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Interconnection of Dirac structures via kernel/image representation

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Abstract— **Dirac structures are used to mathematically formalize the power-conserving interconnection structure of physical systems. For finite-dimensional systems several representations are available and it is known that the composition (or interconnection) of two Dirac structures is again a Dirac structure. It is also known that for infinite-dimensional systems the composition of two Dirac structures may not be a Dirac structure.**

In this paper, the theory of linear relations is used in the first instance to provide different representations of infinite-dimensional Dirac structures (on Hilbert spaces): an orthogonal decomposition, a scattering representation, a constructive kernel representation and an image representation. Some links between scattering and kernel/image representations of Dirac structures are also discussed. The Hilbert space setting is large enough from the point of view of the applications. Further, necessary and sufficient conditions (in terms of the scattering representation and in terms of kernel/image representations) for preserving the Dirac structure on Hilbert spaces under the composition (interconnection) are also presented. Complete proofs and illustrative example(s) will be included in a follow up paper.

I. INTRODUCTION

Port-based modeling leads to port-Hamiltonian systems which are defined with respect to a geometric structure, called Dirac structure (see [2] and the references therein). The study of power conserving interconnections of port-Hamiltonian systems makes use of the interconnection of Dirac structures (up to a certain extent). For finite-dimensional systems it is known that the composition or interconnection of two Dirac structures is again a Dirac structure. It is also known that for infinite-dimensional systems this is not always the case (see [8] for a counterexample). During the last years efforts have been made towards understanding of the Dirac structures and their composition for infinite-dimensional systems (see [8],[10],[9],[11],[16],[24],[25],[17],[27]). A mathematical formulation of the interconnection structure in Hilbert spaces and some properties have been presented for the first time in [10]. The reasons to study Dirac structures on Hilbert spaces was twofold. The Hilbert spaces are generally enough to cover a large class of physical systems and they offer enough tools for analyzing such systems. Necessary and sufficient conditions for the composition of two infinite-dimensional Dirac structures to be again a Dirac structure have been identified for the first time (independently) in [16] and [24] and the connection with the Redheffer star product has been also remarked.

The investigation of Dirac structures on Hilbert spaces and their composition has been continued in [17]. However, the very closed connection between Dirac structures and linear relations on Hilbert spaces deserves (in our opinion) more attention and this is the starting point of the analysis presented in this paper. After a preliminary section, the focus will be on different representations of Dirac structures on Hilbert spaces. It will be first shown in Section III that a Dirac structure can be orthogonally decomposed into three "elementary" Dirac structures. Using techniques from linear relations, a scattering representation (different from the one used in [17]) is then provided in Section IV. In Section V, kernel and image representations of Dirac structures on Hilbert spaces are derived; the kernel representation is constructive. Necessary and sufficient conditions for preserving the Dirac structures under composition are provided in Section VI: one set of two conditions in terms of the scattering representation and another set of two conditions in terms of the kernel/image representations. The paper ends with a concluding section.

II. PRELIMINARIES

A. Linear relations on Hilbert spaces

A linear relation from a Hilbert space $\mathscr{F}$ to a Hilbert space $\mathscr{E}$ is a linear subspace A of the Cartesian product $\mathscr{F}\times\mathscr{E}$. The following self-explanatory notions domain, range, kernel, and multi-valued part of A will be used throughout the paper:

$$\mathrm{dom}A = \{\, f \in \mathscr{F} : (f,e) \in A \,\},$$

$$\mathrm{ran}A = \{\, e \in \mathscr{E} : (f,e) \in A \,\},$$

$$\ker A = \{\, f \in \mathscr{F} : (f,0) \in A \,\},$$

$$\mathrm{mul}A = \{\, e \in \mathscr{E} : (0,e) \in A \,\}.$$

The formal inverse A^{-1} is defined as $A^{-1} = \{\,(e,f) : (f,e) \in A\,\}$; it is a linear relation from $\mathscr{E}$ to $\mathscr{F}$. Observe the following formal identities $\mathrm{dom}A^{-1} = \mathrm{ran}A$ and $\ker A^{-1} = \mathrm{mul}A$.

For relations A_1 and A_2 from $\mathscr{F}$ to $\mathscr{E}$, the operator-like sum $A_1 + A_2$ is the relation from $\mathscr{F}$ to $\mathscr{E}$ defined by

$$A_1 + A_2 = \{\,(f, e_1 + e_2) : (f,e_1) \in A_1, (f,e_2) \in A_2\,\},$$

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Now let A and B be linear relations from $\mathscr{F}$ to $\mathscr{E}$ and from $\mathscr{E}$ to $\mathscr{H}$, respectively. Then the product of B and A is the linear relation BA from $\mathscr{F}$ to $\mathscr{H}$ defined by

$$BA = \{ (f,h) \in \mathscr{F} \times \mathscr{H} : (f,e) \in A, \, (e,h) \in B \text{ for some } e \in \mathscr{E} \}.$$

This definition agrees with the usual one for operators.

The relation A is closed if it is closed as a subspace of $\mathscr{F} \times \mathscr{E}$; the closure of the relation A is the closure of the subspace A in $\mathscr{F} \times \mathscr{E}$. If A is closed then the subspaces $\ker A$ and $\mathrm{mul} A$ are closed. A linear relation A is the graph of an operator if and only if $\mathrm{mul} A = \{0\}$. In the present context a linear operator A from $\mathscr{F}$ to $\mathscr{E}$ is identified with its graph. It is said to be closable if its closure is the graph of an operator.

The adjoint of a linear relation A from $\mathscr{F}$ to $\mathscr{E}$ is the closed linear relation A^* from $\mathscr{E}$ to $\mathscr{F}$ defined by

$$A^* = \{ (e',f') \in \mathscr{E} \times \mathscr{F} : \langle f' \mid f \rangle_{\mathscr{F}} = \langle e' \mid e \rangle_{\mathscr{E}} \text{ for all } (f,e) \in A \}.$$

Observe that $(A^{-1})^* = (A^*)^{-1}$, so that $(\mathrm{dom} A)^{\perp} = \mathrm{mul} A^*$ and $(\mathrm{ran} A)^{\perp} = \ker A^*$. Clearly the double adjoint A^{**} is the closure of the relation A. A linear relation A in a Hilbert space $\mathscr{E}$ (i.e., from $\mathscr{E}$ to $\mathscr{E}$) is said to be skew-symmetric if $A \subset -A^*$, and a linear relation A in a Hilbert space $\mathscr{E}$ is said to be skew-adjoint if $A^* = -A$ (so that it is automatically closed).

