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Second Level Adaptation Using Multiple Models

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Abstract— **The concept of using multiple models to cope with transients which arise in adaptive systems with large parametric uncertainties was introduced in the 1990s. Both switching between multiple fixed models, and switching and tuning between fixed and adaptive models was proposed, and the stability of the resulting schemes was established. In all cases, the number of models needed is generally large, and the models used do not cooperate in any real sense.**

It was recently shown by the authors that if it is known a priori that the unknown plant parameter vector lies in the convex hull of a set of adaptive model parameter vectors at the initial time, it will remain in the convex hull of the parameters even as they evolve with time [1]. Later, a stability result was derived in [2] which decouples the stability and performance issues. In this paper, a new concept of second level adaptation is introduced to develop different stable strategies which improve the performance of the overall system. Simulation results are provided to illustrate the effectiveness of the proposed scheme in a rapidly time-varying environment, and are shown to be far superior to existing schemes.

I. Introduction

Adaptive control theory, dealing with the control of linear time-invariant systems with unknown parameters, has been studied since the 1960s, and an extensive literature currently exists in this area [3], [4], [5], [6], [7], [8]. It is now generally accepted that when parametric errors are small, classical adaptive control assures both stability and robustness.

When parametric errors are large, it has been observed over the years that the transient response of adaptive systems is oscillatory, and numerous efforts have been made to improve the performance in such cases. One such effort, involving multiple models, was introduced in the 1990s. During this period, both fixed models [9], [10] and fixed and adaptive models [11], [12] were proposed for improving the transient response. In [9], [10], a supervisor controller switches into feedback a sequence of linear set-point controllers from a family of candidate controllers. This causes the output of the controlled process to track a constant reference input, provided that the transfer function of the process is in the union of a number of subclasses, each of which is small enough so that one of the candidate controllers would solve the positioning problem. In contrast to the above, both fixed and adaptive models were used in [13], [11], [12] for the identification of the plant, and later for its control. Based on an index of performance, one of the models is chosen at any instant as the "best" and used at that instant to determine the control input. At the same time an adaptive model is initiated from the same point in parameter space, and the process is continued. The qualitative explanation provided is that the index of performance would, in general, choose the model "closest" to the plant in some sense, and consequently adaptation would commence from that model, resulting in improved performance. Switching to the closest model implies fast response in the adaptive context, and tuning from that model improves the system response on a slower time scale (i.e. incrementally as in classical adaptive control). Extensive simulation studies have demonstrated that the methods proposed perform satisfactorily when no limits are placed on the number of models that can be used [14], [15], [16], [17], [1], [18]. From a practical standpoint, the methods proposed above suffer from two major drawbacks. First, it is found that the number of models needed to assure that at least one of the fixed models is sufficiently close to the plant in parameter space is quite large and grows exponentially with the dimension of the unknown parameter vector. Second, the various models do not cooperate in any real sense to make the decision concerning the location of the unknown plant parameter vector. In particular, the performance indices of the different models are merely used to locate a model close to the plant. As a consequence, resources (i.e. the data available at the different models) are not used efficiently. In spite of these shortcomings, the methods are found to perform satisfactorily when the plant is time-invariant and the number of models that can be used is sufficiently large.

It is well known that the demands of a rapidly advancing technology are the prime movers of new theoretical advances. In numerous areas such as medicine, neuroscience, finance and national security, classes of problems are arising where decisions have to be made in the presence of large parametric uncertainty and ambient noise, or rapid variations in parameters. The adaptive methods that are currently available are generally inadequate to deal with such problems. The objective of this paper is to set up a general framework based on multiple models which can address the problems that arise in these contexts. We consider in this paper multiple models for the control of an unknown plant where the control action taken is based collectively on all the models so that resources are used efficiently. Finally, from a practical standpoint, very few models (comparable to the dimension of the unknown parameter vector) are needed for the identification process.

