2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Finite Horizon H^∞ Control for a Class of Linear Quantum Measurement Delayed Systems: A Dynamic Game Approach

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***Abstract*— In this paper, a finite horizon H^∞ control problem is solved for a class of linear quantum systems using a dynamic game approach for the case of delayed measurements. The methodology adopted involves an equivalence between the quantum problem and an auxiliary classical stochastic problem. Then, the finite horizon H^∞ control problem for the class of linear quantum systems under consideration is solved for the case of delayed measurements by solving the finite horizon H^∞ control problem for an equivalent stochastic H^∞ control problem using some results from a corresponding deterministic problem following a dynamic game approach.**

I. Introduction

Most work on tracking and filtering is built on the assumption that measurements are immediately available to an agent. However, it is not difficult to conceive of situations in which measurements are subject to non-negligible delays. The manner in which measurements are delayed, provides a fundamental distinction between different classes of such problems. The problem of constant delay involves every measurement being delayed by the same constant lag. In this way, measurements are never observed out of sequence: they are simply and consistently late. Such behavior could be induced, for example, by a constant bandwidth restriction in a sensor network. In contrast, random delays provide for a number of possibilities, including that measurements are delayed with a constant probability but a fixed lag, or a constant probability and a random lag. Such problems could arise as a result of intermittent bandwidth restrictions on a sensor network. All modes of random delay have the potential to cause out-of-sequence measurements.

This paper aims to extend the finite horizon H^∞ control problem developed in [1] to the case of delayed measurements. The methodology adopted involves an equivalence between the quantum problem and an auxiliary classical stochastic problem. Then, by solving the finite horizon H^∞ control problem for an equivalent classical stochastic system using results from a corresponding deterministic problem following a dynamic game approach, the finite horizon H^∞ control problem is solved for the case of delayed measurements. The proofs of all of the theorems and lemmas will be given in the journal version of this paper.

This work was completed with the support of a University of New South Wales Postgraduate Award and the Australian Research Council.

II. Problem Formulation

A. The Plant Model

We consider a class of linear quantum dynamical systems described in the Heisenberg picture by a set of quantum stochastic differential equations; see [2] and [3]. Also, we assume that the available information at time t is $y_{[0,t-\theta]} = y^{t-\theta}$ where $\theta > 0$ is a time delay. The system is described by the following continuous time-varying quantum stochastic differential equations (QSDEs) defined on the finite time interval $[0, t_f]$ and by the delayed time-varying quantum measurement equation for the measured output:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)du(t) + D_1(t)dw(t) \\ &\quad +G_v(t)dv(t); \\ dy(t-\theta) &= C(t-\theta)x(t-\theta)dt + N_1(t-\theta)dw_1(t-\theta) \\ &\quad +L(t-\theta)dv(t-\theta); \\ z(t) &= H(t)x(t) + G(t)\beta_u(t) + M(t)\beta_w(t); \end{aligned} \quad (1)$$

where

$$\begin{aligned} H(t)^T G(t) &= 0; H(t)^T H(t) = Q(t); \\ G(t)^T G(t) &= I; M(t) = 0. \end{aligned} \quad (2)$$

The initial system variables $x(0) = x_0$ consist of operators (on an appropriate Hilbert space) satisfying the commutation relations: $[x_j(0), x_k(0)] = 2i\Theta_{jk}$where Θ is a real antisymmetric matrix with components Θ_{jk}; see [2]. Also, we write $\phi_q(t) = x(t)$ for $0 \leq t < \theta$. Moreover, we assume that the state of the quantum system is Gaussian with mean $\check{x}_0 \in \mathbb{R}^n$ and covariance matrix Y_0; e.g., see [4]. Then $\langle x_0 \rangle = \check{x}_0$ and

$$Y_0 = \frac{1}{2}\left\langle (x_0 - \check{x}_0)(x_0 - \check{x}_0)^T + ((x_0 - \check{x}_0)(x_0 - \check{x}_0)^T)^T \right\rangle . \quad (3)$$

Here, $\langle . \rangle$ denotes quantum expectation; e.g., see [5]. In the sequel, we will fix Y_0 but $\check{x}_0$ will be taken as part of the disturbance. Also, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times n_u}$, $D_1(t) \in \mathbb{R}^{n\times n_w}$, $G_v(t) \in \mathbb{R}^{n\times n_v}$ and (n, n_w, n_u and n_v are positive integers) for all $t \in [0, t_f]$. Also, $x(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of self-adjoint possibly noncommutative system variables; e.g., see [2] for more details. Furthermore, $C(t) \in \mathbb{R}^{n_y\times n}$, $N_1(t) \in \mathbb{R}^{n_y\times n_{w_1}}$, $L(t) \in \mathbb{R}^{n_y\times n_v}$, $H(t) \in \mathbb{R}^{n_z\times n}$, $G(t) \in \mathbb{R}^{n_z\times n_u}$, $M(t) \in \mathbb{R}^{n_z\times n_w}$ and (n_y, n_z and n_{w_1} are positive integers) for all $t \in [0, t_f]$. The quantity $dw(t)$ represents the input variables or disturbances, $du(t)$ is the control input, $y(t-\theta)$ is the delayed classical measured output and $z(t)$ is the controlled output.

We assume that $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$ where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is assumed to be a square integrable classical disturbance signal. The set of all such $\beta_w(t)$ is denoted $\mathcal{W}$. Also, we assume that $dw_1(t) = \beta_{w_1}(t)dt + d\tilde{w}_1(t)$ where $\tilde{w}_1(t)$ is the noise part of $w_1(t)$ and $\beta_{w_1}(t)$ is assumed to be a square integrable classical disturbance signal. The set of all such $\beta_{w_1}(t)$ is denoted $\mathcal{W}_1$. The noise $\tilde{w}(t)$ is a vector of classical and quantum Wiener processes with Ito table $F_{\tilde{w}}$ and commutation matrix $T_{\tilde{w}}$ which are defined below. Also, the noise $\tilde{w}_1(t)$ is a vector of classical and quantum Wiener processes with Ito table $F_{\tilde{w}_1}$ and commutation matrix $T_{\tilde{w}_1}$ which are defined below. Similarly, we also assume that $du(t) = \beta_u(t)dt + d\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is a self-adjoint adapted process generated by a classical controller. The noise $\tilde{u}(t)$ is a vector of classical and quantum noises with Ito matrix $F_{\tilde{u}}$ and commutation matrix $T_{\tilde{u}}$. Also, the vector $dv(t)$ represents any additional noise in the plant. It has an Ito matrix F_v and commutation matrix T_v. We assume that $\beta_u(t) = 0$ for $0 \le t < \theta$. The non-negative symmetric Ito matrices $F_{\tilde{w}}$, $F_{\tilde{w}_1}$, $F_{\tilde{u}}$, F_v and the commutation matrices $T_{\tilde{w}}$, $T_{\tilde{w}_1}$, $T_{\tilde{u}}$ and $T_{\tilde{v}}$ are as defined in [1].