B. Dirac structures on real vector spaces

Let $\mathscr{F}$ and $\mathscr{E}$ be real vector spaces whose elements are labeled as f and e, respectively. We call $\mathscr{F}$ the space of *flows* and $\mathscr{E}$ the space of *efforts*. The space $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ is called the *bond space* and an element of the space $\mathscr{B}$ is denoted by $b = (f,e)$. The spaces $\mathscr{F}$ and $\mathscr{E}$ are power conjugate. This means that there exists a map

$$\langle \cdot \mid \cdot \rangle : \mathscr{E} \times \mathscr{F} \to \mathbb{R}$$

called the *power product* such that it is linear in each coordinate and it is not degenerate.

Using the power product we define a symmetric *bilinear form*

$$\ll \cdot, \cdot \gg : \mathscr{B} \times \mathscr{B} \to \mathbb{R}$$

defined by

$$\ll (f^1,e^1),(f^2,e^2) \gg = \langle e^1 \mid f^2 \rangle + \langle e^2 \mid f^1 \rangle,$$

for all $(f^1,e^1),(f^2,e^2) \in \mathscr{B}$. We have the following immediate relation between the power product and the bilinear form

$$\langle e \mid f \rangle = \frac{1}{2} \ll b,b \gg$$

for all $b = (f,e) \in \mathscr{B}$.

We recall the notions of a *Tellegen structure* (known also as *power-conserving structure*).

Definition 2.1 (Tellegen structure): Let $\mathscr{Z}$ be a subspace of the vector space $\mathscr{B}$. We say that $\mathscr{Z}$ is a *Tellegen structure* on $\mathscr{B}$ if

$$\langle e \mid f \rangle = 0, \ \forall \, (f,e) \in \mathscr{Z}.$$

We denote $\mathscr{Z}^{[\perp]}$ the orthogonal complement of $\mathscr{Z}$ with respect to the bilinear form $\ll \cdot, \cdot \gg$

$$\mathscr{Z}^{[\perp]} := \{ b \in \mathscr{B} \mid \ll b, \tilde{b} \gg = 0, \, \forall \, \tilde{b} \in \mathscr{Z} \}.$$

Remark 2.1: Let $\mathscr{Z}$ be a subspace of the vector space $\mathscr{B}$. Then $\mathscr{Z}$ is a Tellegan structure on $\mathscr{B}$ if and only if $\mathscr{Z} \subseteq \mathscr{Z}^{[\perp]}$.

We focus on a special class of Tellegan structures called *Dirac structure*.

Definition 2.2 (Dirac structure): Let $\mathscr{D}$ be a subset of $\mathscr{B}$. We say that $\mathscr{D}$ is a Dirac structure on $\mathscr{B}$ if

$$\mathscr{D} = \mathscr{D}^{[\perp]}.$$

It is important to mention that for finite-dimensional spaces a Dirac structure is a Tellegan structure of maximal dimension.

C. Dirac structures on Hilbert spaces

In [9], [10], Dirac structures on Hilbert spaces were defined. For infinite-dimensional Hilbert spaces one can also approach the analysis of Dirac structures using Krein spaces which are not Pontryagin spaces. However, in this paper tools from operator theory and functional analysis will be used to study Dirac structures in Hilbert spaces.

Let $\mathscr{E}$ and $\mathscr{F}$ be two Hilbert spaces, called the space of efforts and the space of flows, respectively. Furthermore, assume that there exists a unitary operator $r_{\mathscr{E},\mathscr{F}}$ from $\mathscr{E}$ to $\mathscr{F}$. The product space $\mathscr{F} \times \mathscr{E}$ equipped with the usual Hilbert-space inner product:

$$\langle (f_1,e_1) \mid (f_2,e_2) \rangle_{\mathscr{F} \oplus \mathscr{E}} = \langle f_1 \mid f_2 \rangle_{\mathscr{F}} + \langle e_1 \mid e_2 \rangle_{\mathscr{E}}, \qquad (1)$$

where $f_1, \ f_2 \in \mathscr{F}, \ e_1, \ e_2 \in \mathscr{E}$ is called the Hilbert space $\mathscr{F} \oplus \mathscr{E}$.

Define a indefinite inner product on $\mathscr{F} \times \mathscr{E}$, by:

$$[(f_1,e_1) \mid (f_2,e_2)]_{\mathscr{B}} = \langle f_1, r_{\mathscr{E},\mathscr{F}} e_2 \rangle_{\mathscr{F}} + \langle e_1, r_{\mathscr{F},\mathscr{E}} f_2 \rangle_{\mathscr{E}}. \quad (2)$$

The Cartesian product $\mathscr{F} \times \mathscr{E}$ equipped with the inner product $[\cdot \mid \cdot]_{\mathscr{B}}$ is called the bond space $\mathscr{B}$. For a linear space $\mathscr{L} \subset \mathscr{B}$ the orthogonal complement $\mathscr{L}^{[\perp]}$ of $\mathscr{L}$ is defined by:

$$\mathscr{L}^{[\perp]} = \{ b' \in \mathscr{B}, [b \mid b']_{\mathscr{B}} = 0, \, \forall b \in \mathscr{L} \}. \qquad (3)$$

It is easily seen that for any linear subspace $\mathscr{L}$ of $\mathscr{B}$ one has:

$$\mathscr{L}^{[\perp]} = \begin{pmatrix} 0 & r_{\mathscr{E},\mathscr{F}} \\ r_{\mathscr{F},\mathscr{E}} & 0 \end{pmatrix} (\mathscr{L}^{\perp}), \qquad (4)$$

where $\mathscr{L}^{\perp}$ denotes the orthogonal complement of $\mathscr{L}$ with respect to the scalar product (1). Therefore any orthogonal complement will be closed, and $\mathscr{B}^{[\perp]} = \{0\}$. The last equality shows that the bond space is non-degenerate.

