2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Nash Strategy for Stochastic Delay Systems

Hiroaki Mukaidani, Toru Yamamoto and Hua Xu

***Abstract*— This paper discusses Nash games for a class of delay systems governed by Itô's stochastic differential equation. Sufficient condition for the existence of Nash strategies is given by means of matrix inequality for the first time. It is shown that the state feedback strategy can be obtained by solving the linear matrix inequality (LMI) recursively.**

I. Introduction

Various engineering systems have the characteristics of time delay in signal transmissions such as communication systems, transmission systems, chemical processing systems, power systems, and so on. So far, the stability analysis and robust control for time-delay systems have been widely investigated over the past years. Particularly, the study that is based on linear matrix inequality (LMI) theory for a class of time-delay systems have received ever greater attention in the past two decades [1].

In the past few decades, stochastic systems governed by Itô-type stochastic differential equations have much received a great deal of research attention [2], [3]. Although a variety of results for the optimal control of linear stochastic systems have been reported, the dynamic games of such systems have received relatively little attention. Moreover, to the best of my knowledge, Nash games for delay stochastic systems have not been fully investigated. Since delays appear in many practical plant, the design of such strategy is an important issue that remains open.

In this paper, we discusses Nash games for a class of delay systems governed by Ito's differential equation. Sufficient condition for the existence of both Nash strategies and the upper bound of the cost is given by means of matrix inequality for the first time. Moreover, it is shown that the state feedback strategy can be obtained by solving the linear matrix inequality (LMI) recursively. As a result, since the LMI is solved independently, the cross-coupled matrix inequalities need not be treated. In order to demonstrate the usefulness, a simple numerical example is examined.

II. Problem Formulation

Throughout this paper, let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \boldsymbol{P})$ be a given filtered probability space. Consider linear stochastic time-delay systems.

$$dx(t) = \left[Ax(t) + A_h x(t-h) + \sum_{j=1}^{N} B_j u_j(t)\right]dt$$
$$+\sum_{p=1}^{M} A_p x(t) dw_p(t),\ x(t) = \phi(t),\ t \in [-h,\ 0], \quad (1)$$

where $x(t) \in \Re^n$ represent the state vectors. $u_j(t) \in \Re^{m_j}$, $j = 1,\ \ldots,N$ represent the j-th control inputs. $w_p(t) \in \Re$, $p = 1,\ \ldots,M$ is a one-dimensional standard Wiener process defined in the filtered probability space [2], [3]. Without loss of generality, it is assumed that $w_r(t)$ and $w_s(t)$ are mutually independent for all $r,\ s = 1,\ \ldots,M$ and $\mathbf{E}[w(t)w^T(t)] = I_M$, where $w(t) := \begin{bmatrix} w_1(t) & \cdots & w_M(t) \end{bmatrix}^T$ and $\mathbf{E}[\cdot]$ denotes the expection operator. Here, the scalar $h > 0$ is the time delay of the system. $\phi(t)$ is a real-valued initial function. The cost function for each strategy subset is defined by

$$\begin{aligned} &J_i(u_1,\ \ldots,u_N,\ x(0)) \\ &\quad = \mathbf{E}\left[\int_0^{\infty}[x^T(t)Q_i x(t) + u_i^T(t)R_{ii}u_i(t)]dt\right], \quad (2) \\ &\quad\quad i = 1,\ \ldots,N,\ Q_i = Q_i^T \geq 0,\ R_{ii} = R_{ii}^T > 0. \end{aligned}$$

The following stabilizability, which is an essential assumption, has been introduced in [3].

Definition 1: [3] A stochastic controlled system governed by the Itô differential equation $dx(t) = [Ax(t) + A_h x(t-h) + Bu(t)]dt + A_p x(t)dw_p(t)$ is considered stabilizable in the mean square sense if there exists a feedback law $u(t) = Kx(t)$ such that the closed- loop system $dx(t) = [(A + BK)x(t) + A_h x(t-h)]dt + A_p x(t)dw_p(t)$ is asymptotically stable in mean square (ASMS), i.e. its trajectories satisfy $\lim_{t\to\infty}\mathbf{E}[\|x(t)\|^2] = 0$ for any initial conditions $\phi(0)$.