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II. Multiple Models for Adaptive Control

We now consider the adaptive control of a linear time-invariant (LTI) plant using multiple models, when the state variables of the plant are accessible. To facilitate the introduction of the principal concepts contained in this paper, we start our discussions with a relatively simple adaptive control problem whose solution can be found in any standard text on adaptive control. After stating the problem, we provide in quick succession the adaptive solution using a single identification model, the solution using a finite number N of adaptive models, and a discussion of the creation of an arbitrary number of virtual adaptive models for control purposes. Throughout the paper, for the sake of completeness, well known arguments in adaptive literature are included in the discussions, but details are omitted to conserve space.

A. Statement of the Adaptive Control Problem

An LTI plant Σ_p is described by the state equations

$$\Sigma_p : \qquad \dot{x}_p(t) = A_p x_p(t) + bu(t) \tag{1}$$

where $x_p(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}$. $A_p \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are in companion form. The elements of the last row of the matrix A_p are $[a_{p(1)}, a_{p(2)}, \dots, a_{p(n)}] = \theta_p^T$ and are assumed to be unknown. $b = [0, \dots, 0, 1]^T$. A reference model Σ_m is described by the differential equation

$$\Sigma_m : \qquad \dot{x}_m(t) = A_m x_m(t) + br(t) \tag{2}$$

where $r(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is a known bounded piecewise continuous reference signal. The matrix A_m is also in companion form, is stable, and has the last row θ_m^T. Assuming that $\theta_p \in \mathcal{S}_\theta$ where $\mathcal{S}_\theta$ is a compact set in parameter space, the objective is to determine the input $u(\cdot)$ to the plant such that all signals in the system are bounded and $\lim_{t\to\infty}[x_p(t) - x_m(t)] = 0$.

B. Single Model

Assuming that an indirect approach is used to control Σ_p, an identification model Σ_i is set up which is described by the differential equation

$$\Sigma_i : \qquad \dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m]x_p(t) + bu(t) \tag{3}$$

where $A_i(t)$ is a matrix in companion form, whose last row $\theta_i^T(t) = [a_{i(1)}(t), a_{i(2)}(t), \dots, a_{i(n)}(t)]$ (the estimate of the plant parameters) can be adjusted adaptively. Defining $\theta_i(t) - \theta_p = \tilde{\theta}_i(t)$ and $x_i(t) - x_p(t) = e_i(t)$, the error equation can be written as $\dot{e}_i(t) = A_m e_i(t) + b\tilde{\theta}_i^T(t)x_p(t)$. $e_i(t)$ will be referred to as the identification error and $\tilde{\theta}_i(t)$ as the parameter error.

Using a Lyapunov function candidate $V(e_i, \tilde{\theta}_i) = e_i^T P e_i + \tilde{\theta}_i^T \tilde{\theta}_i$ where P is the positive definite matrix solution of the Lyapunov equation $A_m^T P + PA_m = -Q$, $Q = Q^T > 0$, it follows directly from well known results in adaptive control that the adaptive law

$$\dot{\theta}_i(t) = -e_i^T(t)Pbx_p(t) \tag{4}$$

results in $\dot{V}(e_i, \tilde{\theta}_i) = -e_i^T Q e_i \le 0$. This assures the stability of the identifier and consequently the boundedness of both the identification error $e_i(t)$ and the parameter error $\tilde{\theta}_i(t)$ (and hence $\theta_i(t)$). To assure the stability of the plant, and hence the boundedness of $x_p(t)$, feedback control is used so that $u(t) = -k^T(t)x_p(t) + r(t)$ where $k(t) = \theta_i(t) - \theta_m$. This yields the (control) error equation $\dot{e}_c(t) = A_m e_c(t) + b\tilde{k}^T(t)x_p(t)$. Following the same arguments as before, it follows that $e_c \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ which assures the boundedness of $x_p(t)$ and $\dot{e}_c(t)$. From Barbalat's lemma it follows that $\lim_{t\to\infty} e_c(t) = 0$. Or the state $x_p(t)$ of the plant follows the state $x_m(t)$ of the reference model asymptotically.