Definition 2.3: Let $\mathscr{E}$ and $\mathscr{F}$ be the spaces of efforts and flows, respectively, and let $\mathscr{B}$ the associated bond space and let $\mathscr{D}$ be a linear subspace of $\mathscr{B}$. $\mathscr{D}$ is said to be a Tellegen structure on $\mathscr{B}$ if $\mathscr{D} \subset \mathscr{D}^{[\perp]}$ and $\mathscr{D}$ is said to be a Dirac structure on $\mathscr{B}$ if $\mathscr{D} = \mathscr{D}^{[\perp]}$.

Lemma 2.1: Let $\mathscr{D}$ be a subspace of $\mathscr{B}$. Equivalent are:
(i) $\mathscr{D}$ is a Tellegen structure;

(ii) $[d_1, d_2]_{\mathscr{B}} = 0$ for all $d_1,\, d_2 \in \mathscr{D}$;

(iii) $[d,d]_{\mathscr{B}} = 0$ for all $d \in \mathscr{D}$.

Lemma 2.2: Let $\mathscr{D}$ be a Tellegen structure. Then:

(i) $\ker \mathscr{D} \subset \left(\operatorname{ran}(r_{\mathscr{E},\mathscr{F}}\mathscr{D})\right)^{\perp} = \ker (r_{\mathscr{E},\mathscr{F}}\mathscr{D})^*$;

(ii) $\operatorname{mul} \mathscr{D} \subset \left(r_{\mathscr{F},\mathscr{E}}(\operatorname{dom} \mathscr{D})\right)^{\perp}$.

Proof: (i) Let $f_1 \in \ker \mathscr{D}$, so that $(f_1, 0) \in \mathscr{D}$. Then $[(f_1,0) \mid (f_2,e_2)]_{\mathscr{B}} = 0$ for all $(f_2,e_2) \in \mathscr{D}$, so that $\langle f_1, r_{\mathscr{E},\mathscr{F}} e_2 \rangle_{\mathscr{F}} = 0$ for all $e_2 \in \operatorname{ran} \mathscr{D}$. This shows that $f_1 \perp \operatorname{ran}(r_{\mathscr{E},\mathscr{F}}\mathscr{D})$ and the inclusion $\ker \mathscr{D} \subset \left(\operatorname{ran}(r_{\mathscr{E},\mathscr{F}}\mathscr{D})\right)^{\perp}$ follows. The well known identity $(\operatorname{ran} T)^{\perp} = \ker T^*$ for any linear relation T (see for instance [1]) leads to the latter identity in (i).

(ii) Assume now that $e_1 \in \operatorname{mul} \mathscr{D}$ so that $(0, e_1) \in \mathscr{D}$. Then $\langle e_1 \mid r_{\mathscr{F},\mathscr{E}} f_2 \rangle_{\mathscr{E}} = 0$ for all $f_2 \in \operatorname{dom} \mathscr{D}$. Thus $e_1 \perp r_{\mathscr{F},\mathscr{E}}(\operatorname{dom} \mathscr{D})$, and the inclusion $\operatorname{mul} \mathscr{D} \subset \left(r_{\mathscr{F},\mathscr{E}}(\operatorname{dom} \mathscr{D})\right)^{\perp}$ follows. □

Lemma 2.3: Let $\mathscr{D}$ be a Dirac structure. Then the inclusions in Lemma 2.2 become equalities.

Lemma 2.4: Let $\mathscr{L}$ be a linear subspace of $\mathscr{B}$. Then:

$$\mathscr{L}^{[\perp]} = \begin{pmatrix} 0 & -r_{\mathscr{E},\mathscr{F}} \\ r_{\mathscr{F},\mathscr{E}} & 0 \end{pmatrix} (\mathscr{L}^*)^{-1}. \qquad (5)$$

Proof: It follows from the definition of the adjoint relation that

$$(\mathscr{L}^*)^{-1} = \begin{pmatrix} I_{\mathscr{F}} & 0 \\ 0 & -I_{\mathscr{E}} \end{pmatrix} (\mathscr{L}^{\perp}), \qquad (6)$$

so that

$$\mathscr{L}^{\perp} = \begin{pmatrix} I_{\mathscr{F}} & 0 \\ 0 & -I_{\mathscr{E}} \end{pmatrix} (\mathscr{L}^*)^{-1}. \qquad (7)$$

A combination of (4) and (7) leads to (5). □

Lemma 2.5: (i) $\mathscr{D}$ is a Tellegen structure if and only if

$$\mathscr{D} \subset \begin{pmatrix} 0 & -r_{\mathscr{E},\mathscr{F}} \\ r_{\mathscr{F},\mathscr{E}} & 0 \end{pmatrix} (\mathscr{D}^*)^{-1};$$

(i) $\mathscr{D}$ is a Dirac structure if and only if

$$\mathscr{D} = \begin{pmatrix} 0 & -r_{\mathscr{E},\mathscr{F}} \\ r_{\mathscr{F},\mathscr{E}} & 0 \end{pmatrix} (\mathscr{D}^*)^{-1}.$$

Remark 2.2: In the sense of the product of linear relations the conditions in Lemma 2.5 can be stated as follows:

a) $\mathscr{D}$ is a Tellegen structure if and only if $\mathscr{D} \subset -r_{\mathscr{F},\mathscr{E}} \mathscr{D}^* r_{\mathscr{F},\mathscr{E}}$;

b) $\mathscr{D}$ is a Dirac structure if and only if $\mathscr{D} = -r_{\mathscr{F},\mathscr{E}} \mathscr{D}^* r_{\mathscr{F},\mathscr{E}}$.

Remark 2.3: In the case $\mathscr{E} = \mathscr{F}$ a Tellegen structure is nothing but a skew-symmetric structure in the Hilbert space $\mathscr{E}$, while a Dirac structure is a skew-adjoint structure in the same Hilbert space $\mathscr{E}$.

III. An orthogonal decomposition of Dirac structures

It will be shown that any Dirac structure can be orthogonally decomposed into three "elementary" Dirac structures. The following example offers a class of Dirac structures.

Example 3.1: Let $\mathscr{E}$ be a Hilbert space and let A be a skew-adjoint (unbounded in general) operator from dom $A \subseteq \mathscr{E}$ to $\mathscr{E}$, that is

$$\langle Ax \mid y \rangle + \langle x \mid Ay \rangle = 0,$$

for all $x,\, y \in \operatorname{dom} A = \operatorname{dom} A^*$. Then the graph of A,

$$\mathscr{G}(A) = \{(x, Ax) : x \in \operatorname{dom} A\}$$

is a Dirac structure. Indeed, the definition of a skew-adjoint operator leads to

$$(\mathscr{G}(A))^{[\perp]} = \mathscr{G}(-A)^* = \mathscr{G}(A),$$

so that the conclusion follows.