Definition 2: [6] The zero solution of $dx(t) = [Ax(t) + A_h x(t-h) + Bu(t)]dt + A_p x(t)dw_p(t),\ x(t) = \phi(t)$, $-h \leq t \leq 0$ is said to be exponentially mean-square stable (EMSS) if there is a pair of positive constants α and β such that for any solution $x(t,\ \phi)$ with the initial condition ϕ, $\mathbf{E}[\|x(t,\ \phi)\|^2] \leq \beta\mathbf{E}[\|\phi\|^2]e^{-\alpha t}$, $t \geq 0$.

It is noteworthy that in this study, the strategies u_i^* are restricted as linear feedback strategies such as $u_i(t) := F_i x(t)$.

Let $\mathcal{F}_N$ denote the set of all $(F_1,\ \ldots,F_N)$ such that the following closed-loop stochastic system

$$dx(t) = \left[A + \sum_{j=1}^{N} B_j F_j\right]x(t)dt + A_h x(t-h)dt$$

H. Mukaidani is with Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima, 739-8524 Japan. mukaida@hiroshima-u.ac.jp

T. Yamamoto is with Div. of Elec. Sys. & Math. Eng, Faculty of Engineering Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hirohsima, 739-8527 Japan. yama@hiroshima-u.ac.jp.

H. Xu is with Graduate School of Business Sciences, The University of Tsukuba, 3-29-1, Otsuka, Bunkyo-ku, Tokyo, 112-0012 Japan. xu@mbaib.gsbs.tsukuba.ac.jp

$$+\sum_{p=1}^{M} A_p x(t) dw_p(t) \quad (3)$$

is asymptotically mean-square stable.

According to the feedback information structure, a set of equilibrium strategies should be independent of the initial state. Furthermore, the strategies should satisfy the usual equilibrium inequalities. A formal definition is given below.

Definition 3: [4] The strategy set $(u_1^*, \ldots, u_N^*)$, $u_i^*(t) := F_i^* x(t)$ is a stochastic Nash equilibrium strategy set if for each $i = 1, \ldots, N$, the following inequality holds:

$$\begin{aligned} &J_i(u_1^*, \ldots, u_N^*, x(0)) \\ &\le J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*, x(0)), \end{aligned} \quad (4)$$

for all $x(0)$ and for all $(F_1, \ldots, F_N)$ that satisfy $(F_1, \ldots, F_N) \in \mathcal{F}_N$.

Lemma 1: [2] The trivial solution of a stochastic differential equation is as follows:

$$dx(t) = f(t, x)dt + g(t, x)dw(t), \quad (5)$$

where $f(t, x)$ and $g(t, x)$ which are sufficiently differentiable maps, are exponentially mean-square stable if there exists a function $V(x(t))$, which satisfies the following inequalities:

$$a_1\|x(t)\|^2 \le V(x(t)) \le a_2\|x(t)\|^2, \ a_1, \ a_2 > 0, \quad (6a)$$

$$\begin{aligned} \mathcal{D}V(x(t)) &:= \frac{\partial V(x(t))}{\partial x} f(t, x) \\ &\quad + \frac{1}{2}\mathbf{Tr}\left[g^T(t, x)\frac{\partial^2 V(x(t))}{\partial x^2} g(t, x)\right] \\ &\le -c\|x(t)\|^2, \ c > 0 \end{aligned} \quad (6b)$$

for $x(t) \neq 0$.

The stochastic Nash games are given below.