C. Multiple Models

In adaptive control it is well known that the designer can use an arbitrary number of models to identify the plant, but only one controller to control it. It therefore follows that N identification models $\Sigma_1, \Sigma_2, \dots, \Sigma_N$ can be set up to provide N estimates of the parameter vector. The model Σ_i ($i \in \Omega = \{1, 2, \dots, N\}$) includes the parameter estimate $\theta_i(t)$ which can be updated adaptively, i.e.

$$\Sigma_i : \qquad \dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m]x_p(t) + bu(t) \tag{5}$$

with $x_i(t_0) = x_p(t_0)$.

Comment: The N adaptive models are consequently described by identical differential equations with the same initial state as the plant but with different initial values of the parameter vectors. The former condition is realizable since it is assumed that the plant states are accessible.

From the above assumptions it follows that the identification errors $e_i(t) = x_i(t) - x_p(t)$ satisfy the error differential equations

$$\dot{e}_i(t) = A_m e_i(t) + b\tilde{\theta}_i^T(t)x_p(t) \tag{6}$$

with $\theta_i(t_0) = \theta_{it_0}$ and $e_i(t_0) = 0$ $i \in \Omega$.

Assuming that the N models are operating in parallel, the question arises as to how the information obtained is to be used to control the system at every instant. This becomes particularly relevant when the plant is unstable. Using classical theory, any one of the estimates can be used to stabilize the system. In [13], [11], [12], it was suggested that different performance indices of the form $J_i(t) = \int_{t_0}^t \|e_i(\tau)\|^2 d\tau$ could be used to compare the different estimates and provide a basis for the choice of the control parameter vector. If one of the models is chosen as the "best" at any instant according to one of the criteria, it can, in turn, be used to select the controller parameter. It was shown in [12] that if an arbitrarily small but finite dwell time is used, switching between different parameters results in the stability of the overall system and the asymptotic convergence of the control error to zero. As in classical adaptive control, the parameter estimates need not converge to the plant parameter vector θ_p but do so if the reference input is persistently exciting.

Comment: The efficacy of the control depends upon how rapidly (and how accurately) the plant parameter can be estimated. When the number of models N is small and the region of uncertainty $\mathcal{S}_\theta$ is large, the improvement in the transient behavior of the system, over that realized using a single model, may not be significant. Additional properties of the multiple models need to be exploited, and this is treated in the following section.

D. Convex Hull Property

The following discussion concerns the convex hull in parameter space formed by the set of parameter vectors $\theta_i(t)$ $(i \in \Omega)$ corresponding to the N models Σ_i. In the previous subsection N estimates $\theta_1, \theta_2, \ldots, \theta_N$ of the plant parameter vector θ_p were generated. Since the plant Σ_p is linear, it follows that any convex combination of the estimates is also an estimate of θ_p so that $\theta_0(t) = \sum_{i=1}^N \beta_i \theta_i(t)$ can be considered as a virtual model, where β_i are nonnegative constant coefficients satisfying $\sum_{i=1}^N \beta_i = 1$. Further, $\tilde{\theta}_0(t) = \theta_0(t) - \theta_p = \sum_{i=1}^N \beta_i[\theta_i(t) - \theta_p] = \sum_{i=1}^N \beta_i \tilde{\theta}_i(t)$. More specifically, let an additional identification model Σ_0 be set up as

$$\Sigma_0 : \quad \dot{x}_0(t) = A_m x_0(t) + [A_0(t) - A_m]x_p(t) + bu(t) \quad (7)$$

with parameter vector $\theta_0(t)$. Let the initial condition of $\theta_0(t)$ be $\theta_0(t_0) = \sum_{i=1}^N \beta_i \theta_i(t_0)$. Further, let the adaptive law governing the adjustment of $\theta_0(t)$ be

$$\begin{aligned}\dot{\theta}_0(t) &= \sum_{i=1}^N \dot{\theta}_i(t)\beta_i = -\sum_{i=1}^N x_p(t) b^T P e_i(t) \beta_i \\ &= -x_p(t) b^T P \int_{t_0}^t \Phi(t,\tau) b \left[\sum_{i=1}^N \beta_i \tilde{\theta}_i^T(\tau)\right] x_p(\tau) d\tau \\ &= -x_p(t) b^T P e_0(t).\end{aligned} \quad (8)$$

The law is observed to be identical to that of the N real models. Therefore, any arbitrary convex combination of the N models can be considered as a virtual model with the same properties as the real models.

