of the following OPF problem:

$$\begin{aligned} &\min \sum_{l=1}^{m} P_{G_l}^{(c)} \\ \text{s.t.} \quad & P_{k,\text{inj}}^{(c)} = \text{trace}\left\{\mathbf{Y}_k^{(c)}W_{\text{opt}}^{(c)}\right\}, \quad k \in \{1, ..., n\} \\ & Q_{k,\text{inj}}^{(c)} = \text{trace}\left\{\bar{\mathbf{Y}}_k^{(c)}W_{\text{opt}}^{(c)}\right\}, \quad k \in \{1, ..., n\} \\ & \left|V_k^{(c)}\right|^2 = \text{trace}\left\{M_k W_{\text{opt}}^{(c)}\right\}, \quad k \in \{1, ..., n\} \\ & \left|S_{kl}^{(c)}\right|^2 \leq \left(S_{kl,\max}^{(c)}\right)^2, \quad k, l \in \{1, ..., n\} \end{aligned}$$

Since the duality gap is zero for this OPF problem (due to the assumption $Y^{(c)} \in \mathcal{D}$), the dual of its dual has a rank-one solution (see Theorem 1 and the diagram given in Figure 1). In other words, there exists a rank-one matrix W_f such that $W^{(c)} = W_f$ satisfies the constraints given in (10). Following the discussion made earlier, this result simply implies that $W_{\text{opt}}^{(c)}$ can be taken as the rank-one matrix W_f. The same argument can be continued for other matrices $W_{\text{opt}}^{(0)}, ..., W_{\text{opt}}^{(c-1)}$ to conclude that the dual of the dual of the SCOPF problem has a solution $(W_{\text{opt}}^{(0)}, W_{\text{opt}}^{(1)} ..., W_{\text{opt}}^{(c)})$, where each of the matrices $W_{\text{opt}}^{(0)}, ..., W_{\text{opt}}^{(c)}$ has rank one. This completes the proof. ■

B. Optimization of Shunt Elements in Power Systems

A popular method towards a better steady-state control of a power system is to exploit variable reactive/capacitive shunt elements (e.g. capacitor banks or static VAR compensators) at some designated buses. To optimize these shunt parameters, they should be incorporated into the classical OPF problem. In order to formulate the underlying problem, assume that each bus $k \in \{1, 2, ..., n\}$ is equipped with a variable shunt element with the admittance $b_k\text{i}$, where b_k must lie between two given lower and upper bounds $b_{k,\min}$ and $b_{k,\max}$ (these bounds can take both positive and negative values). Note that if some bus k does not have such a shunt element, the bounds $b_{k,\min}$ and $b_{k,\max}$ are set to zero. As before, with no loss of generality, assume that the objective function to be minimized is the total generation. The elements $b_1, ..., b_n$ can be directly incorporated into the admittance matrix of the power system, which makes some of the elements of this matrix unknown and therefore adds another source of non-convexity to the OPF problem. Alternatively, one can use the fact that the shunt element of bus $k \in \{1, 2, ..., n\}$ injects no active power but the reactive power $b_k|V_k|^2$ to its corresponding bus. Hence, the resulting OPF problem with variable shunt elements can be obtained from the classical OPF problem by replacing the constraints

$$Q_{k,\min} - Q_{D_k} \leq Q_{k,\text{inj}} \leq Q_{k,\max} - Q_{D_k}, \quad k = 1, ..., n$$

with the new constraints

$$Q_{k,\min} - Q_{D_k} + b_k|V_k|^2 \leq Q_{k,\text{inj}} \tag{11a}$$
$$Q_{k,\text{inj}} \leq Q_{k,\max} - Q_{D_k} + b_k|V_k|^2 \tag{11b}$$
$$b_{k,\min} \leq b_k \leq b_{k,\max} \tag{11c}$$

where $b_1, ..., b_n$ are a part of the variables of the new optimization problem. Since $|V_k|$ is a nonnegative number, the change of variable $Q_{k,b} := b_k|V_k|^2$ equivalently converts the OPF problem with variable shunt elements into the following:

$$\begin{aligned} &\min \sum_{l=1}^{m} P_{G_l} \\ \text{s.t.} \quad & P_{k,\min} - P_{D_k} \leq P_{k,\text{inj}} \\ & P_{k,\text{inj}} \leq P_{k,\max} - P_{D_k} \\ & Q_{k,\min} - Q_{D_k} + Q_{k,b} \leq Q_{k,\text{inj}} \\ & Q_{k,\text{inj}} \leq Q_{k,\max} - Q_{D_k} + Q_{k,b} \\ & V_{k,\min} \leq |V_k| \leq V_{k,\max} \\ & |S_{kl}| \leq S_{kl,\max} \\ & b_{k,\min}|V_k|^2 \leq Q_{k,b} \leq b_{k,\max}|V_k|^2 \end{aligned} \tag{12}$$

$\forall k, l \in \{1, 2, ..., n\}$, where $Q_{1,b}, ..., Q_{n,b}$ are the extra variables of the optimization problem. In this subsection, we study this variant of the OPF problem with unknown shunt elements. The dual of this problem can be expressed as a semidefinite program. The next theorem proves that the solution of this dual problem can be used to find a solution to the original primal problem.