Theorem 1: Assume that for all $u_i(t) = F_i x(t)$, $(F_1, \ldots, F_N) \in \mathcal{F}_N$, $i = 1, \ldots, N$ the closed-loop system is asymptotically mean-square stable. Suppose that N real symmetric matrices $P_i > 0$ and N real symmetric matrices $W_i > 0$ exist such that

$$\boldsymbol{F}_i(P_1, \ldots, P_N) := \begin{bmatrix} \Xi_i & P_i A_h \\ A_h^T P_i & -W_i \end{bmatrix} \le 0, \quad (7)$$

where $i = 1, \ldots, N$,

$$\Xi_i := P_i \boldsymbol{A}_{-i} + \boldsymbol{A}_{-i}^T P_i + \sum_{p=1}^{M} A_p^T P_i A_p - P_i S_i P_i + Q_i + W_i,$$

$$\boldsymbol{A}_{-i} := A - \sum_{j=1, \ j \neq i}^{N} S_j P_j, \ S_i := B_i R_{ii}^{-1} B_i^T.$$

Define the set $(F_1^*, \ldots, F_N^*)$ by

$$u_i^*(t) := F_i^* x(t) = -R_{ii}^{-1} B_i^T P_i x(t), \ i = 1, \ldots, N. \quad (8)$$

Then, $(F_1^*, \ldots, F_N^*) \in \mathcal{F}_N$, and this strategy set denotes the stochastic Nash equilibrium. Furthermore,

$$\begin{aligned} &J_i(F_1^* x, \ldots, F_N^* x, x(0)) \\ &\le \mathbf{E}[x^T(0) P_i x(0)] + \mathbf{E}\left[\int_{-h}^{0} \phi^T(\tau) W_i \phi(\tau) d\tau\right]. \end{aligned} \quad (9)$$

Proof: The proof can be demonstrated by using completion of squares. First, define the following quadratic function.

$$V_i(t) := x^T(t) P_i x(t) + \int_{t-h}^{t} x^T(\tau) W_i x(\tau) d\tau, \quad (10)$$

where $W_i = W_i^T > 0$.

Let us consider the following stochastic system with $u_j(t) := u_j^*(t)$, $j \neq i$.

$$\begin{aligned} dx(t) &= \left[\boldsymbol{A}_{-i} x(t) + A_h x(t-h) + B_i u_i(t)\right] dt \\ &\quad + \sum_{p=1}^{M} A_p x(t) dw_p(t), \ x(t) = \phi(t), \ t \in [-h, 0]. \end{aligned} \quad (11)$$

By using the Itô formula, the weak infinitesimal generator along with the stochastic system (11) can be obtained.

$$\begin{aligned} &\mathcal{D}[V_i(t)] + x^T(t) Q_i x(t) + u_i^T(t) R_{ii} u_i(t) \\ &:= x^T(t) \Xi_i x(t) + 2x^T(t) P_i A_h x(t-h) \\ &\quad + [u_i(t) + R_{ii}^{-1} B_i^T P_i x_i(t)]^T R_{ii} [u_i(t) + R_{ii}^{-1} B_i^T P_i x_i(t)] \\ &\quad - x^T(t-h) W_i x(t-h). \end{aligned} \quad (12)$$

According to the assumption that the closed-loop system is asymptotically mean-square stable, $\mathbf{E}[V_i(\infty)] = 0$. Thus, integrating both sides of the above equation and using $\mathbf{E}[V_i(\infty)] = 0$ results in

$$\begin{aligned} &J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*, x(0)) - \mathbf{E}[V_i(0)] \\ &= \mathbf{E}\left[\int_0^\infty \eta^T(t) \boldsymbol{F}_i(P_1, \ldots, P_N) \eta(t) dt\right] \\ &\quad + \mathbf{E}\left[\int_0^\infty [u_i(t) + R_{ii}^{-1} B_i^T P_i x_i(t)]^T \right. \\ &\qquad \left. \times R_{ii} [u_i(t) + R_{ii}^{-1} B_i^T P_i x_i(t)] dt\right] \\ &\ge \mathbf{E}\left[\int_0^\infty \eta^T(t) \boldsymbol{F}_i(P_1, \ldots, P_N) \eta(t) dt\right] \\ &= J_i(u_1^*, \ldots, u_N^*, x(0)) - \mathbf{E}[V_i(0)], \end{aligned} \quad (13)$$

where $\eta^T(t) := \begin{bmatrix} x^T(t) & x^T(t-h) \end{bmatrix}$.