Note that (1) can also be rewritten as

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)\beta_u(t)dt + D(t)\check{\beta}_w(t)dt \\ &\quad + G_{v_t}(t)dv_t(t); \\ dy(t-\theta) &= C(t-\theta)x(t-\theta)dt + N(t-\theta)\check{\beta}_w(t-\theta)dt \\ &\quad + L_t(t-\theta)dv_t(t-\theta); \\ z(t) &= H(t)x(t) + G(t)\beta_u(t) + M(t)\beta_w(t); \end{aligned} \quad (4)$$

where $D(t) = \begin{bmatrix} D_1(t) & 0 \end{bmatrix}$, $N(t) = \begin{bmatrix} 0 & N_1(t) \end{bmatrix}$, $G_{v_t}(t) = \begin{bmatrix} B(t) & D(t) & G_v(t) \end{bmatrix}$, $dv_t(t) = \begin{bmatrix} d\tilde{u}(t) \\ d\tilde{w}(t) \\ d\tilde{w}_1(t) \\ dv(t) \end{bmatrix}$, $L_t(t) = \begin{bmatrix} 0 & N(t) & L(t) \end{bmatrix}$ and $\check{\beta}_w(t) = \begin{bmatrix} \beta_w(t) \\ \beta_{w_1}(t) \end{bmatrix}$. This implies $D(t)N(t)^T = 0$.

B. The Controller Model

We consider a classical controller $\mathcal{K}$ of the following form with a delay θ on the finite time interval $[0, t_f]$:

$$\begin{aligned} d\psi(t) &= F_c(t)\psi(t)dt + G_c(t)dy(t-\theta) \\ &\quad + \tilde{F}_c(t)\psi(t-\theta)dt; \\ \psi(0) &= \psi_0; \quad \psi(t) = \check{\phi}(t) \quad \text{for} \quad 0 \le t < \theta; \\ \beta_u(t) &= H_c(t)\psi(t) \quad \text{for} \quad t \ge \theta \end{aligned} \quad (5)$$

where $\psi(t)$ is the classical controller state. Here, $F_c(t) \in \mathbb{R}^{n_c \times n_c}$, $G_c(t) \in \mathbb{R}^{n_c \times n_y}$, $\tilde{F}_c(t) \in \mathbb{R}^{n_c \times n_c}$ and $H_c(t) \in \mathbb{R}^{n_u \times n_c}$ (n_c is a positive integer). The set of admissible controllers $\mathcal{K}$ will be denoted by $\mathfrak{M}$. These are controllers of the form (5) under which the closed-loop system defined by (1) and (5) has a solution for every $\hat{\beta}_w(.) = \left(\check{x}_0, \tilde{\beta}_w(t)\right) \in \mathbb{R}^n \times \mathcal{W} \times \mathcal{W}_1$ where $\tilde{\beta}_w(t) = \begin{bmatrix} \beta_w(t) \\ \beta_{w_1}(t-\theta) \end{bmatrix}$.

C. The Closed-Loop System

The closed-loop system is obtained by making the identification $\beta_u(t) = H_c(t)\psi(t)$ and interconnecting equations (1) and (5) to give a quantum-classical system described by the following quantum-classical stochastic differential equations

$$\begin{aligned} d\eta(t) &= \tilde{A}(t)\eta(t)dt + \tilde{A}_\theta(t)\eta(t-\theta)dt + \tilde{B}(t)d\tilde{u}(t) \\ &\quad + \tilde{D}(t)\tilde{\beta}_w(t)dt + \tilde{D}(t)d\check{w}(t) + \tilde{G}_v(t)d\tilde{v}(t); \\ \eta_0 &= \eta(0); \\ z(t) &= \tilde{H}(t)\eta(t) \end{aligned} \quad (6)$$

where $\eta(t) = \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix}$, $\tilde{A}(t) = \begin{bmatrix} A(t) & B(t)H_c(t) \\ 0 & F_c(t) \end{bmatrix}$, $\tilde{A}_\theta(t) = \begin{bmatrix} 0 & 0 \\ G_c(t)C(t-\theta) & \tilde{F}_c(t) \end{bmatrix}$, $\tilde{B}(t) = \begin{bmatrix} B(t) \\ 0 \end{bmatrix}$, $\tilde{D}(t) = \begin{bmatrix} D_1(t) & 0 \\ 0 & G_c(t)N_1(t-\theta) \end{bmatrix}$, $\tilde{\beta}_w(t) = \begin{bmatrix} \beta_w(t) \\ \beta_{w_1}(t-\theta) \end{bmatrix}$, $\tilde{G}_v(t) = \begin{bmatrix} G_v(t) & 0 \\ 0 & G_c(t)L(t-\theta) \end{bmatrix}$, $d\tilde{v}(t) = \begin{bmatrix} dv(t) \\ dv(t-\theta) \end{bmatrix}$, $d\check{w}(t) = \begin{bmatrix} d\tilde{w}(t) \\ d\tilde{w}_1(t-\theta) \end{bmatrix}$ and $\tilde{H}(t) = \begin{bmatrix} H(t) & G(t)H_c(t) \end{bmatrix}$. Such quantum stochastic differential equations with time delays have been used previously in the control literature, for instance see [6].

Let $\tilde{G}_{v_t}(t) = \begin{bmatrix} \tilde{B}(t) & \tilde{D}(t) & \tilde{G}_v(t) \end{bmatrix}$ and $d\tilde{v}_t(t) = \begin{bmatrix} d\tilde{u}(t) \\ d\check{w}(t) \\ d\tilde{v}(t) \end{bmatrix}$. Hence, the closed-loop system (6) becomes

$$\begin{aligned} d\eta(t) &= \tilde{A}(t)\eta(t)dt + \tilde{A}_\theta(t)\eta(t-\theta)dt + \tilde{D}(t)\tilde{\beta}_w(t)dt \\ &\quad + \tilde{G}_{v_t}(t)d\tilde{v}_t(t); \quad \eta_0 = \eta(0); \\ z(t) &= \tilde{H}(t)\eta(t). \end{aligned} \quad (7)$$

D. The cost function

We take the overall disturbance as $\hat{\beta}_w(t) = \left(\check{x}_0, \tilde{\beta}_w(t)\right)$. We therefore have to determine, whether, under the given measurement scheme, the upper value of the game

$$\inf_{\mathcal{K}\in\mathfrak{M}} \sup_{\hat{\beta}_w \in \mathbb{R}^n \times \mathcal{W} \times \mathcal{W}_1} L_\gamma(\mathcal{K}, \hat{\beta}_w)$$

with cost function

$$\begin{aligned} L_\gamma(\mathcal{K}, \hat{\beta}_w) &= \left\langle x(t_f)^T Q_f x(t_f) \right\rangle + \int_0^{t_f} \left\langle z(t)^T z(t) \right\rangle dt \\ &\quad - \gamma^2 \check{x}_0^T Q_0 \check{x}_0 - \gamma^2 \int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt \\ &= \left\langle x(t_f)^T Q_f x(t_f) \right\rangle \\ &\quad + \int_0^{t_f} \left\langle x(t)^T Q(t) x(t) \right\rangle dt \\ &\quad + \int_0^{t_f} \left\langle \beta_u(t)^T \beta_u(t) \right\rangle dt - \gamma^2 \check{x}_0^T Q_0 \check{x}_0 \\ &\quad - \gamma^2 \int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt \end{aligned} \quad (8)$$