Theorem 3: Assume that the duality gap is zero for every classical OPF problem associated with the configuration Y (i.e. $Y \in \mathcal{D}$). Then, the duality gap is zero for the OPF problem with variable shunt elements as well.

Sketch of Proof: The dual of the dual of the OPF problem with variable shunt elements minimizes $\sum_{l=1}^{m}(\text{trace}\{\mathbf{Y}_l W\} + P_{D_l})$ over the positive semidefinite matrix W and the scalars $Q_{1,b}, ..., Q_{n,b}$ subject to

$$\begin{aligned} & P_{k,\min} - P_{D_k} \leq \text{trace}\{\mathbf{Y}_k W\} \leq P_{k,\min} - P_{D_k} \\ & Q_{k,\min} - Q_{D_k} + Q_{k,b} \leq \text{trace}\{\bar{\mathbf{Y}}_k W\} \\ & \text{trace}\{\bar{\mathbf{Y}}_k W\} \leq Q_{k,\min} - Q_{D_k} + Q_{k,b} \\ & \text{trace}\{\mathbf{Y}_{kl} W\}^2 + \text{trace}\{\bar{\mathbf{Y}}_{kl} W\}^2 \leq (S_{kl,\max})^2 \\ & (V_{k,\min})^2 \leq \text{trace}\{M_k W\} \leq (V_{k,\max})^2 \\ & b_{k,\min}\text{trace}\{M_k W\} \leq Q_{k,b} \leq b_{k,\max}\text{trace}\{M_k W\} \end{aligned}$$

$\forall k, l \in \{1, 2, ..., n\}$. Similar to the technique used in the proof of Theorem 2, it suffices to show that this optimization problem has a solution $(W_{\text{opt}}, Q_{1,b}^{\text{opt}}, ..., Q_{n,b}^{\text{opt}})$ such that $\text{rank}\{W_{\text{opt}}\} = 1$. To this end, given an arbitrary solution $(W_{\text{opt}}, Q_{1,b}^{\text{opt}}, ..., Q_{n,b}^{\text{opt}})$ to the above problem, consider the following optimization:

$$\begin{aligned}
\min_{W} \quad & \sum_{l=1}^{m} \text{trace}\left\{\mathbf{Y}_l W\right\} \\
\text{s.t.} \quad & \text{trace}\left\{\mathbf{Y}_k W\right\} = \text{trace}\left\{\mathbf{Y}_k W_{\text{opt}}\right\} \\
& \text{trace}\left\{\bar{\mathbf{Y}}_k W\right\} = \text{trace}\left\{\bar{\mathbf{Y}}_k W_{\text{opt}}\right\} \\
& \text{trace}\left\{\mathbf{Y}_{kl} W\right\}^2 + \text{trace}\left\{\bar{\mathbf{Y}}_{kl} W\right\}^2 \leq (S_{kl,\max})^2 \\
& \text{trace}\left\{M_k W\right\} = \text{trace}\left\{M_k W_{\text{opt}}\right\}
\end{aligned} \tag{13}$$

$\forall k, l \in \{1, 2, ..., n\}$. It can be observed that

i) $W = W_{\text{opt}}$ is a solution to the optimization (13).

ii) The feasibility region of the optimization (13) is a subset of the feasibility region of the dual of the dual of the OPF problem with variable shunt elements after fixing $Q_{k,b}$ as $Q_{k,b}^{\text{opt}}$, $k = 1, ..., n$.

These two properties imply that $(W_f, Q_{1,b}^{\text{opt}}, ..., Q_{n,b}^{\text{opt}})$ is a solution to the dual of the dual of the OPF problem with variable shunt elements for any arbitrary minimizer W_f of the optimization (13). On the other hand, the optimization (13) is the dual of the dual of some classical OPF problem with respect to the configuration Y. Hence, this optimization problem has a rank-one solution W_f. As a result, the minimizer W_{opt} can be taken as W_f. This completes the proof. ■

C. Optimization of Transformer Ratios in Power Systems

Practical power systems are often accompanied by a number of transformers whose (tap) ratios are controllable within certain limits. To optimize the performance of a power system, these ratios are often considered as some controllable parameters in the corresponding OPF problem. This subsection aims to study how the OPF problem with variable transformer ratios can be convexified using the duality theory. To this end, consider a transformer installed on some transmission line of the system. The most common method is to replace the transformer with a two-port Π block in order to be able to have an equivalent circuit model for the power system with only resistors, capacitors and inductors. However, if the transformer ratio is unknown, this parameter appears in a nonlinear way in the admittance matrix of the equivalent circuit model.