Comment: From the error equation (6) with initial conditions $e_i(t_0) = 0$, it follows that $e_i(t) = \int_{t_0}^t \Phi(t,\tau) b \tilde{\theta}_i^T(\tau) x_p(\tau) d\tau$ where $\Phi(t,\tau) = e^{A_m(t-\tau)}$ is the transition matrix of $\dot{e}_i = A_m e_i$.

The above discussion concerning virtual models defined as the convex combination of the N real models leads to the following theorem in this section, which was first introduced in [1]. Let the initial values of the parameter vectors $\theta_i(t_0)$ be chosen such that $\mathcal{S}_\theta \subset \mathcal{K}(t_0)$ where $\mathcal{K}(t_0)$ is the convex hull of the set $\{\theta_i(t_0)\}$, i.e. $\theta_p \in \mathcal{K}(t_0)$.

Comment: If $\theta_p \in \mathbb{R}^n$, $N = n+1$ is sufficient to satisfy the above condition. In practice N can be chosen to be greater than $(n+1)$ for convenience or efficiency.

Theorem 1: If N adaptive identification models described in (5) are adjusted using adaptive laws (4) with initial conditions $\theta_i(t_0)$ and initial states $x_i(t_0) = x_p(t_0)$, and if the plant parameter vector θ_p lies in the convex hull $\mathcal{K}(t_0)$ of $\theta_i(t_0)$ $(i \in \Omega)$, then θ_p lies in the convex hull $\mathcal{K}(t)$ of $\theta_i(t)$ $(i \in \Omega)$ for all $t \geq t_0$.

Proof: Since θ_p lies in the convex hull of $\theta_i(t_0)$ $(i \in \Omega)$, it follows that it satisfies the equation $\theta_p = \sum_{i=1}^N \alpha_i \theta_i(t_0), \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0$. Assuming that a virtual model $\theta_0(t)$ is initiated with $\theta_0(t_0) = \theta_p$, it follows that $\theta_0(t) = \theta_p$ for all $t \geq t_0$ if it is adjusted adaptively using (8). Therefore $\theta_p = \theta_0(t_0) = \theta_0(t) = \sum_{i=1}^N \alpha_i \theta_i(t)$ for all $t \geq t_0$. ■

From Theorem 1 it follows that if θ_p lies in the convex hull $\mathcal{K}(t_0)$, it also lies in the convex hull $\mathcal{K}(t)$. If it lies outside the convex hull, it remains outside the convex hull, and if it lies on the boundary of the convex hull, it will remain on the boundary of the convex hull, for all $t \geq t_0$. A crucial assumption made in deriving the above results is that all the identification models have initial conditions $x_i(t_0) = x_p(t_0)$ so that $e_i(t_0) = 0$. Since it was assumed that $x_p(t)$ is accessible, all models can be chosen to satisfy this condition.

III. Second Level Adaptation

Speed, accuracy, stability, and robustness are the features sought after in any efficient adaptive system. If the indirect approach is used, this depends upon the speed and accuracy with which the unknown plant parameter vector θ_p can be determined.

A. Second Level Adaptation

In the previous section it was shown that the plant parameter vector θ_p can be expressed as $\theta_p = \sum_{i=1}^N \alpha_i \theta_i(t_0) = \sum_{i=1}^N \alpha_i \theta_i(t) t \geq t_0$ for $\sum_{i=1}^N \alpha_i = 1$ and $\alpha_i \geq 0$.

This can be expressed in matrix form as

$$[\theta_1(t), \theta_2(t), \ldots, \theta_N(t)]\, \alpha = \Theta(t)\alpha = \theta_p \quad (9)$$

where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ and the columns of the $(n \times N)$ matrix $\Theta(t)$ are the parameter estimates $\theta_i(t)$. Equation (9) can be used to determine the vector α. It was also stated earlier that the minimum value that can be chosen for N (so that the convex hull of $\{\theta_i(t_0)\}$ $(i \in \Omega)$ contains $\mathcal{S}_\theta$) is $(n+1)$. In the following discussion, we use $N = n+1$ for computing the value of α.