Three classes of Dirac structures are introduced in the sequel

1) *Completely multivalued Dirac structures* which are of the form
$$\mathscr{D}_{mul} = \{(0,e) : e \in \mathscr{E}\};$$
2) *Completely kernel Dirac structures* which are of the form
$$\mathscr{D}_{ker} = \{(f,0) : f \in \mathscr{F}\};$$
3) *Completely skew-adjoint Dirac structures* which are determined by the graphs of injective skew-adjoint (not necessarily bounded) operators from $\mathscr{F}$ to $\mathscr{E}$.

It can be easily seen that the linear subspaces of type (1.) and type (2.) are Dirac structures, while Example (3.1) shows that the linear subspaces of type (3.) are Dirac structures as well. These particular Dirac structures are called fundamental Dirac structures. Under some conditions it can be shown that a Dirac structure can be decomposed as an orthogonal sum of the previous introduced fundamental Dirac structures. The idea of the construction of such decomposition is as follows. Define the linear subspace $\mathscr{D}_{mul} = \mathscr{D} \cap (\{0\} \times \mathscr{E})$ in $\mathscr{B}$ and the linear subspace $\mathscr{E}_{mul} = \{e \in \mathscr{E} : (0,e) \in \mathscr{D}\}$ in $\mathscr{E}$. Clearly, they are closed in $\mathscr{B}$ and in $\mathscr{E}$, respectively. Let $\mathscr{E}_1$ be the orthogonal complement of $\mathscr{E}_{mul}$ in $\mathscr{E}$, so that $\mathscr{E} = \mathscr{E}_{mul} \oplus \mathscr{E}_1$. Now define in $\mathscr{F}$ the linear subspace $\mathscr{F}_{mul}$ as

$$\mathscr{F}_{mul} = \{f \in \mathscr{F} : \langle r_{\mathscr{E},\mathscr{F}} e \mid f \rangle = 0, \forall e \in \mathscr{E}_1\},$$

and then it is easy to see that $\mathscr{F}_{mul}$ has an orthogonal complement $\mathscr{F}_1$ in $\mathscr{F}$. Assume now that $r_{\mathscr{E},\mathscr{F}}(\mathscr{E}_{mul}) = \mathscr{F}_{mul}$, so that $r_{\mathscr{E},\mathscr{F}}(\mathscr{E}_1) = \mathscr{F}_1$. Therefore $\mathscr{D}_{mul}$ is a completely multivalued Dirac structure on the bond space $\mathscr{B}_{mul} := \mathscr{F}_{mul} \times \mathscr{E}_{mul}$ and there exists a Dirac structure $\mathscr{D}_1$ on the bond space $\mathscr{B}_1 := \mathscr{F}_1 \times \mathscr{E}_1$ such that

$$\mathscr{D} = \mathscr{D}_{mul} \oplus \mathscr{D}_1.$$

Furthermore, the Dirac structure $\mathscr{D}_1$ is the graph of a skew-adjoint (not necessarily bounded) operator from the Hilbert space $\mathscr{F}_1$ to the Hilbert space $\mathscr{E}_1$. Define the linear subspace $\mathscr{D}_{ker} = \mathscr{D}_1 \cap (\mathscr{F}_1 \times \{0\})$ in $\mathscr{B}_1$ and the linear subspace $\mathscr{F}_{ker} = \{f \in \mathscr{F}_1 : (f,0) \in \mathscr{D}_1\}$ in $\mathscr{F}_1$. These subspaces are closed in $\mathscr{B}_1$ and in $\mathscr{F}_1$, respectively. Let $\mathscr{F}_{skew}$ be the orthogonal complement of $\mathscr{F}_{ker}$ in $\mathscr{F}_1$, so that $\mathscr{F}_1 = \mathscr{F}_{ker} \oplus \mathscr{F}_{skew}$. Now define in $\mathscr{E}_1$ the linear subspace $\mathscr{E}_{ker}$ as

$$\mathscr{E}_{ker} = \{e \in \mathscr{E}_1 : \langle r_{\mathscr{E},\mathscr{F}} e \mid f \rangle = 0, \forall f \in \mathscr{F}_{skew}\},$$

and then it follows that $\mathscr{E}_{ker}$ has an orthogonal complement $\mathscr{E}_{skew}$ in $\mathscr{E}_1$. Assume now that $r_{\mathscr{E},\mathscr{F}}(\mathscr{E}_{ker}) = \mathscr{F}_{ker}$, so that $r_{\mathscr{E},\mathscr{F}}(\mathscr{E}_{skew}) = \mathscr{F}_{skew}$. Then $\mathscr{D}_{ker}$ is a completely kernel Dirac structure on the bond space $\mathscr{B}_{ker} := \mathscr{F}_{ker} \times \mathscr{E}_{ker}$ and there exists a Dirac structure $\mathscr{D}_{skew}$ on the bond space $\mathscr{B}_{skew} := \mathscr{F}_{skew} \times \mathscr{E}_{skew}$ such that

$$\mathscr{D}_1 = \mathscr{D}_{ker} \oplus \mathscr{D}_{skew}.$$

Clearly, the Dirac structure $\mathscr{D}_{skew}$ is the graph of an closed injective skew-adjoint (not necessarily bounded) operator from the Hilbert space $\mathscr{F}_{skew}$ to the Hilbert space $\mathscr{E}_{skew}$. Conclude that under the assumptions imposed above, a Dirac structure can be written down as an orthogonal sum of three fundamental Dirac structures on the "smaller" bond Hilbert spaces $\mathscr{B}_{mul}$, $\mathscr{B}_{ker}$ and $\mathscr{B}_{skew}$, respectively. Moreover, this decomposition is given by

$$\mathscr{D} = \mathscr{D}_{mul} \oplus \mathscr{D}_{ker} \oplus \mathscr{D}_{skew},$$

and is comparable to the so called "constrained input-output representation" of a Dirac structure in finite-dimensional spaces, see [23].

IV. A SCATTERING REPRESENTATION OF DIRAC STRUCTURES

The scattering representation of Dirac structures for infinite dimensional spaces was basically introduced in [8] (see also [9]). For simplicity one considers the Hilbert space $\mathscr{E}$ to be the scattering variable space.