Thus, the strategy set (8) satisfies the stochastic Nash equilibrium (4). On the other hand,

$$\begin{aligned} &J_i(u_1^*, \ldots, u_N^*, x(0)) - \mathbf{E}[V_i(0)] \\ &= \mathbf{E}\left[\int_0^\infty \eta^T(t) \boldsymbol{F}_i(P_1, \ldots, P_N) \eta(t) dt\right] \le 0. \end{aligned} \quad (14)$$

Thus, if (7) holds, then the desired result is obtained. ■

III. Numerical Algorithms

Nash strategy F_i of (8) can be obtained by solving the matrix inequalities (7). It should be noted that the matrix inequalities (7) are cross-coupled equations and it cannot be assessed by applying the LMI Control Toolbox with Matlab directly. We now propose a numerical approach for the matrix inequalities (7).

Let us consider the following new algorithm that is based on the optimization problems.

$$\min_{\mathcal{X}_i^{(n)}}\left(\mathbf{Tr}[x(0)x^T(0)P_i^{(n+1)}]+\mathbf{Tr}[MM^TW_i^{(n+1)}]\right),\ (15)$$
$$\mathcal{X}_i^{(n)}\in(P_i^{(n+1)},\ W_i^{(n+1)}),$$

where $MM^T:=\mathbf{E}\left[\int_{-h}^{0}\phi(\tau)\phi^T(\tau)d\tau\right]$ and

$$\begin{bmatrix} \Xi_i^{(n)} & P_i^{(n+1)}A_h & P_i^{(n+1)}B_i & I_n \\ A_h^TP_i^{(n+1)} & -W_i^{(n+1)} & 0 & 0 \\ B_i^TP_i^{(n+1)} & 0 & -R_{ii} & 0 \\ I_n & 0 & 0 & -Q_i^{-1} \end{bmatrix}\leq 0,\ (16)$$

$i=1,\ \dots,N$ with $\Xi_i^{(n)}:=P_i^{(n+1)}\boldsymbol{A}^{(n)}+\boldsymbol{A}^{(n)T}P_i^{(n+1)}+\sum_{p=1}^{M}A_p^TP_i^{(n+1)}A_p+W_i^{(n+1)}$, $\boldsymbol{A}^{(n)}:=A-\sum_{j=1}^{N}S_jP_j^{(n)}$, $S_i:=B_iR_{ii}^{-1}B_i^T$. Moreover, the matrices $P_i^{(0)}$, $i=1,\ \dots,N$ are chosen as the initial conditions such that the reduced-order closed-loop system is mean square stable.

The iterative procedure for solving the Semi-Definite Programming (SDP) is now summarized. The basic concept consists of subsequently solving each problem, repeating inequality (15). The algorithm is as follows:

Step 1. *Initialization: Set $P_i^{(0)}=I_n$ and $W_i^{(0)}=I_n$, for all i, $i=1,\ \dots,N$.*

Step 2. *For $i=1,\ \dots,N$ repeat the following steps: Solve the SDP problem, with respect to $\mathcal{X}_i^{(n)}$, subject to* (16).

Step 3. *If the algorithm converges, then $\mathcal{X}_i^{(n+1)}$ is the solution of SDP, STOP. Otherwise, increment $n\to n+1$ and go to Step 2, until all LMIs* (16) $i=1,\ \dots,N$ *are simultaneously satisfied.*

It should be noted that convergence of the above algorithm cannot be guaranteed [5]. However, we found the proposed algorithm to work well in practice.