In equation (9), the vector θ_p is unknown. Subtracting θ_p from both sides of the equation and noting that $\sum_{i=1}^{n+1} \alpha_i = 1$, we have the error equation

$$\tilde{\Theta}(t)\alpha = 0 \quad (10)$$

where the columns of $\tilde{\Theta}(t)$ are $\theta_i(t) - \theta_p = \tilde{\theta}_i(t)$.

Equation (10) cannot be used to determine α since $\tilde{\theta}_i(t)$ $(i \in \Omega)$ are unknown. However, more relevant for our purposes, and easier to implement is the derivative of equation (10) i.e.

$$\dot{\tilde{\Theta}}(t)\alpha = 0. \quad (11)$$

The columns $\dot{\hat{\theta}}_i$ represent the adaptive laws and are readily available from (4). Expressing $\alpha \in \mathbb{R}^{n+1}$ as $\alpha = [\bar{\alpha}^T, \alpha_{n+1}]^T$, where $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ it follows from the convexity condition that the scalar $\alpha_{n+1} = 1 - \sum_{i=1}^{n} \alpha_i$. This permits equation (11) to be written as

$$M(t)\bar{\alpha} = \ell(t) \tag{12}$$

where the i^{th} column of the matrix $M(t) \in \mathbb{R}^{n \times n}$ is $M_i(t) = \dot{\theta}_i(t) - \dot{\theta}_{n+1}(t)$ and the vector $\ell(t) \in \mathbb{R}^n$ is $\ell(t) = -\dot{\theta}_{n+1}(t)$.

Comment: It is worth noting that the matrix $M(t)$ is not obtained by differentiation but is directly obtained from the adaptive laws of the real models. It also underscores the dependence of $\bar{\alpha}$ on the rules by which the n parameters of the real models are adjusted, rather than on their values.

Equation (12) may be made the starting point of a second level adaptation method for estimating $\bar{\alpha}$. In such a case, an estimation model is set up as $M(t)\hat{\bar{\alpha}}(t) = \hat{\ell}(t)$, where $\hat{\bar{\alpha}}(t)$ is the estimate of $\bar{\alpha}$ and is adjusted using the adaptively law

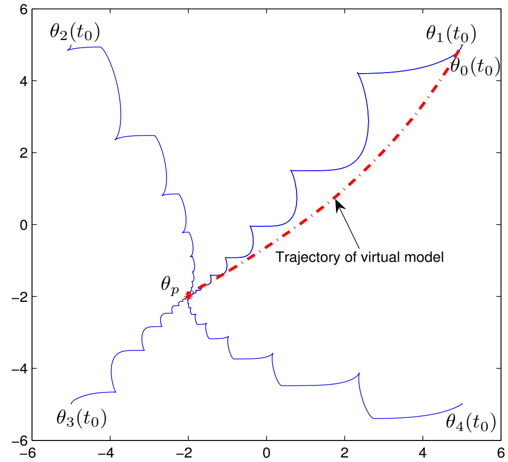
$$\dot{\hat{\bar{\alpha}}}(t) = -M^T(t)\tilde{\ell}(t) = -M^T(t)M(t)\hat{\bar{\alpha}}(t) + M^T(t)\ell(t). \tag{13}$$

It is known a priori that the unique equilibrium sate is the desired value $\bar{\alpha}$.