For any linear subspace $\mathscr{V}$ of $\mathscr{B}$ define the linear relation $\mathscr{O}$ in $\mathscr{E}$ by

$$\mathscr{O}_{\mathscr{V}} = I_{\mathscr{E}} - 2r_{\mathscr{F},\mathscr{E}}(\mathscr{V} + r_{\mathscr{F},\mathscr{E}})^{-1}. \qquad (8)$$

One can show that the following results hold.

Lemma 4.1: Let $\mathscr{D}$ be a Dirac structure on the Hilbert space $\mathscr{B}$. Then $\mathscr{O}_{\mathscr{D}}$ is a unitary operator in $\mathscr{E}$.

Lemma 4.2: Let $\mathscr{O}$ be a unitary operator in $\mathscr{E}$. Then the linear relation

$$\mathscr{D}_{\mathscr{O}} := \left\{ \left(\frac{1}{2} r^{-1}_{\mathscr{F},\mathscr{E}}(I_{\mathscr{E}} - \mathscr{O})e, \frac{1}{2}(I_{\mathscr{E}} + \mathscr{O})e \right) : e \in \mathscr{E} \right\} \qquad (9)$$

is a Dirac structure on $\mathscr{B}$.

Lemmas 4.1 and 4.2 lead to the following characterization of Dirac structures on Hilbert spaces.

Theorem 4.1: There exists a one-to-one correspondence between the class of Dirac structures on the Hilbert space $\mathscr{B}$ and the class of unitary operators in $\mathscr{E}$.

V. THE KERNEL / IMAGE REPRESENTATIONS OF DIRAC STRUCTURES

Let $\mathscr{H}$ be a Hilbert space isometrically isomorphic to $\mathscr{E}$ and $\mathscr{F}$. A Dirac structure $\mathscr{D}$ is said to be represented in *kernel representation* if

$$\mathscr{D} = \{(f,e) \in \mathscr{F} \times \mathscr{E}, Ff + Ee = 0\} \qquad (10)$$

for certain linear maps $F : \mathscr{F} \to \mathscr{H}$ and $E : \mathscr{E} \to \mathscr{H}$ satisfying the conditions

$$Fr_{\mathscr{E},\mathscr{F}}E^* + Er_{\mathscr{F},\mathscr{E}}F^* = 0, \qquad (11)$$

$$\overline{\operatorname{ran}[F\ E]} = \mathscr{H}, \qquad (12)$$

where $\overline{\operatorname{ran}[F\ E]}$ stands for the closure of the range of the operator $[F\ E]$. This definition agrees with the one in the finite-dimensional case (see for instance [2]). Next it will be proven that a kernel representation of a Dirac structure can always be done.

Lemma 5.1: Let $\mathscr{D}$ be a Dirac structure. Then the linear relations $(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}$ and $(\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}$ are the graphs of two bounded operators from $\mathscr{E}$ to $\mathscr{F}$.

Proof: Let $(e,f) \in (\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}$, so that $(f,e) \in (\mathscr{D} + r_{\mathscr{F},\mathscr{E}})$. Since $(f, r_{\mathscr{F},\mathscr{E}}f) \in r_{\mathscr{F},\mathscr{E}}$ one has $(f, e - r_{\mathscr{F},\mathscr{E}}f) \in \mathscr{D}$. Using Cauchy inequality one can obtain

$$\|f\| \le \|e\|, \qquad (13)$$

for all $(f,e) \in \mathscr{D}$. If $e = 0$ then $f = 0$ so that $(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}$ is the graph of an operator. Furthermore, the inequality (13) shows that the operator is bounded. Using similar arguments it can be shown that $(\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}$ is the graph of a bounded operator from $\mathscr{E}$ to $\mathscr{F}$. $\square$

Lemma 5.2: Let $\mathscr{D}$ be a Dirac structure. Then one has:

$$(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1} \cdot (\mathscr{D}^* + r_{\mathscr{E},\mathscr{F}})^{-1} = (\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1} \cdot (\mathscr{D}^* - r_{\mathscr{E},\mathscr{F}})^{-1}. \qquad (14)$$

Proof: The identity (14) is equivalent to

$$(\mathscr{D}^* + r_{\mathscr{E},\mathscr{F}}) \cdot (\mathscr{D} + r_{\mathscr{F},\mathscr{E}}) = (\mathscr{D}^* - r_{\mathscr{E},\mathscr{F}}) \cdot (\mathscr{D} - r_{\mathscr{F},\mathscr{E}}).$$

By a direct computation the last identity is equivalent to the identity $\mathscr{D}^* = -r_{\mathscr{E},\mathscr{F}}\mathscr{D}r_{\mathscr{E},\mathscr{F}}$, which expresses the fact that $\mathscr{D}$ is a Dirac structure. $\square$

A. A construction for the kernel representation

Assume that $\mathscr{D}$ is a Dirac structure, and let $(f,e) \in \mathscr{D}$. Since $(f, r_{\mathscr{F},\mathscr{E}}f) \in r_{\mathscr{F},\mathscr{E}}$ it follows that $(f, e + r_{\mathscr{F},\mathscr{E}}) \in \mathscr{D} + r_{\mathscr{F},\mathscr{E}}$, so that $(e + r_{\mathscr{F},\mathscr{E}}, f) \in (\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}$. This implies that $(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}(e + r_{\mathscr{F},\mathscr{E}}) = f$, so that

$$(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}e + (\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1}r_{\mathscr{F},\mathscr{E}}f = f. \qquad (15)$$

Similarly one gets:

$$(\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}e - (\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}r_{\mathscr{F},\mathscr{E}}f = f. \qquad (16)$$

Define now the operators E and F by:

$$E = (\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1} - (\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}, \qquad (17)$$

and by:

$$F = [(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1} + (\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1}]r_{\mathscr{F},\mathscr{E}}. \qquad (18)$$

Take $\mathscr{H} = \mathscr{F}$. Clearly, E and F are bounded everywhere defined linear operators $E \in [\mathscr{E}, \mathscr{F}]$ and $F \in [\mathscr{F}]$. Using the operators E and F, a combination of (15) and (16) leads to the representation (10).