To bypass the foregoing issue, we exploit a different modeling method here. First, we replace every transformer with an ideal transformer and some lumped elements (accounting for the leakage reactance, series resistance, etc.). Then, we add some virtual buses to the set of the real (existing) buses in such a way that every ideal transformer is connected directly to two real/virtual buses (this may need defining a virtual bus for every transformer). For the sake of simplifying the presentation, we present the ideas for the case when there is only one tap-changing transformer in the system that connects bus 1 to bus 2. The generalization to multi-transformer case is straightforward.

Assume that bus 1 is connected to bus 2 via an ideal transformer. Let $P_{12} + Q_{12}\mathrm{i}$ denote the power transferred from bus 1 to the rest of the network through the transformer and η denote the transformer ratio bounded by the given nonnegative numbers $\eta_{\min}$ and $\eta_{\max}$. With a slight abuse of notation, define Y as the admittance matrix of the power system after removing the transformer (i.e. after disconnecting the line $(1, 2)$). By virtue of having no power loss in the transformer, one can write the power flow equations at buses 1 and 2 as follows:

$$\begin{aligned}
\text{trace}\left\{\mathbf{Y}_1 \mathbf{X}\mathbf{X}^T\right\} &= P_{1,\text{inj}} - P_{12} \\
\text{trace}\left\{\mathbf{Y}_2 \mathbf{X}\mathbf{X}^T\right\} &= P_{2,\text{inj}} + P_{12} \\
\text{trace}\left\{\bar{\mathbf{Y}}_1 \mathbf{X}\mathbf{X}^T\right\} &= Q_{1,\text{inj}} - Q_{12} \\
\text{trace}\left\{\bar{\mathbf{Y}}_2 \mathbf{X}\mathbf{X}^T\right\} &= Q_{2,\text{inj}} + Q_{12}
\end{aligned} \tag{14}$$

On the other hand, the voltages at the two ports of the transformer are related as

$$\text{Re}\{V_1\} = \eta \times \text{Re}\{V_2\} \tag{15a}$$

$$\text{Im}\{V_1\} = \eta \times \text{Im}\{V_2\} \tag{15b}$$

$$\eta_{\min} \leq \eta \leq \eta_{\max} \tag{15c}$$

In order to remove the nonlinearity caused by the product of η and the components of V_2, we eliminate the variable η. For this purpose, consider the relations

$$\eta_{\min}^2 |V_2|^2 \leq |V_1|^2 \leq \eta_{\max}^2 |V_2|^2 \tag{16a}$$

$$\text{Re}\{V_1\} \times \text{Im}\{V_2\} = \text{Re}\{V_2\} \times \text{Im}\{V_1\} \tag{16b}$$

$$\text{Re}\{V_1\} \times \text{Re}\{V_2\} \geq 0 \tag{16c}$$

$$\text{Im}\{V_1\} \times \text{Im}\{V_2\} \geq 0 \tag{16d}$$

It can be shown that the relations in (16) are satisfied if and only if there exists a nonnegative number η satisfying the relations in (15). Notice that all of the constraints given in (16) are quadratic in $\mathbf{V}$, which is a useful property for studying the duality gap. To formulate the OPF problem with the variable tap ratio η, the following actions should be taken:

- Write the power flow equations and physical limit constraints for every bus $k \in \{3, 4, ..., n\}$.
- Write the line flow constraints for all lines except for the line $(1, 2)$.
- Add the extra constraints given in (14) and (16), where P_{12} and Q_{12} are considered as scalar variables.
- Add the condition $P_{12}^2 + Q_{12}^2 \leq (S_{12,\max})^2$ associated with the flow constraint of the line $(1, 2)$.