is bounded, and to obtain a corresponding min-sup controller. Here, $Q_f = Q_f^T \ge 0$, Q_0 is a weighting matrix taken to be positive definite, $Q(t) = Q(t)^T \geq 0$ and $\langle . \rangle$ represents the quantum and classical expectation over all initial variables and noises; see [2], [3], [5]. The solution to this problem can be obtained by an extension of the method used in [1] for the continuous measurement case. Also, note that

$$\begin{aligned}\left\langle x(t)^T Q(t) x(t) \right\rangle + \left\langle \beta_u(t)^T \beta_u(t) \right\rangle &= \left\langle z(t)^T z(t) \right\rangle \\ &= \left\langle \eta(t)^T R(t) \eta(t) \right\rangle\end{aligned}$$

since (2) are satisfied and $\tilde{H}(t)^T \tilde{H}(t) = R(t) \geq 0$. Also,

$$\begin{aligned}\left\langle x(t_f)^T Q_f x(t_f) \right\rangle &= \left\langle \eta(t_f)^T \begin{bmatrix} Q_f^{\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} Q_f^{\frac{1}{2}} & 0 \end{bmatrix} \eta(t_f) \right\rangle \\ &= \left\langle \eta(t_f)^T \hat{Q}_f \eta(t) \right\rangle\end{aligned}$$

where $\hat{Q}_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. Similarly,

$$\check{x}_0^T Q_0 \check{x}_0 = \check{\eta}_0^T \begin{bmatrix} Q_0^{\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} Q_0^{\frac{1}{2}} & 0 \end{bmatrix} \check{\eta}_0 = \check{\eta}_0^T \hat{Q}_0 \check{\eta}_0$$

where $\hat{Q}_0 = \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ and $\check{\eta}_0 = \langle \eta_0 \rangle$.

Hence, the cost function (8) can be rewritten as

$$\begin{aligned}L_\gamma(\mathcal{K}, \hat{\beta}_w) &= \left\langle \eta(t_f)^T \hat{Q}_f \eta(t_f) \right\rangle + \int_0^{t_f} \langle \eta(t)^T R(t) \eta(t) \rangle dt \\ &\quad -\gamma^2 \left\{ \check{\eta}_0^T \hat{Q}_0 \check{\eta}_0 + \int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt \right\}. \quad (9)\end{aligned}$$

E. Explicit Expression for L_γ

For the quantum-classical closed-loop system (7), we define the covariance matrix P by

$$P(t) = \frac{1}{2} \left\langle \eta(t)\eta(t)^T + \left(\eta(t)\eta(t)^T\right)^T \right\rangle. \quad (10)$$

Then,

$$\begin{aligned}dP(t) &= \frac{1}{2} \left\{ \left\langle d\eta(t)\eta(t)^T \right\rangle + \left\langle \left(d\eta(t)\eta(t)^T\right)^T \right\rangle \right\} \\ &\quad + \frac{1}{2} \left\{ \left\langle \eta(t) d\eta(t)^T \right\rangle + \left\langle \left(\eta(t) d\eta(t)^T\right)^T \right\rangle \right\} \\ &\quad + \frac{1}{2} \left\{ \left\langle d\eta(t) d\eta(t)^T \right\rangle + \left\langle \left(d\eta(t) d\eta(t)^T\right)^T \right\rangle \right\}\end{aligned}$$

An expression for $dP(t)$ using the quantum Ito rule is

$$\begin{aligned}dP(t) &= \tilde{A}(t)P(t)dt + P(t)\tilde{A}(t)^T dt + \tilde{A}_\theta(t)P_1(t)dt \\ &\quad + P_2(t)\tilde{A}_\theta^T dt + \tilde{D}(t)\tilde{\beta}_w(t) \left\langle \eta(t)^T \right\rangle dt \\ &\quad + \langle \eta(t) \rangle \tilde{\beta}_w(t)^T \tilde{D}(t)^T dt + \tilde{B}(t) S_{\tilde{u}}(t) \tilde{B}(t)^T dt \\ &\quad + \tilde{D}(t) S_{\check{w}}(t) \tilde{D}(t)^T dt + \tilde{G}_v(t) S_{\check{v}}(t) \tilde{G}_v(t)^T dt \\ &= \tilde{A}(t)P(t)dt + P(t)\tilde{A}(t)^T dt + \tilde{A}_\theta(t)P_1(t)dt \\ &\quad + P_2(t)\tilde{A}_\theta^T dt + \tilde{D}(t)\tilde{\beta}_w(t) \left\langle \eta(t)^T \right\rangle dt \\ &\quad + \langle \eta(t) \rangle \tilde{\beta}_w(t)^T \tilde{D}(t)^T dt \\ &\quad + \tilde{G}_{v_t}(t) S_{\tilde{v}_t}(t) G_{v_t}(t)^T dt \quad (11)\end{aligned}$$

where

$$\begin{aligned}P_1(t) &= \frac{1}{2} \left\langle \eta(t-\theta)\eta(t)^T + \left(\eta(t)\eta(t-\theta)^T\right)^T \right\rangle; \\ P_2(t) &= \frac{1}{2} \left\langle \eta(t)\eta(t-\theta)^T + \left(\eta(t-\theta)\eta(t)^T\right)^T \right\rangle; \\ S_{\tilde{u}}(t)dt &= \frac{1}{2} \left\langle d\tilde{u}(t)d\tilde{u}(t)^T + \left(d\tilde{u}(t)d\tilde{u}(t)^T\right)^T \right\rangle; \\ S_{\check{w}}(t)dt &= \frac{1}{2} \left\langle d\check{w}(t)d\check{w}(t)^T + \left(d\check{w}(t)d\check{w}(t)^T\right)^T \right\rangle; \\ S_{\check{v}}(t)dt &= \frac{1}{2} \left\langle d\check{v}(t)d\check{v}(t)^T + \left(d\check{v}(t)d\check{v}(t)^T\right)^T \right\rangle; \\ S_{\tilde{v}_t}(t)dt &= \frac{1}{2} \left\langle d\tilde{v}_t(t)d\tilde{v}_t(t)^T + \left(d\tilde{v}_t(t)d\tilde{v}_t(t)^T\right)^T \right\rangle.\end{aligned}$$

Note that $S_{\tilde{v}_t}(t)dt = \begin{bmatrix} S_{\tilde{u}}(t)dt & 0 & 0 \\ 0 & S_{\check{w}}(t)dt & 0 \\ 0 & 0 & S_{\check{v}}(t)dt \end{bmatrix}$.