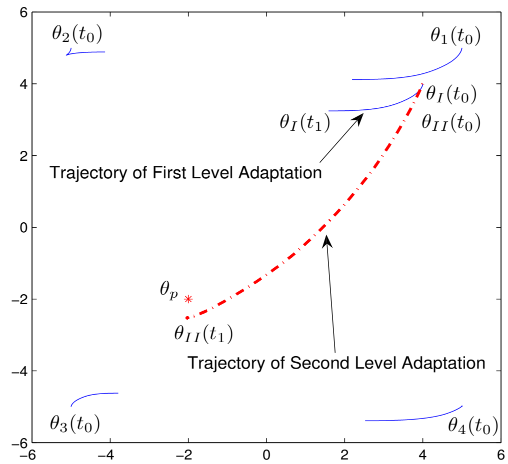
Comment: Since first level adaptation involves the real identification models and the estimates $\theta_i(t)$, we refer to the above as second level adaptation. The stability of such a procedure is discussed later in the paper.

Comment: By introducing the concepts of virtual model and second level adaptation, the identification of the unknown parameter vector of a dynamical system is converted into the identification of the unknown coefficient vector $\bar{\alpha}$.

Example 1: Figure 1 indicates the evolution of the parameter estimates (of θ_p) using both first level and second level adaptation. A stable plant θ_p was chosen to be $[-2, -2]^T$ and four real models $\theta_i(t)$ $(i = 1, 2, 3, 4)$ were used to estimate θ_p as described earlier. The initial location of the virtual model can be chosen anywhere in $\mathcal{K}(t_0)$, and was chosen to have two different values in Figures 1(a) and 1(b) i.e. $[5, 5]^T$ and $[4, 4]^T$ respectively. Second level adaptation was used, as described earlier, to estimate $\bar{\alpha}$ and hence the evolution of the virtual models. The trajectories of the real models are indicated in solid lines and those of virtual models in dotted lines. In Figure 1(a), trajectories of the virtual model $(\theta_0(t))$ and the real models $\theta_i(t)$ $(i = 1, 2, 3, 4)$ are plotted for 50 units of time. In Figure 1(b), an additional real model $\theta_I(t)$ is initialized at the same point as the virtual model $\theta_{II}(t)$ for comparison purposes. Trajectories in this figure are plotted for 5 units of time. In both cases the convergence of the virtual models are found to be faster and smoother than those of the real models. The analytical justification for these observations is discussed in the following section.



(a) Virtual Model Starting from the Boundary



(b) Comparison of First and Second Level Adaptation

Fig. 1: Trajectory of Second Level Adaptation

B. Stability

From the discussions in the previous section, any convex combination of the real models $\theta_i(t)$ $(i \in \Omega)$ is a virtual model. Depending on whether the convex combination is time-invariant or time-varying, we will refer to the virtual models as time-invariant virtual models or time-varying virtual models respectively. As shown earlier, a time-invariant virtual model has the same dynamical property as a real model.

As stated earlier, using results in [12], that switching between a finite number of real models (with a finite dwell time) does not affect the stability of the overall system. It directly follows that switching between a finite number of time-invariant virtual models (with a finite dwell time) will also be stable. Since the convex hull $\mathcal{K}(t)$ is a set of uncountably infinitely many time-invariant virtual models, the question naturally arises whether switching between this set of time-invariant virtual models will also be stable. Furthermore, when second level adaptation is used as in the previous section, a time-varying virtual model is produced, and the stability of such a model needs to be established. This question was discussed in [2] and a theoretical proof was provided there. In the following we include the main stability theorem in [2] for the sake of completeness.

Theorem 2: If the assumptions in Theorem 1 are satisfied, and the control signal is generated algebraically based on the plant parameter estimate

$$\bar{\theta}(t) = \text{Proj}_{\bar{\theta}(t)\in\mathcal{S}_\theta}\left\{\sum_{i=1}^{N}\alpha_i(t)\theta_i(t)\right\} \tag{14}$$

for nonnegative piecewise differentiable $\alpha_i(t)$ which satisfy the condition $\sum_{i=1}^{N}\alpha_i(t) = 1$, then the overall system is asymptotically stable.

Comment: From the above theorem, any convex combination of the N estimates results in a control parameter vector which assures stability. This decouples the stability and performance issues, and $\alpha_i(t)$ can be chosen primarily to improve performance.