In order to prove (11) one remarks that

$$E^* = (\mathscr{D}^* + r_{\mathscr{E},\mathscr{F}})^{-1} - (\mathscr{D}^* - r_{\mathscr{E},\mathscr{F}})^{-1},$$

and

$$F^* = r_{\mathscr{E},\mathscr{F}} \cdot [(\mathscr{D}^* + r_{\mathscr{E},\mathscr{F}})^{-1} + (\mathscr{D}^* - r_{\mathscr{E},\mathscr{F}})^{-1}].$$

Using the obvious identity $r_{\mathscr{E},\mathscr{F}} \cdot r_{\mathscr{F},\mathscr{E}} = I$ and (14), one gets (11) by a direct computation. Finally, one can show that (12) holds, and so the construction of the kernel representation has been done. We call this kernel representation the "*canonical kernel representation*" of $\mathscr{D}$.

B. The image representation of a Dirac structure

It follows from (10) and (11) that $\mathscr{D}$ can be also written in *image representation* as

$$\mathscr{D} = \{(f,e) \in \mathscr{F} \times \mathscr{E}, f = r_{\mathscr{E},\mathscr{F}} E^* h, e = r_{\mathscr{F},\mathscr{E}} F^* h, h \in \mathscr{H}\}$$

or equivalently:

$$\mathscr{D} = \{(r_{\mathscr{E},\mathscr{F}} E^* h, r_{\mathscr{F},\mathscr{E}} F^* h), h \in \mathscr{H}\}. \tag{19}$$

C. The links between scattering and kernel/image representations of Dirac structures

Assume that $\mathscr{D}$ is a Dirac structure and let $\mathscr{O}$ its canonical scattering operator, and (E,F) its canonical kernel/image representation. It follows from (8) that

$$(\mathscr{D} + r_{\mathscr{F},\mathscr{E}})^{-1} = \frac{1}{2} r_{\mathscr{E},\mathscr{F}} (I_{\mathscr{E}} - \mathscr{O}). \tag{20}$$

Simple computations leads to the following identity:

$$(\mathscr{D} - r_{\mathscr{F},\mathscr{E}})^{-1} = \frac{1}{2} r_{\mathscr{E},\mathscr{F}} (\mathscr{O}^{-1} - I_{\mathscr{E}}). \tag{21}$$

Using now (17) and (18) it is easily seen that:

$$E = \frac{1}{2} r_{\mathscr{E},\mathscr{F}} (2I_{\mathscr{E}} - \mathscr{O} - \mathscr{O}^{-1}), \tag{22}$$

while

$$F = \frac{1}{2} r_{\mathscr{E},\mathscr{F}} (\mathscr{O}^{-1} - \mathscr{O}) r_{\mathscr{F},\mathscr{E}}. \tag{23}$$

Conversely, a combination of (22) and (23) implies that

$$\mathscr{O} = I_{\mathscr{E}} - r_{\mathscr{F},\mathscr{E}} E - r_{\mathscr{F},\mathscr{E}} F r_{\mathscr{E},\mathscr{F}}, \tag{24}$$

and

$$\mathscr{O}^* = \mathscr{O}^{-1} = I_{\mathscr{E}} - r_{\mathscr{F},\mathscr{E}} E + r_{\mathscr{F},\mathscr{E}} F r_{\mathscr{E},\mathscr{F}}. \tag{25}$$

VI. Composition of Dirac structures via kernel/image representation

In this section the concept of the composition of Dirac structures is briefly studied via kernel/image representation. In order to define the composition, one needs two Dirac structures which have a joint pair of variables that can be used for interconnection. Hence one assumes that the efforts and flows of both Dirac structures can be split into a *proper* pair and a *joint* pair. More precisely, the following definition is used:

Definition 6.1: Assume that the spaces of efforts and flows are decomposed as $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$, $\mathscr{F} = \mathscr{H} = \mathscr{F}_1 \oplus \mathscr{F}_2$, and that there exist unitary mappings $r_{\mathscr{E}_i,\mathscr{F}_i}$ from $\mathscr{E}_i$ onto $\mathscr{F}_i$, $i = 1, 2$. A linear subspace $\mathscr{D} \subset \mathscr{B} = (\mathscr{F}_1 \oplus \mathscr{F}_2) \times (\mathscr{E}_1 \oplus \mathscr{E}_2)$ is called a split Tellegen structure (split Dirac structure) if it is a Tellegen structure (Dirac structure, respectively), with

$$r_{\mathscr{E},\mathscr{F}} = \begin{bmatrix} r_{\mathscr{E}_1,\mathscr{F}_1} & 0 \\ 0 & r_{\mathscr{E}_2,\mathscr{F}_2} \end{bmatrix}.$$

Furthermore, the way of composing two Dirac structures is given as follows:

Definition 6.2: Let $\mathscr{F}_i$ and $\mathscr{E}_i$, $i = 1, 2, 3$ be Hilbert spaces and let $\mathscr{D}^A \subset (\mathscr{F}_1 \oplus \mathscr{F}_2) \times (\mathscr{E}_1 \oplus \mathscr{E}_2)$ and $\mathscr{D}^B \subset (\mathscr{F}_2 \oplus \mathscr{F}_3) \times (\mathscr{E}_2 \oplus \mathscr{E}_3)$ be two split Tellegen or Dirac structures. Then the composition $\mathscr{D}^A o \mathscr{D}^B$ of $\mathscr{D}^A$ and $\mathscr{D}^B$ (through $\mathscr{F}_2 \times \mathscr{E}_2$) is defined by:

$$\begin{aligned} \mathscr{D}^A o \mathscr{D}^B \quad = \quad & \{(f_1, f_3, e_1, e_3) : \\ & (f_1, f_2, e_1, e_2) \in \mathscr{D}^A, (-f_2, f_3, e_2, e_3) \in \mathscr{D}^B\} \end{aligned} \tag{26}$$

If $\mathscr{D}^A$ and $\mathscr{D}^B$ in (26) are split Tellegen structures, then their composition is a split Tellegen structure with

$$r_{\mathscr{E},\mathscr{F}} = \begin{bmatrix} r_{\mathscr{E}_1,\mathscr{F}_1} & 0 \\ 0 & r_{\mathscr{E}_3,\mathscr{F}_3} \end{bmatrix}.$$