Also, we define

$$\begin{aligned}S_{\tilde{w}}(t)dt &= \frac{1}{2} \left\langle d\tilde{w}(t)d\tilde{w}(t)^T + \left(d\tilde{w}(t)d\tilde{w}(t)^T\right)^T \right\rangle; \\ S_{\tilde{w}_1}(t)dt &= \frac{1}{2} \left\langle d\tilde{w}_1(t)d\tilde{w}_1(t)^T + \left(d\tilde{w}_1(t)d\tilde{w}_1(t)^T\right)^T \right\rangle; \\ S_v(t)dt &= \frac{1}{2} \left\langle dv(t)dv(t)^T + \left(dv(t)dv(t)^T\right)^T \right\rangle; \\ \hat{S}_{\tilde{w}}(t)dt &= \frac{1}{2} \left\langle d\tilde{w}(t)d\tilde{w}_1(t-\theta)^T \right. \\ &\quad \left. + \left(d\tilde{w}_1(t-\theta)d\tilde{w}(t)^T\right)^T \right\rangle;\end{aligned}$$

$$\begin{aligned}\tilde{S}_{\tilde{w}}(t-\theta)dt &= \frac{1}{2} \left\langle d\tilde{w}_1(t-\theta)d\tilde{w}(t)^T \right. \\ &\quad \left. + \left(d\tilde{w}(t)d\tilde{w}_1(t-\theta)^T\right)^T \right\rangle; \\ \hat{S}_v(t)dt &= \frac{1}{2} \left\langle dv(t)dv(t-\theta)^T \right. \\ &\quad \left. + \left(dv(t-\theta)dv(t)^T\right)^T \right\rangle; \\ \tilde{S}_v(t-\theta)dt &= \frac{1}{2} \left\langle dv(t-\theta)dv(t)^T \right. \\ &\quad \left. + \left(dv(t)dv(t-\theta)^T\right)^T \right\rangle.\end{aligned}$$

Note that,

$$\begin{aligned}S_{\check{w}}(t)dt &= \begin{bmatrix} S_{\tilde{w}}(t)dt & \hat{S}_{\tilde{w}}(t)dt \\ \tilde{S}_{\tilde{w}}(t-\theta)dt & S_{\tilde{w}_1}(t-\theta)dt \end{bmatrix}; \\ S_{\check{v}}(t)dt &= \begin{bmatrix} S_v(t)dt & \hat{S}_v(t)dt \\ \tilde{S}_v(t-\theta)dt & S_v(t-\theta)dt \end{bmatrix}.\end{aligned}$$

Hence, we obtain the matrix differential equation

$$\begin{aligned}\dot{P}(t) &= \tilde{A}(t)P(t) + P(t)\tilde{A}(t)^T + \tilde{A}_\theta(t)P_1(t) + P_2(t)\tilde{A}_\theta^T \\ &\quad + \tilde{D}(t)\tilde{\beta}_w(t) \left\langle \eta(t)^T \right\rangle + \langle \eta(t) \rangle \tilde{\beta}_w(t)^T \tilde{D}(t)^T \\ &\quad + \tilde{G}_{v_t}(t) S_{\tilde{v}_t}(t) G_{v_t}(t)^T. \quad (12)\end{aligned}$$

Note that $P(0) = P_0 = diag(Y_0 + \check{x}_0 \check{x}_0^T, 0)$.

We now find an expression for L_γ. In fact,

$$\begin{aligned}&\left\langle \eta(t_f)^T \hat{Q}_f \eta(t_f) \right\rangle \\ &\quad = \frac{1}{2} tr \left\langle \hat{Q}_f \left(\eta(t_f)\eta(t_f)^T + \left(\eta(t_f)\eta(t_f)^T\right)^T \right) \right\rangle \\ &\quad = tr \left(\hat{Q}_f P(t_f) \right).\end{aligned}$$

On the other hand, $\langle \eta(t)^T R(t)\eta(t)\rangle = tr\,(R(t)P(t))$.

Hence,

$$\begin{aligned} L_\gamma(\mathcal{K},\hat{\beta}_w) &= tr\left(\hat{Q}_f P(t_f)\right) + \int_0^{t_f} tr\,(R(t)P(t))dt \\ &\quad -\gamma^2\left\{\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\tilde{\beta}_w(t)^T\tilde{\beta}_w(t)dt\right\}. \end{aligned} \tag{13}$$

F. The Finite Horizon H^∞ problem

We will consider, as the standard problem, the case where $\check{x}_0$ is a part of the unknown disturbance. Let

$$\left(\check{x}_0, \tilde{\beta}_w(.)\right) := \hat{\beta}_w(.) \in \Omega_q = \mathbb{R}^n \times \mathcal{W} \times \mathcal{W}_1. \tag{14}$$

$L_\gamma(\mathcal{K},\hat{\beta}_w)$ defined in (8) can be written in terms of the closed-loop system variable $\eta(t)$ as

$$\begin{aligned} L_\gamma(\mathcal{K},\hat{\beta}_w) &= \left\langle \eta(t_f)^T\hat{Q}_f\eta(t_f)\right\rangle \\ &\quad + \int_0^{t_f}\langle\eta(t)^T R(t)\eta(t)\rangle dt \\ &\quad -\gamma^2\left\{\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\tilde{\beta}_w(t)^T\tilde{\beta}_w(t)dt\right\} \\ &= tr\left(\hat{Q}_f P(t_f)\right) + \int_0^{t_f} tr\,(R(t)P(t))dt \\ &\quad -\gamma^2\left\{\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\tilde{\beta}_w(t)^T\tilde{\beta}_w(t)dt\right\}. \end{aligned}$$

The disturbance attenuation problem to be solved is the following:

Problem $\mathcal{P}_\gamma$. Determine necessary and sufficient conditions on γ such that the quantity

$$\inf_{\mathcal{K}\in\mathfrak{M}}\ \sup_{\hat{\beta}_w\in\Omega_q} L_\gamma(\mathcal{K},\hat{\beta}_w)$$

is finite, and for each such γ find a controller $\mathcal{K}$ that achieves the minimum. The infimum of all γ's that satisfy these conditions will be denoted by γ_q^*.

III. Auxiliary Classical Stochastic and Deterministic Systems

The Auxiliary Classical Stochastic System

We define the following classical linear stochastic system with delayed measurements

$$\begin{aligned} d\xi(t) &= A(t)\xi(t)dt + B(t)\beta_u(t)dt + D_1(t)\beta_w(t)dt \\ &\quad + B(t)S_{\tilde{u}}^{1/2}(t)d\tilde{u}(t) + D_1(t)S_{\tilde{w}}^{1/2}(t)d\tilde{w}(t) \\ &\quad + D_1(t)\hat{S}_{\tilde{w}}^{1/2}(t)d\tilde{w}_1(t-\theta) \\ &\quad + G_v(t)S_v^{1/2}(t)dv(t) \\ &\quad + G_v(t)\hat{S}_v^{1/2}(t)dv(t-\theta); \quad t\geq 0; \\ dy(t-\theta) &= C(t-\theta)\xi(t-\theta)dt + N_1(t-\theta)\beta_{w_1}(t-\theta)dt \\ &\quad + N_1(t-\theta)\tilde{S}_{\tilde{w}}^{1/2}(t-\theta)d\tilde{w}(t) \\ &\quad + N_1(t-\theta)S_{\tilde{w}_1}^{1/2}(t-\theta)d\tilde{w}_1(t-\theta) \\ &\quad + L(t-\theta)S_v^{1/2}(t-\theta)dv(t-\theta) \\ &\quad + L(t-\theta)\tilde{S}_v^{1/2}(t-\theta)dv(t); \\ z(t) &= H(t)\xi(t) + G(t)\beta_u(t) \end{aligned} \tag{15}$$

where equations (2) are satisfied and $\xi(0)=\xi_0$ is a Gaussian random vector with mean $\check{x}_0$ and covariance matrix Y_0 and $\xi(t) = \langle\phi_q(t)\rangle = \tilde{\phi}(t)$ for $0\leq t<\theta$.