IV. Simulation Studies

We conclude the paper by considering simulation studies on the adaptive control of an unstable second order system with rapidly time-varying unknown parameters. Three different methods are used for comparison, i.e. adaptive control using (i) switching [9], (ii) switching and tuning [12], and (iii) second level adaptation. In all cases measurement noise with standard deviation $\sigma = 0.1$ was added to illustrate the robustness properties of the methods. The unstable plant to be controlled is described by equation (1) where $n = 2$ and $\theta_p = [\theta_{p(1)}(t), \theta_{p(2)}(t)]^T$ is time-varying and unknown as shown in Figure 2. More specifically, three types of time-variations are included, i.e. small but rapid variation, large but slow variation, and piecewise constant variation.

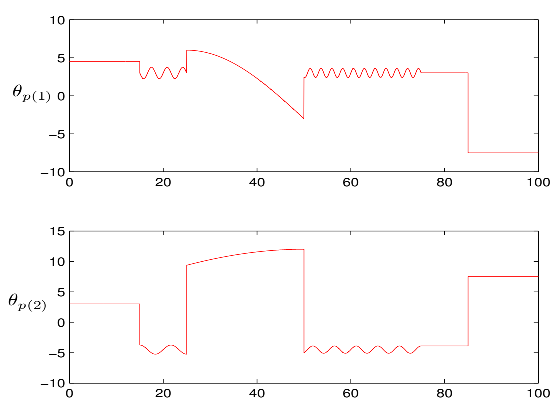


Fig. 2: Rapidly Time-Varying Plant Parameters

The reference model is stable with $\theta_m = [-6, -5]^T$ and has poles at -3 and -2. θ_p is known to belong to the set $\mathcal{S}_\theta$ where $\mathcal{S}_\theta = [-15, 15] \times [-15, 15] \in \mathbb{R}^2$. For convenience, the convex hull $\mathcal{K}(t_0)$ which contains $\mathcal{S}_\theta$ is chosen to be the same as $\mathcal{S}_\theta$, i.e. $\mathcal{K}(t_0) = \mathcal{S}_\theta$.

In method (i) 17 fixed models are uniformly distributed in $\mathcal{S}_\theta$, in method (ii) 9 fixed models are uniformly distributed in $\mathcal{S}_\theta$ and 1 re-initialized adaptive model and 1 free-running adaptive model are used, and in method (iii) only 3 adaptive models are used. In each case the desired state variable x_{m1} (the first element of the state) and the corresponding state x_{p1} of the plant are plotted together, and the output error is plotted separately on the same scale in Figure 3. The plant parameters and their corresponding estimates generated using each method are given in Figure 4. The responses in the last case are seen to be significantly better than the corresponding responses in the first two methods.

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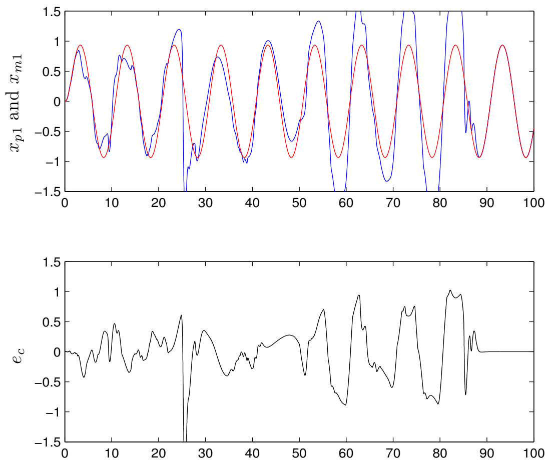
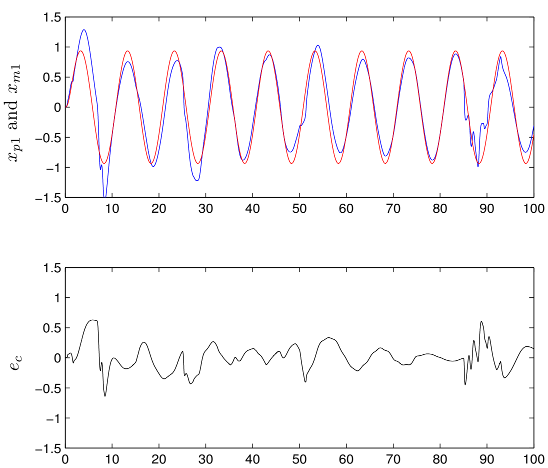
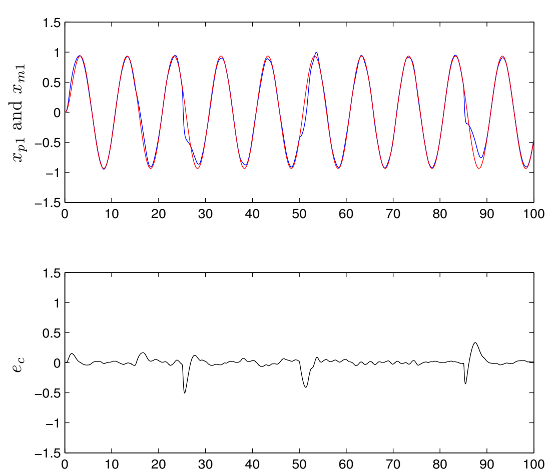