We refer to [25], [24], [16], [17] for further details. Consider now the corresponding scattering operators:

$$\mathscr{O}^A = \begin{bmatrix} \mathscr{O}^A_{11} & \mathscr{O}^A_{12} \\ \mathscr{O}^A_{21} & \mathscr{O}^A_{22} \end{bmatrix} : \mathscr{E}_1 \oplus \mathscr{E}_2 \to \mathscr{E}_1 \oplus \mathscr{E}_2,$$

and

$$\mathscr{O}^B = \begin{bmatrix} \mathscr{O}^B_{22} & \mathscr{O}^B_{23} \\ \mathscr{O}^A_{32} & \mathscr{O}^B_{33} \end{bmatrix} : \mathscr{E}_2 \oplus \mathscr{E}_3 \to \mathscr{E}_2 \oplus \mathscr{E}_3.$$

Consider also the kernel/image operators

$$E^A = \begin{bmatrix} E^A_{11} & E^A_{12} \\ E^A_{21} & E^A_{22} \end{bmatrix} : \mathscr{E}_1 \oplus \mathscr{E}_2 \to \mathscr{F}_1 \oplus \mathscr{F}_2,$$

and

$$F^A = \begin{bmatrix} F^A_{11} & F^A_{12} \\ F^A_{21} & F^A_{22} \end{bmatrix} : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \oplus \mathscr{F}_2.$$

corresponding to the Dirac structure $\mathscr{D}^A$, and the kernel/image operators

$$E^B = \begin{bmatrix} E^B_{22} & E^B_{23} \\ E^B_{32} & E^B_{33} \end{bmatrix} : \mathscr{E}_2 \oplus \mathscr{E}_3 \to \mathscr{F}_2 \oplus \mathscr{F}_3,$$

and

$$F^B = \begin{bmatrix} F^B_{22} & F^B_{23} \\ F^B_{32} & F^B_{33} \end{bmatrix} : \mathscr{F}_2 \oplus \mathscr{F}_3 \to \mathscr{F}_2 \oplus \mathscr{F}_3,$$

corresponding to the Dirac structure $\mathscr{D}^B$.

For unitary operators $\mathscr{O}^A$ and $\mathscr{O}^B$, denote by $\mathscr{O}^A \star \mathscr{O}^B$ the Redheffer star product of them (see for instance [26], [6]). One also needs the following bounded linear operators, defined on appropriate spaces:

$$(EF)^1_{23} = E^A_{22} r_{\mathscr{F}_2,\mathscr{E}_2} E^B_{23} + F^A_{22} E^B_{23} - E^B_{23},$$

$$(EF)^2_{23} = E^A_{22} r_{\mathscr{F}_2,\mathscr{E}_2} F^B_{23} + F^A_{22} F^B_{23} - F^B_{23},$$

$$(EF)^1_{21} = E^A_{21} + F^A_{21} r_{\mathscr{E}_1,\mathscr{F}_1}.$$

$$(EF)^1_{22} = E^A_{22} r_{\mathscr{F}_2,\mathscr{E}_2} E^B_{22} + F^A_{22} E^B_{22} - E^A_{22} - E^B_{22},$$

$$(EF)^2_{22} = E^A_{22} r_{\mathscr{F}_2,\mathscr{E}_2} E^B_{22} + F^A_{22} F^B_{22} - F^A_{22} - F^B_{22},$$

$$(EF)^{1(*)}_{12} = E^{B*}_{22} r_{\mathscr{E}_2,\mathscr{F}_2} E^{A*}_{12} + E^{B*}_{22} F^{A*}_{12} - E^{A*}_{12},$$

$$(EF)^{2(*)}_{12} = F^{B*}_{22} r_{\mathscr{E}_2,\mathscr{F}_2} E^{A*}_{12} + F^{B*}_{22} F^{A*}_{12},$$

$$(EF)^{1(*)}_{33} = E^{B*}_{33} + r_{\mathscr{F}_3,\mathscr{E}_3} F^{B*}_{33}.$$

The main result of this section is a complete characterization of the case when $\mathscr{D}^A o \mathscr{D}^B$ is a Dirac structure. More precisely, the following result holds true:

Theorem 6.1: Let $\mathscr{D}^A$ and $\mathscr{D}^B$ be split Dirac structures on $(\mathscr{F}_1 \oplus \mathscr{F}_2) \times (\mathscr{E}_1 \oplus \mathscr{E}_2)$ and on $(\mathscr{F}_2 \oplus \mathscr{F}_3) \times (\mathscr{E}_2 \oplus \mathscr{E}_3)$, respectively. Let $\mathscr{O}^A$ and $\mathscr{O}^B$ be corresponding scattering operators, and let (E^A, F^A) and (E^B, F^B) be the corresponding kernel/image operators. The following four statements are equivalent

(i) $\mathscr{D}^A o \mathscr{D}^B$ is a split Dirac structure on $(\mathscr{F}_1 \oplus \mathscr{F}_3) \times (\mathscr{E}_1 \oplus \mathscr{E}_3)$;

(ii) $\mathscr{O}^B * \mathscr{O}^A$ is a unitary operator in $\mathscr{E}_1 \oplus \mathscr{E}_3$;

(iii) The following two conditions hold true:

$$\operatorname{ran}\left[\mathscr{O}_{21}^A \mid \mathscr{O}_{22}^A \mathscr{O}_{23}^B\right] \subset \operatorname{ran}\left(\mathscr{O}_{22}^A \mathscr{O}_{22}^B - \mathscr{I}_{\mathscr{E}_2}\right)$$

and

$$\operatorname{ran}\left[\mathscr{O}_{22}^{B*} \mathscr{O}_{12}^{A*} \mid \mathscr{O}_{32}^{B*}\right] \subset \operatorname{ran}\left(\mathscr{O}_{22}^{B*} \mathscr{O}_{22}^{A*} - \mathscr{I}_{\mathscr{E}_2}\right)$$