A. Closed-Loop System

The classical controller $\mathcal{K}$ is given by (5) and the corresponding closed-loop classical stochastic system is obtained by making the identification $\beta_u(t) = H_c(t)\psi(t)$ and interconnecting equations (15) and (5) to give:

$$\begin{aligned} d\mu(t) &= \tilde{A}(t)\mu(t)dt + \tilde{A}_\theta(t)\mu(t-\theta)dt \\ &\quad + \tilde{B}(t)S_{\tilde{u}}^{1/2}(t)d\tilde{u}(t) \\ &\quad + \tilde{D}(t)\tilde{\beta}_w(t)dt + \tilde{D}(t)S_{\check{w}}^{1/2}(t)d\check{w}(t) \\ &\quad + \tilde{G}_v(t)S_{\tilde{v}}^{1/2}(t)d\tilde{v}(t); \quad \mu(0)=\mu_0; \\ z(t) &= \tilde{H}(t)\mu(t) \end{aligned} \tag{16}$$

where $\mu(t) = \begin{bmatrix}\xi(t)\\ \psi(t)\end{bmatrix}$.

Note that the existence of solutions to stochastic differential equations with time delays has been studied extensively in the literature, for instance see [7], [8] and [9].

The closed-loop system (16) can also be rewritten as

$$\begin{aligned} d\mu(t) &= \tilde{A}(t)\mu(t)dt + \tilde{A}_\theta(t)\mu(t-\theta)dt + \tilde{D}(t)\tilde{\beta}_w(t)dt \\ &\quad + \tilde{G}_{v_t}(t)S_{\tilde{v}_t}^{1/2}(t)d\tilde{v}_t(t); \quad \mu(0)=\mu_0; \\ z(t) &= \tilde{H}(t)\mu(t) \end{aligned} \tag{17}$$

where $S_{\tilde{v}_t}(t)dt = \begin{bmatrix} S_{\tilde{u}}(t)dt & 0 & 0 \\ 0 & S_{\check{w}}(t)dt & 0 \\ 0 & 0 & S_{\tilde{v}}(t)dt\end{bmatrix}$.

B. Cost Function

We define the classical cost function

$$\hat{L}(\mathcal{K},\hat{\beta}_w) = E\left(\xi(t_f)^T Q_f\xi(t_f)\right) + \int_0^{t_f} E\left(z(t)^T z(t)\right)dt$$

where $Q_f = Q_f^T \geq 0$ and

$$\begin{aligned} \hat{L}_\gamma(\mathcal{K},\hat{\beta}_w) &= E\left(\xi(t_f)^T Q_f\xi(t_f)\right) \\ &\quad + \int_0^{t_f} E\left(z(t)^T z(t)\right)dt \\ &\quad -\gamma^2\left\{\check{x}_0^T Q_0\check{x}_0 + \int_0^{t_f}\tilde{\beta}_w(t)^T\tilde{\beta}_w(t)dt\right\} \end{aligned} \tag{18}$$

where $E(.)$ denotes the classical stochastic expectation.

C. An Explicit Expression for the Closed-Loop Cost Function

For the stochastic closed-loop system (17), we define the covariance matrix

$$\tilde{P}(t) = E(\mu(t)\mu(t)^T). \tag{19}$$

Using the classical Ito rule, we can write

$$d\tilde{P}(t) = E(d\mu(t)\mu(t)^T) + E(\mu(t)d\mu(t)^T) + E(d\mu(t)d\mu(t)^T).$$

Note that

$$\begin{aligned} &E\left(\xi(t)^T Q(t)\xi(t)\right) + E\left(\beta_u(t)^T\beta_u(t)\right) \\ &= E\left(z(t)^T z(t)\right) \\ &= E\left(\mu(t)^T R(t)\mu(t)\right) \end{aligned}$$

since the equations (2) are satisfied and $\tilde{H}(t)^T\tilde{H}(t) = R(t) \geq 0$.

Also,

$$\begin{aligned} &E\left(\xi(t_f)^T Q_f \xi(t_f)\right) \\ &= E\left(\mu(t_f)^T \begin{bmatrix} Q_f^{\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} Q_f^{\frac{1}{2}} & 0 \end{bmatrix} \mu(t_f)\right) \\ &= E\left(\mu(t_f)^T \hat{Q}_f \mu(t)\right) \end{aligned}$$

where $\hat{Q}_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} \geq 0$.

Similarly,

$$\check{x}_0^T Q_0 \check{x}_0 = \check{\mu}_0^T \begin{bmatrix} Q_0^{\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} Q_0^{\frac{1}{2}} & 0 \end{bmatrix} \check{\mu}_0 = \check{\mu}_0^T \hat{Q}_0 \check{\mu}_0$$

where $\hat{Q}_0 = \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} > 0$ and $\check{\mu}_0 = E(\mu_0)$.

Hence, the cost function (18) can be rewritten as

$$\begin{aligned} \hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w) &= E\left(\mu(t_f)^T \hat{Q}_f \mu(t_f)\right) \\ &\quad + \int_0^{t_f} E\left(\mu(t)^T R(t)\mu(t)\right) dt \\ &\quad - \gamma^2 \left\{ \check{\mu}_0^T \hat{Q}_0 \check{\mu}_0 + \int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt \right\}. \end{aligned} \tag{20}$$

D. Equivalence Between $P(.)$ and $\tilde{P}(.)$

Theorem 3.1: Given any admissible controller $\mathcal{K} \in \mathfrak{M}$ and any $\tilde{\beta}_w(.) \in \mathcal{W}\times\mathcal{W}_1$, the covariance matrices $P(t)$ given by (10) and $\tilde{P}(t)$ given by (19) are equal for all $t \in [0, t_f]$.

As a consequence of Theorem 3.1, the resulting quantum closed-loop system (6) and the resulting stochastic closed-loop system (16) will have the same cost values for all disturbance inputs $\tilde{\beta}_w(t) \in \mathcal{W} \times \mathcal{W}_1$; i.e, $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ will have the same value as $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$.

E. Reformulation of the Auxiliary Classical Stochastic Closed-Loop System

In this subsection, we reformulate the stochastic worst case performance problem for the closed-loop system. The closed-loop system (17) can also be rewritten as:

$$\begin{aligned} d\mu(t) &= \tilde{A}(t)\mu(t)dt + \tilde{A}_\theta(t)\mu(t-\theta)dt + \tilde{D}(t)\tilde{\beta}_w(t)dt \\ &\quad + dv_n(t); \\ z(t) &= \tilde{H}(t)\mu(t) \end{aligned} \tag{21}$$

where $dv_n(t) = \tilde{G}_{v_t}(t)S_{\tilde{v}_t}^{1/2}(t)d\tilde{v}_t(t)$. We now assume that the initial condition random variable $\mu(0) = \mu_0$ for the closed-loop system (21) is normal with mean m and covariance matrix R_0. The stochastic process $v_n(t)$ has zero mean and covariance matrix $R_1(t)$. We assume that the process $v_n(t)$ is independent of μ_0 and that the matrices R_0 and $R_1(t)$ are symmetric and nonnegative definite for all $t \in [0, t_f]$.