(a) Switching

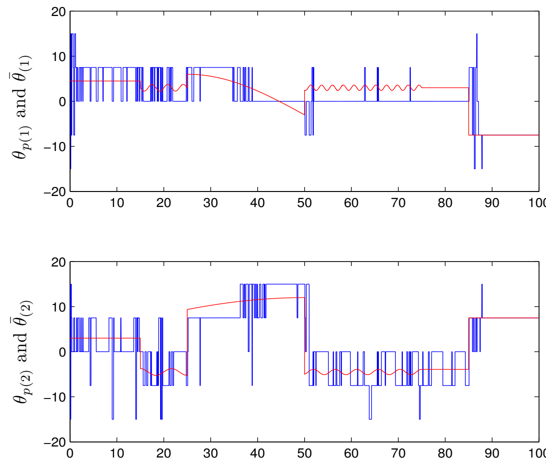


(b) Switching and Tuning

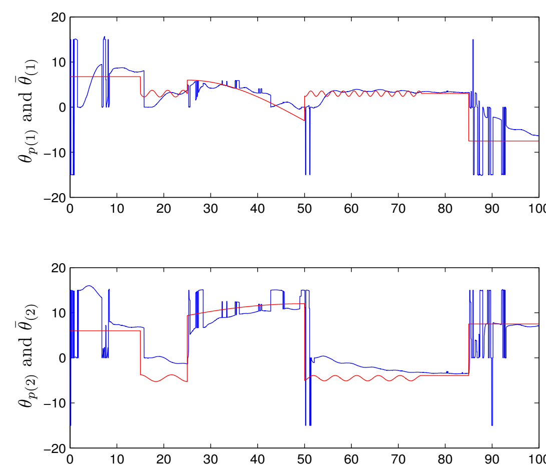


(c) Second Level Adaptation

Fig. 3: Reference Output, Plant Output and Control Error For The Three Methods



(a) Switching



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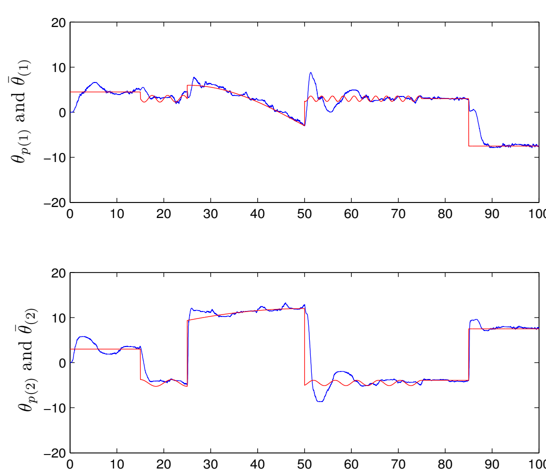

(c) Second Level Adaptation

Fig. 4: Plant Parameters and Corresponding Estimates For The Three Methods