IV. Numerical Example

In order to demonstrate the efficiency of our proposed control, we have run a simple numerical example. The system matrices are given as follows.

$$A=\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix},\ h=1,\ A_h=\begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$
$$p=1,\ A_1=\begin{bmatrix} 0 & 0.01 \\ 0.01 & 0 \end{bmatrix},$$
$$B_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2=\begin{bmatrix} 0 \\ 2 \end{bmatrix},\ Q_1=I_2,\ Q_2=\begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix},$$
$$R_{11}=1,\ R_{22}=2,\ \phi(t)=\begin{bmatrix} 1 \\ 2 \end{bmatrix},-1\leq t\leq 0.$$

By solving the corresponding optimization problem (15), we obtain the linear state feedback strategies

$$F_1=\begin{bmatrix} -7.6954e-01 & -5.5458e-02 \end{bmatrix},$$
$$F_2=\begin{bmatrix} 5.7913e-02 & -2.2602e-01 \end{bmatrix}.$$

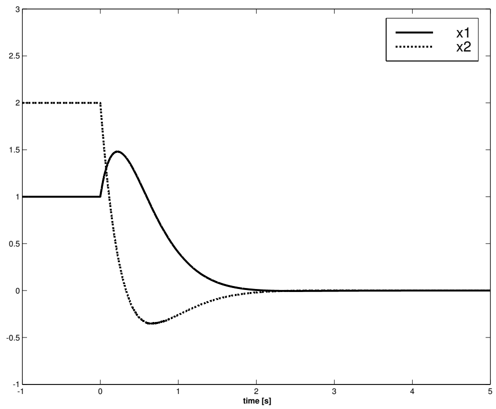

Fig. 1. Simulation results by means of the proposed strategies.

It may be noted that the proposed algorithm in section III converges to the exact solution with an accuracy of $\sum_{i=1}^{2}\boldsymbol{F}_i(P_1^{(n)},\ P_2^{(n)})<1.0e-10$ after 158 iterations.

Finally, the result of the simulation of this example is depicted in Fig. 1. It is shown from Fig. 1 that the closed-loop stochastic systems are asymptotically mean-square stable.

V. Conclusion

In this paper, Nash games for a class of delay systems governed by Itô's differential equation have been investigated. Sufficient condition for the existence of Nash strategies has been obtained by applying the matrix inequality for the first time. In order to calculate the strategy set, the new algorithm that is based on the LMI was considered. As a result, the exact Nash strategy can be easily computed because the numerical algorithm is decoupled from other optimization problems.

Finally, the convergence is not considered in this paper. Such proof is more important to guarantee the reliability. This problem will be addressed in future investigations.

References

[1] K. Gu, V. Kharitonov, and J. Chen, Stability of Time-Delay Systems, Birkhäuser : Boston, 2003.

[2] V. N. Afanas'ev, V. B. Kolmanowskii, V. R. Nosov, Mathematical Theory of Control Systems Design, Dordrecht : Kluwer Academic, 1996.

[3] B. S. Chen and W. Zhang, Stochastic H_2/H_∞ Control with State-Dependent Noise, *IEEE Trans. Automatic Control*, vol. 49, no. 1, 2004, pp 45-57.

[4] A. W. Starr and Y. C. Ho. Nonzero-sum Differential Games, *J. Optimization Theory and Application*, vol. 3, no. 3, 1969, pp 184-206.

[5] M. Jungers, E. B. Castelan, E. R. De Pieri and H. Abou-Kandil, Bounded Nash Type Controls for Uncertain Linear Systems, *Automatica*, vol. 44, no. 7, 2008, pp 1874-1879.

[6] L. Xua and D. Xu, Mean Square Exponential Stability of Impulsive Control Stochastic Systems with Time-Varying Delay, *Physics Letters A*, vol. 373, no.3, 2009, pp 328-333.