1) Reformulating the Closed-Loop Cost Function:: Let

$$\begin{aligned} \check{J}_\gamma(\mathcal{K}, \hat{\beta}_w) &= -\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\ &= E\left(\mu(t_f)^T \tilde{Q}_f \mu(t_f)\right) \\ &\quad + \int_0^{t_f} E\left(\mu(t)^T \tilde{R}(t)\mu(t)dt\right) \\ &\quad + \gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t)dt + \check{\mu}_0^T \hat{Q}_0 \check{\mu}_0\right) \end{aligned} \tag{22}$$

where $\tilde{Q}_f = \tilde{Q}_f^T = -Q_f \leq 0$ and $\tilde{R}(t) = \tilde{R}(t)^T = -R(t) \leq 0$.

Also, we define

$$\begin{aligned} &\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) \\ &= E\left(\mu(t_f)^T \tilde{Q}_f \mu(t_f)\right) + \int_0^{t_f} E\left(\mu(t)^T \tilde{R}(t)\mu(t)dt\right) \\ &\quad + \gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t)dt\right). \end{aligned}$$

We want to minimize $\check{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ which is equivalent to maximizing $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$.

Using Theorem 3.1, $P(.)$ and $\tilde{P}(.)$ are equal. Thus, minimizing $\check{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ is equivalent to maximizing $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$.

By taking $\check{x}_0$ as a part of the unknown disturbance, the quantum cost function $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ defined in Problem P_γ is equal to the stochastic cost function $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ since $P(.)$ and $\tilde{P}(.)$ are equal.

Hence, minimizing $\check{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ is equivalent to maximizing $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ in Problem P_γ.

The Auxiliary Classical Deterministic System

We now consider a deterministic system corresponding to the auxiliary classical stochastic system (15) defined as:

$$\begin{aligned} \dot{\xi}(t) &= A(t)\xi(t) + B(t)\beta_u(t) + D_1(t)\beta_w(t); \\ \xi_0 &= \check{x}_0; \\ y(t-\theta) &= C(t-\theta)\xi(t-\theta) + N_1(t-\theta)\beta_{w_1}(t-\theta); \\ z(t) &= H(t)\xi(t) + G(t)\beta_u(t) \end{aligned} \tag{23}$$

where equations (2) are satisfied. Also, $\xi(t) = \tilde{\phi}(t)$ for $0 \leq t < \theta$.

The deterministic closed-loop system corresponding to the auxiliary stochastic closed-loop system (17) is given by

$$\begin{aligned} \dot{\mu}(t) &= \tilde{A}(t)\mu(t) + \tilde{A}_\theta(t)\mu(t-\theta) + \tilde{D}(t)\tilde{\beta}_w(t); \quad \mu_0 = m \\ z(t) &= \tilde{H}(t)\mu(t). \end{aligned} \tag{24}$$

Note that the solution to these deterministic differential equations with time delay has been studied extensively in the literature, for instance see [10].

The standard problem we consider is the case where $\breve{x}_0$ is a part of the unknown disturbance. The set of admissible controllers $\mathcal{K}$ will be denoted by $\mathfrak{M}$. These controllers are of the form given by (5) and such that the problem defined by (23) and (5) has a unique solution for every $\breve{x}_0$ and every $\hat{\beta}_w(.) \in \mathcal{W} \times \mathcal{W}_1$.

We also introduce the extended cost function

$$\begin{aligned}\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w) &= \xi(t_f)^T Q_f \xi(t_f) + \int_0^{t_f} z(t)^T z(t) dt \\ &\quad -\gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt + \breve{x}_0^T Q_0 \breve{x}_0 \right) \\ &= \xi(t_f)^T Q_f \xi(t_f) \\ &\quad + \int_0^{t_f} \left(\xi(t)^T Q(t) \xi(t) + \beta_u(t)^T \beta_u(t) \right) dt \\ &\quad -\gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt + \breve{x}_0^T Q_0 \breve{x}_0 \right)\end{aligned}$$

where Q_0 is a weighting matrix, taken to be positive definite and $\gamma > 0$.

Also, $\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ can be rewritten in terms of the closed-loop variable $\mu(t)$ and $\hat{\mu}(t)$ as

$$\begin{aligned}\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w) &= \mu(t_f)^T \hat{Q}_f \mu(t_f) + \int_0^{t_f} \mu(t)^T R(t) \mu(t) dt \\ &\quad -\gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt + \breve{\mu}_0^T \hat{Q}_0 \breve{\mu}_0 \right).\end{aligned} \tag{25}$$

The corresponding disturbance attenuation problem to be solved is the following:

Problem $\tilde{\mathcal{P}}_\gamma$. Determine necessary and sufficient conditions on γ such that the quantity

$$\inf_{\mathcal{K} \in \mathfrak{M}} \sup_{\hat{\beta}_w \in \Omega_q} \tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$$

is finite, and for each such γ find a controller $\mathcal{K}$ (or family of controllers) that achieves the minimum. The infimum of all γ's that satisfy these conditions will be denoted by γ_c^*.

IV. An Equivalent Deterministic Worst Case Performance Problem for The Closed-Loop System

A. The Closed-Loop System and the performance index

In the deterministic case, the closed-loop system corresponding to (23) and (5) is given by (24). Also, the closed-loop deterministic performance index is given by

$$\begin{aligned}\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) &= \mu(t_f)^T \tilde{Q}_f \mu(t_f) + \int_0^{t_f} \mu(t)^T \tilde{R}(t) \mu(t) dt \\ &\quad + \gamma^2 \int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt.\end{aligned} \tag{26}$$

B. Solution to the Deterministic Worst Case Performance Problem

The deterministic worst case performance problem can be stated as follows:

Problem: Consider the closed-loop deterministic system described by (24). Find an admissible strategy $\tilde{\beta}_w(.)$ such that the cost function (26) is minimized.

We define the following Riccati partial differential equations

$$\begin{aligned}&\dot{\Pi}(t) + \tilde{A}(t)^T \Pi(t) + \Pi(t) \tilde{A}(t) + \tilde{A}_\theta(t)^T Q(t, -\theta)^T \\ &+ Q(t, -\theta) \tilde{A}_\theta(t) + \Pi(t) S(t) \Pi(t) + \tilde{R}(t) = 0; \\ &\frac{\partial Q(t, \tilde{\xi})}{\partial t} + \frac{\partial Q(t, \tilde{\xi})}{\partial \tilde{\xi}} = -\left[\tilde{A}(t)^T + \Pi(t) S(t) \right] Q(t, \tilde{\xi}) \\ &-\tilde{A}_\theta(t)^T R(t, -\theta, \tilde{\xi}); \\ &\frac{\partial R(t, \tilde{\xi}, s)}{\partial t} + \frac{\partial R(t, \tilde{\xi}, s)}{\partial \tilde{\xi}} + \frac{\partial R(t, \tilde{\xi}, s)}{\partial s} \\ &\quad = -Q(t, \tilde{\xi})^T S(t) Q(t, s)\end{aligned} \tag{27}$$

where $\tilde{\xi} \in [0, \theta]$, $s \in [0, \theta]$ and

$$\begin{aligned}\Pi(t_f) &= \tilde{Q}_f; \\ \Pi(t) &= Q(t, 0); \\ Q(t, \tilde{\xi}) &= R(t, 0, \tilde{\xi}); \\ R(t, \tilde{\xi}, s) &= R(t, s, \tilde{\xi})^T; \\ Q(t_f, \tilde{\xi}) &= 0; \\ R(t_f, \tilde{\xi}, s) &= 0; \\ S(t) &= -\gamma^{-2} \tilde{D}(t) \tilde{D}(t)^T; \\ \tilde{R}(t) &= -\tilde{H}(t)^T \tilde{H}(t).\end{aligned} \tag{28}$$

Theorem 4.1: The deterministic linear quadratic problem (24)-(26) has a finite solution for every initial condition $\mu_0 = m$ if and only if the Riccati partial differential equations (27) with the terminal conditions (28) have solutions on $[0, t_f]$.