It can be verified that the dual of this problem is a semidefinite program with the same structure as the dual of the classical OPF problem (partially due to the quadratic nature of the constraints in (16)). Now, one can write the dual of the dual of the OPF problem with the variable tap ratio η in terms of the matrix variable W and the scalar variables P_{12}, Q_{12}. In this optimization problem, the constraints corresponding to the ones given in (16) are

$$\eta_{\min}^2 \text{trace}\left\{M_2 W\right\} \leq \text{trace}\left\{M_1 W\right\}$$
$$\text{trace}\left\{M_1 W\right\} \leq \eta_{\max}^2 \text{trace}\left\{M_2 W\right\}$$

and

$$W_{1,n+2} = W_{2,n+1}, \quad W_{1,2} \geq 0, \quad W_{n+1,n+2} \geq 0 \tag{17}$$

where $W_{i,j}$ denotes the (i, j) entry of W for every $i, j \in \{1, 2, ..., 2n\}$. Now, it can be observed that the constraints corresponding to the unknown transformer ratio have appeared linearly in terms of the entries of W. If the conditions in (17) are removed from the dual of the dual of the OPF problem with the extra variable η, the technique used in the proof of Theorem 3 can be simply applied to this problem to show the existence of no duality gap. The removal of these two constraints corresponds to designing a complex-valued transformer ratio η such that $\eta_{\min} \leq |\eta| \leq \eta_{max}$. However, for the case when the ratio η is a real number (as considered in this work), the above-mentioned technique is not sufficient and, indeed, the long proof developed in [7] for the classical OPF problem should be followed closely. The details are omitted here for brevity.

IV. Simulation Results

Let the results of this paper be applied to the IEEE test systems with 14 and 30 buses. The specifications of these benchmark systems can be found in the library of the toolbox [13] and the online database [14].

The IEEE 30-bus system has 6 generators at buses 1, 2, 13, 22, 23 and 27. Assume that the controllable parameters of the system are the active powers supplied by the generators and the voltage magnitudes at the generator buses. If the classical OPF problem is solved to minimize the total generation (or equivalently the active power loss), the optimal values of the controllable parameters will be obtained as

$$\begin{aligned} &P_{G_1} = 7.69, \quad P_{G_2} = 48.57, \quad P_{G_{13}} = 40.00, \\ &P_{G_{22}} = 32.17, \quad P_{G_{23}} = 16.66, \quad P_{G_{27}} = 45.99, \\ &|V_1| = 1.028, \quad |V_2| = 1.027, \quad |V_{13}| = 1.090, \\ &|V_{22}| = 1.032, \quad |V_{23}| = 1.048, \quad |V_{27}| = 1.069 \end{aligned} \tag{18}$$

Suppose that while the controllable parameters of the power system are controlled continuously in order to be kept at their optimal values, a fault happens in the transmission line $(2, 6)$ leading to its disconnection. It can be shown that some of the line flow constraints will be violated in this case. To avoid this issue, one can solve an SCOPF problem to optimize the controllable parameters in such a way that the total generation is minimized and that the power flow and physical constraints are satisfied in the normal and contingency states. Due to the non-convexity of the SCOPF problem, this paper suggests solving the dual of the SCOPF problem, which is a semidefinite problem. The duality gap is zero for this problem and, therefore, a globally optimal solution to the SCOPF problem can be obtained as

$$\begin{aligned} &P_{G_1} = 12.66, \quad P_{G_2} = 43.06, \quad P_{G_{13}} = 40.00, \\ &P_{G_{22}} = 31.16, \quad P_{G_{23}} = 18.89, \quad P_{G_{27}} = 45.50, \\ &|V_1| = 1.031, \quad |V_2| = 1.030, \quad |V_{13}| = 1.094, \\ &|V_{22}| = 1.021, \quad |V_{23}| = 1.048, \quad |V_{27}| = 1.068 \end{aligned} \tag{19}$$

Now, consider the problem of the loss minimization for the IEEE 14-bus system, where the tap ratios of the transformers in the lines $(4, 7)$ and $(4, 9)$ are to be optimized as well. Assume that these unknown tap ratios must lie in the range $(0.8, 1.2)$. The duality gap for the OPF problem with these two variable tap ratios turns out to be zero, which makes it possible to globally optimize the parameters of the system. The optimal tap ratios for the transformers $(4, 7)$ and $(4, 9)$ are both equal to 0.9157. If the transformers are equipped with phase shifters, the optimal complex ratios of these transformers will be obtained as $0.9158 + 0.0066\mathrm{i}$ and $0.9157 - 0.0146\mathrm{i}$.

V. Conclusions

The classical optimal power flow (OPF) problem is one of the most fundamental optimization problems in power systems, which has been extensively studied in the past several years. Although the dual of the OPF problem is a semidefinite program that can be solved efficiently, the lack of strong duality might not allow for recovering a solution to the OPF problem. However, we showed in our recent work that the duality gap is zero for IEEE test systems and, more importantly, this gap is very likely to be zero for practical power systems due to the physical properties of a power network. The present paper shows how this result can be generalized to a great extent. More precisely, it is proved that zero duality gap for the classical OPF problem implies zero duality gap for more complicated power problems with other sources of non-convexity, such as variable shunt elements, variable transformer ratios and security constraints.

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