(iv) The following two conditions hold true:

$$\operatorname{ran}\left[-r_{\mathscr{F}_2,\mathscr{E}_2}(EF)_{21}^1 \mid r_{\mathscr{F}_2,\mathscr{E}_2}\left((EF)_{23}^1 + (EF)_{23}^2 r_{\mathscr{E}_3,\mathscr{F}_3}\right)\right] \subset \operatorname{ran}\left(r_{\mathscr{F}_2,\mathscr{E}_2}\left((EF)_{22}^1 + (EF)_{22}^2 r_{\mathscr{F}_2,\mathscr{E}_2}\right)\right)$$

$$\begin{aligned}
&\operatorname{ran}\Big[\left((EF)_{12}^{1(*)} + r_{\mathscr{F}_2,\mathscr{E}_2}(EF)_{12}^{2(*)}\right) r_{\mathscr{E}_1,\mathscr{F}_1} \\
&\mid \mathscr{I}_{\mathscr{E}_3} - (EF)_{33}^{1(*)} r_{\mathscr{E}_3,\mathscr{F}_3}\Big] \\
&\subset \operatorname{ran}\left(\left((EF)_{22}^{1(*)} + r_{\mathscr{F}_2,\mathscr{E}_2}(EF)_{22}^{2(*)}\right) r_{\mathscr{E}_2,\mathscr{F}_2}\right)
\end{aligned}$$

Proof: Different arguments for the proof of the equivalence of (i), (ii) and (iii) can be found in [25], [24], [16], [17]. Although a direct proof can be done (it will be presented elsewhere), the equivalence of (i) and (iv) can be proved by rewriting the conditions in (iii) with the help of (24) and (25). □

VII. Conclusions

In this paper standard tools from linear relations have been used to derive different representations for Dirac structures on Hilbert spaces: an orthogonal decomposition, a scattering representation, a constructive kernel representation and an image representation. The Hilbert space setting is large enough from the point of view of the possible applications.

The composition for Dirac structures on Hilbert spaces has been also studied. It is known that for infinite-dimensional systems the composition of two Dirac structures may not be a Dirac structure ([8]). Necessary and sufficient conditions for preserving the Dirac structure under composition of systems have been presented. One set of conditions is in terms of kernel/image representations. The other set of conditions is using a certain scattering representation. The complete proofs and illustrative example(s) will be presented in a follow up paper.

References

[1] R. Arens, Operational calculus of linear relations. Pacific J. Math. Vol. 11(1), pp. 9–23, 1961.

[2] J. Cervera, A.J. van der Schaft, and A. Baños, Interconnection of port-Hamiltonian systems and composition of Dirac structures, Automatica 43, pp. 212–225, 2007.

[3] T. Courant, Dirac manifolds, Tr. Am. Math. Soc. 319, pp. 631–661, 1990.

[4] I. Dorfman, Dirac structures of integrable evolution equations, Phys. Lett. A 125, pp. 240–246, 1987.

[5] N. Dunford and J.T. Schwartz Linear Operators (Vol I-III). *Interscience Piblishers, Inc.*, New York, 1958.

[6] C. Foias and A.E. Frazho, Redheffer products and the lifting of contractions on Hilbert space, J. Operator Theory, 11, pp. 193–196, 1984.

[7] S. Goldberg Unbounded Linear Operators: Theory and Applications. *McGraw-Hill, Inc.*, New York, 1966.

[8] G. Golo. Interconnection Structures in Port-Based Modeling: Tools for Analysis and Simulation. *Ph.D. Thesis*, University of Twente, September 2002.

[9] G. Golo, O.V. Iftime, H. Zwart and A. van der Schaft, Tools for analysis of Dirac structures on Hilbert spaces. Memorandum no. 1729, Faculty of Applied Mathematics, University of Twente, July 2004.

[10] G. Golo, O.V. Iftime and A. van der Schaft, Interconnection Structures in Physical Systems: a Mathematical Formulation. In Proceedings of the 15th MTNS Conference Notre Dame University, USA. Editors D.S. Gilliam and J. Rosenthal, 2002.

[11] Y. Gorrec Le, H.J. Zwart and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *Memorandum Faculteit TW no. 1730*, Enschede, University of Twente, 32 pp., 2004.

[12] O. V. Iftime, A. Sandovici and G. Golo, Tools for analysis of Dirac Structures on Banach Spaces, Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference, pp. 3856-3861, ISBN 0-7803-9567-0, December 12-15, 2005, Sevilla, Spain.

[13] D. Karnopp, D. Margolis, R. Rosenberg, System Dynamics: Modeling and Simulation of Mechatronic System, John Wiley and Sons, Inc. New York, 2000.

[14] T. Kato. Perturbation Theory for Linear Operators. *Springer-Verlag*, Berlin Heidelberg New York, 1995.

[15] E. Kreyszig, Introductory functional analysis with applications, John Wiley and Sons, Inc. New York, 1978.

[16] M. Kurula, A. van der Schaft and H. Zwart, Composition of Infinite-Dimensional Linear Dirac-Type Structures, Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems, 2006.

[17] M. Kurula, H. J. Zwart, A. van der Schaft and J. Behrndt, Dirac structure and their composition on Hilbert spaces, Journal of Mathematical Analysis and Applications 372, pp. 402–422, 2010.

[18] H. Paynter, Analysis and Design of Engineering Systems, The M.I.T. Press, Cambridge, Massachusetts, 1961.

[19] A. van der Schaft, B. Maschke, The Hamiltonian formulation of energy conserving physical systems with external ports, Archiv fur Elektronik und Ubertragungstechnik 49, pp. 362–371, 1995.

[20] A. van der Schaft, B. Maschke, Modeling and Control of Mechanical Systems, Imperial College Press, 1997, Ch. Interconnected Mechanical Systems, part I: geometry of interconnection and implicit Hamiltonian systems, pp. 1–15.

[21] A. van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, Springer-Verlag, 2000.

[22] A. van der Schaft, B. Maschke, Hamiltonian formulation of distributed-parameter systems with boundary energy flow, Journal of Geometry and Physics 42, pp. 166–194, 2002.

[23] A. van der Schaft, The Mathematics of Systems and Control: from Intelligent Control to Behavioral Systems, University of Groningen, Ch. Interconnection and Geometry, pp. 203–217, 1999.

[24] A. Sandovici and O. V. Iftime, Dirac structures for infinite-dimensional systems: representations and composition, manuscript, 2005.

[25] A. Sandovici and O. V. Iftime, Composition of Dirac Structures for Infinite-Dimensional Systems, The 7th Worksop on Operator Theory in Krein Spaces, 13-16 December 2007, abstract.

[26] R. Redheffer, On Certain linear fractional transformations, Journal of Mathematics and Physics, Vol. 39, pp. 269–286, 1960.

[27] H. Yoshimura, H. Jacobs, J.E. Marsden, Interconnection of Dirac Structures and Lagrange-Dirac Dynamical Systems, In Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems, Budapest, Hungary, pp. 5–9, July 2010.