If the deterministic linear quadratic problem has a solution, then it is unique and the optimal disturbance signal $\tilde{\beta}_w(t)$ is given by

$$\begin{aligned}\tilde{\beta}_w^*(t) &= -\gamma^{-2} \tilde{D}(t)^T \left(\Pi(t) \mu(t) + \int_0^\theta Q(t, \tilde{\xi}) \tilde{A}_\theta(t) \right. \\ &\qquad \left. \mu(t - \theta - \tilde{\xi}) d\tilde{\xi} \right).\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w^*) &= \min_{\tilde{\beta}_w \in \mathcal{W} \times \mathcal{W}_1} \tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) \\ &= m^T \Pi(0) m 0 \\ &\quad + m^T \int_0^\theta Q(0, \tilde{\xi}) \tilde{A}_\theta(0) \mu(\theta - \tilde{\xi}) d\tilde{\xi} \\ &\quad + \left[\int_0^\theta \mu(\theta - \tilde{\xi})^T \tilde{A}_\theta(0)^T Q(0, \tilde{\xi})^T d\tilde{\xi} \right] m \\ &\quad + \int_0^\theta \int_0^\theta \mu(\theta - s)^T \tilde{A}_\theta(0)^T R(0, s, \tilde{\xi}) \\ &\quad\quad \tilde{A}_\theta(0) \mu(\theta - \tilde{\xi}) ds d\tilde{\xi}.\end{aligned}$$

V. A Relationship Between $\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ and $\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$

The following theorem shows the relationship between the optimum values of the stochastic cost function $\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ and the deterministic cost function $\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ where $m \in \mathbb{R}^{(n+n_c)}$ defines the initial condition of the deterministic system (24) and the mean of the initial condition in the stochastic system (21). Let

$$\hat{V}(m) = \inf_{\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1} \hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) \qquad (29)$$

and

$$\tilde{V}(m) = \inf_{\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1} \tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w). \qquad (30)$$

Theorem 5.1: Given any $m \in \mathbb{R}^{(n+n_c)}$, the infimum $\hat{V}(m)$ in the stochastic case is related to the corresponding infimum $\tilde{V}(m)$ in the deterministic case by the following equation

$$\hat{V}(m) = \tilde{V}(m) + \alpha \qquad (31)$$

where α is independent of m and depends on the variances of the noises.

VI. Solution to the Stochastic Worst case Performance Problem

The stochastic worst case performance problem can be stated as follows:

Problem: Consider the closed-loop stochastic system described by (21). Find an admissible strategy $\tilde{\beta}_w(.) \in \mathcal{W}\times\mathcal{W}_1$ such that the following cost function is minimized

$$\begin{aligned}\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) &= E\left(\mu(t_f)^T \tilde{Q}_f \mu(t_f)\right) \\ &\quad + \int_0^{t_f} E\left(\mu(t)^T \tilde{R}(t)\mu(t)dt\right) \\ &\quad + \gamma^2 \left(\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt\right).\end{aligned}$$

Theorem 6.1: Assume that the Riccati partial differential equations (27) with the terminal conditions (28) have solutions on $[0, t_f]$. Then, the minimal value of the cost function in the stochastic worst case performance problem (32) satisfies

$$\begin{aligned}&\min_{\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1} \hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w) \\ &\geq m^T \Pi(0) m + E\left(\mu_0^T \int_0^\theta Q(0,\tilde{\xi})\tilde{A}_\theta(0)\mu(\theta-\tilde{\xi})d\tilde{\xi}\right) \\ &\quad + E\left(\left[\int_0^\theta \mu(\theta-\tilde{\xi})^T \tilde{A}_\theta(0)^T Q(0,\tilde{\xi})^T d\tilde{\xi}\right]\mu_0\right) \\ &\quad + E\left(\int_0^\theta \int_0^\theta \mu(\theta-s)^T \tilde{A}_\theta(0)^T R(0,s,\tilde{\xi}) \right. \\ &\qquad \left. \tilde{A}_\theta(0)\mu(\theta-\tilde{\xi}) ds d\tilde{\xi}\right) \\ &\quad + \alpha_1 \end{aligned} \qquad (32)$$

where

$$\alpha_1 = tr(\Pi(0)R_0) + \int_0^{t_f} tr(R_1(t)\Pi(t))dt. \qquad (33)$$

Also, the optimal signal $\tilde{\beta}_w(t)$ is given by

$$\begin{aligned}\tilde{\beta}_w^*(t) &= -\gamma^{-2}\tilde{D}(t)^T \left(\Pi(t)\mu(t) + \int_0^\theta Q(t,\tilde{\xi})\tilde{A}_\theta(t)\right. \\ &\qquad \left. \mu(t-\theta-\tilde{\xi})d\tilde{\xi}\right).\end{aligned}$$

In that case, $\Pi(t)$ is the solution of Riccati partial differential equations (27), R_0 is the covariance matrix of μ_0 and $R_1(t)$ is the covariance matrix of $v_n(t)$.

Theorem 6.2: The stochastic linear quadratic control problem has a finite solution, for every initial condition $\mu_0 = m$, if and only if the Riccati partial differential equations (27) with the terminal conditions (28) have solutions on $[0, t_f]$.

VII. Equivalence Between the Deterministic Worst Case Performance Problem and the Stochastic Worst Case Performance Problem

The following theorems lead to the equivalence between the deterministic worst case performance problem and the stochastic worst case performance problem.

Theorem 7.1: In the deterministic case, $\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1$ for all $m \in \mathbb{R}^{(n+n_c)}$ if and only only if $\gamma > \hat{\gamma}$.

Theorem 7.2: In the stochastic case, $\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1$ for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\gamma > \hat{\gamma}$.

Theorem 7.3: $\hat{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1$ for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w$ for all $m \in \mathbb{R}^{(n+n_c)}$.

VIII. Equivalence Between the Quantum Worst Case Performance Problem and the Deterministic Worst Case Performance Problem

Let

$$\begin{aligned}J_\gamma(\mathcal{K}, \tilde{\beta}_w) &= -L_\gamma(\mathcal{K}, \tilde{\beta}_w) \\ &= \left\langle \eta(t_f)^T \tilde{Q}_f \eta(t_f)\right\rangle \\ &\quad + \int_0^{t_f} \left\langle \eta(t)^T \tilde{R}(t)\eta(t)\right\rangle dt \\ &\quad + \gamma^2 \left\{\int_0^{t_f} \tilde{\beta}_w(t)^T \tilde{\beta}_w(t) dt\right\}.\end{aligned} \qquad (34)$$

The following theorem leads to the equivalence between the quantum worst case performance problem and the deterministic worst case performance problem.

Theorem 8.1: In the quantum case, $J_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1$ for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\tilde{J}_\gamma(\mathcal{K}, \tilde{\beta}_w)$ has a finite infimum over $\tilde{\beta}_w \in \mathcal{W}\times\mathcal{W}_1$ for all $m \in \mathbb{R}^{(n+n_c)}$.

IX. Solution to the Finite Horizon H^∞ Control Problem for Delayed Measurement Systems

In order to solve the finite horizon quantum H^∞ problem for delayed measurement systems, we now introduce the following GRDE (Generalized Riccati Differential Equations):

$$\dot{S}(t)+S(t)A(t)+A(t)^TS(t)-S(t)\left(B(t)B(t)^T -\gamma^{-2}D(t)D(t)^T\right)S(t)+Q(t)=0; S(t_f)=Q_f; \quad (35)$$

$$\dot{\Sigma}(t)=A(t)\Sigma(t)+\Sigma(t)A(t)^T-\Sigma(t)\left(C(t)E(t)^{-1}C(t) -\gamma^{-2}Q(t)\right)\Sigma(t)+D(t)D(t)^T; \Sigma(0)=Q_0^{-1}; \quad (36)$$

$$\dot{L}(t)=A(t)L(t)+L(t)A(t)^T+\gamma^{-2}L(t)Q(t)L(t) +D(t)D(t)^T; L(t-\theta)=\Sigma(t-\theta); \quad (37)$$

$$K(t)A(t)+A(t)^TK(t)+\gamma^{-2}K(t)D(t)D(t)^TK(t) +Q(t)-\gamma^2C(t)^TE(t)C(t)=0;$$
$$K(t-\theta)=\gamma^2\Sigma(t-\theta)^{-1} \quad (38)$$

$$\dot{W}(t)+W(t)A(t)+A(t)^TW(t)+\gamma^{-2}W(t)D(t) D(t)^TW(t)+Q(t)=0; W(t-\theta)=K(t-\theta); \quad (39)$$

$$\frac{d}{dt}\Psi_L(t)=\left(A(t)+\gamma^{-2}L(t)Q(t)\right)\Psi_L(t) +\Psi_L(t)\left(A(t-\theta)+\gamma^{-2}\Sigma(t-\theta)Q(t-\theta)\right); \quad (40)$$

$$\frac{dL(t)}{dt}=A(t)L(t)+L(t)A(t)^T+\gamma^{-2}L(t)Q(t)L(t) +D(t)D(t)^T, \quad t<\theta, L(0)=Q_0^{-1}; \quad (41)$$

$$\frac{dL(t)}{dt}=A(t)L(t)+L(t)A(t)^T+\gamma^{-2}L(t)Q(t)L(t) +D(t)D(t)^T-\gamma^{-2}\Psi_L(t) \quad (42)$$
$$\left(\Sigma(t)C(t)^TE(t)^{-1}C(t)\Sigma(t)\right)|_{t-\theta}\Psi_L(t), \quad t>\theta \quad (43)$$

where $E(t)=N(t)N(t)^T$.

In addition, we introduce the following conditions

$$\forall t\in[-\theta,0], \qquad \Sigma(t)=Q_0^{-1}, \qquad \tilde{x}(t)=0; \quad (44)$$
$$\forall \tau\in[0,t_f], \qquad \rho\left(L(\tau)S(\tau)\right)<\gamma^2 \quad (45)$$

where

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \left(A(t)+W(t)^{-1}Q(t)\right)\tilde{x}(t)+B(t)\beta_u(t); && (46)\\
\tilde{x}(t-\theta) &= \check{x}(t-\theta);\\
\dot{\check{x}}(t) &= \left(A(t)+\gamma^{-2}\Sigma(t)Q(t)\right)\check{x}(t)\\
&\quad +\Sigma(t)C(t)^TE(t)^{-1}(y(t)-C(t)\check{x}(t))\\
&\quad +B(t)\hat{u}(t),\\
\check{x}(0) &= 0; && (47)\\
\hat{u}(t) &= -B(t)^TS(t)\left[I-\gamma^{-2}\Sigma(t)S(t)\right]^{-1}\check{x}(t) && (48)
\end{aligned}$$

A. Solution to the Finite Horizon H^∞ Control Problem for the Quantum System

Let

$$\begin{aligned}
\hat{u}_1(t) &= -B(t)^TS(t)\left(I-\gamma^{-2}L(t)S(t)\right)^{-1}\tilde{x}_1(t); && (49)\\
\dot{\tilde{x}}_1(t) &= \left[A(t)+\gamma^{-2}L(t)Q(t)\right]\tilde{x}_1(t)+B(t)\hat{u}_1(t)\\
&\quad +\Psi_L(t)\Sigma(t-\theta)C(t-\theta)^T\\
&\quad \times E(t-\theta)^{-1}\left[y(t-\theta)-C(t-\theta)\tilde{x}_1(t-\theta)\right]. && (50)
\end{aligned}$$

Theorem 9.1: Consider the disturbance attenuation problem with delayed output measurement with a fixed delay θ given by $\mathcal{P}_\gamma$. Let the infimum of the feasible attenuation levels be γ_q. If

(a) Equation (35) has a solution over $[0,t_f]$,
(b) Equation (36) has a solution over $[0,t_f-\theta]$,
(c) for every $\tau\in[0,t_f]$, equation (37), with $\Sigma(t)$ extended to negative values of t as in (44), has a solution over $[\tau-\theta,\tau]$ satisfying (45),

then necessarily $\gamma\geq\gamma_q$, and an optimal controller achieving the attenuation level γ_q is given by (49) with $L(\tau)$ given by (37) or (41) and (42) and $\tilde{x}(\tau)$ by (46) or (50).

If any one of conditions (a)-(c) above fails, then $\gamma\leq\gamma_q$.

X. Conclusion

This paper shows that solving the finite horizon H^∞ control problem for delayed measurement systems is equivalent to solving a corresponding classical deterministic continuous-time problem with imperfect delayed state measurements. From this, the solution to the finite horizon quantum H^∞ control problem for delayed measurement systems can be obtained in terms of GRDEs.

References

[1] A. I. Maalouf and I. R. Petersen, "Finite horizon H^∞ control for a class of linear quantum systems: A dynamic game approach," *Proceedings of the American Control Conference ACC 2010*, pp. 1904 – 1911, 2010.

[2] M. R. James, H. I. Nurdin, and I. R. Petersen, "H^∞ control of linear quantum stochastic systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.

[3] H. Nurdin, M. R. James, and I. R. Petersen, "Coherent quantum LQG control," *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009.

[4] P.-L. Meyer, *Quantum Probability for Probabilists*. Germany: Springer, 1995.

[5] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*. Berlin: Birkhauser, 1992.

[6] K. Kashima and N. Yamamoto, "Control of quantum systems despite feedback delay," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 876–881, 2009.

[7] S. E. A. Mohammed, *Stochastic functional differential equations*. Pitman, 1984.

[8] X. Mao, *Stability of stochastic differential equations with respect to semi-martingales*. New York: Springer-Verlag, 1995.

[9] ——, *Exponential Stability of Stochastic Differential Equations*. New York: M. Dekker, 1994.

[10] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.