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Mean-Square Filter Design for Nonlinear Polynomial Systems with Poisson Noise

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***Abstract*— This paper presents the mean-square filtering problem for incompletely measured polynomial system states, confused with white Poisson noises, over linear observations. The problem is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance. As a result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are first derived. The procedure for obtaining a closed system of the filtering equations for any polynomial system state with white Poisson noises over linear observations is then established, which yields the explicit closed form of the filtering equations in the particular case of a third-order state equation. In the example, performance of the designed optimal filter is verified against the conventional mean-square polynomial filter designed for systems with white Gaussian noises.**

I. INTRODUCTION

It is well known that the mean-square optimal solution of the filtering problem for nonlinear state and observation equations confused with Gaussian white noises is given by the Kushner equation for the conditional density of an unobserved state with respect to observations [1]. There are a very few known examples of nonlinear systems where the Kushner equation can be reduced to a finite-dimensional closed system of filtering equations for a certain number of lower conditional moments. The most famous result, the Kalman-Bucy filter [2], is related to the case of linear state and observation equations, where only two moments, the estimate itself and its variance, form a closed system of filtering equations. Some other mean-square nonlinear finite-dimensional filters can be found in [3], [4], [5]. There also exists a considerable bibliography on robust filtering for the linear and nonlinear systems corrupted with white Gaussian noises (see, for example, [6]–[23] and references therein).

On the other hand, the number of publications about mean-square filtering for systems with white Poisson noises is relatively small. It is known that the mean-square filter for linear systems with white Poisson noises coincides with the Kalman-Bucy filter [24], [25]. A few results related to nonlinear Poisson systems can be found in [26]-[32]. However, the mean-square filters for nonlinear polynomial systems with white Poisson noises, similar to those obtained in [33], [34], [35], have not been yet designed.

This paper presents the optimal finite-dimensional filter for incompletely measured polynomial system states, confused with white Poisson noises, over linear observations. Designing the mean-square filter for polynomial systems with white Poisson noises presents a significant advantage in the filtering theory and practice, since it enables one to address the mean-square estimation problems for nonlinear system states confused with other than Gaussian white noises. The optimal filtering problem is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance ([25], Section 5.10). As the first result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are derived. Next, a transformation of the observation equation is introduced to reduce the original problem to the one with an invertible observation matrix. It is then proved, using the technique of representing the superior moments of a Poisson random variable as functions of its expectation and variance, that a closed finite-dimensional system of the optimal filtering equations with respect to a finite number of filtering variables can be obtained for a polynomial state equation and linear observations with an arbitrary observation matrix. In this case, the corresponding procedure for designing the optimal filtering equations is established. Finally, the closed system of the optimal filtering equations with respect to two variables, the optimal estimate and the error variance, is derived in the explicit form in the particular case of a third-order state equation.

In the illustrative example, performance of the designed optimal filter is verified for a third-order bi-dimensional state over scalar linear observations against a conventional mean-square filter for stochastic polynomial systems with white Gaussian noises. The simulation results show a definite advantage in favor of the designed optimal filter. Indeed, it can be observed that the estimation error produced by the optimal filter rapidly reaches and then maintains the zero mean value. On the contrary, the estimation error given by the conventional mean-square filter for stochastic polynomial systems with Gaussian noises behaves unstably and diverges to infinity before the asymptotic time of the reference state.

The paper is organized as follows. Section 2 presents the filtering problem statement for incompletely measured polynomial system states, confused with white Poisson noises, over linear observations. The Ito differentials for the optimal estimate and the error variance are derived in Section 3. A transformation of the observation equation is then introduced to reduce the original problem to a one with an invertible observation matrix. Section 3 also establishes the procedure for obtaining a closed system of the filtering equations for any polynomial system state with white Poisson noise,

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which yields the explicit result in the case of a third-order state equation. Performance of the obtained optimal filter is verified in Section 4. The simulation results show a definite advantage in favor of the designed optimal filter against the conventional mean-square polynomial filter for stochastic polynomial systems with white Gaussian noises.

II. Filtering Problem for Incompletely Measured Polynomial States over Linear Observations

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent centralized Poisson processes. The F_t-measurable random process $(x(t), y(t))$ is described by a nonlinear differential equation with a polynomial drift term for the system state

$$dx(t) = f(x,t)dt + b(t)dW_1(t), \quad x(t_0) = x_0, \tag{1}$$

and a linear differential equation for the observation process

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t). \tag{2}$$

Here, $x(t) \in R^n$ is the state vector and $y(t) \in R^m$ is the linear observation vector, $m \leq n$. The initial condition $x_0 \in R^n$ is a Poisson vector such that x_0, $W_1(t) \in R^p$, and $W_2(t) \in R^q$ are independent. The observation matrix $A(t) \in R^{m \times n}$ is not supposed to be invertible or even square. It is assumed that $B(t)B^T(t)$ is a positive definite matrix, therefore, $m \leq q$. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The nonlinear function $f(x,t)$ is considered polynomial of n variables, components of the state vector $x(t) \in R^n$, with time-dependent coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [33], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + \ldots + a_p(t)x \ldots_{p\ times} \ldots x, \tag{3}$$

where $a_0(t)$ is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, a_p is an $(p+1)$D tensor of dimension $n \times \cdots_{(p+1)\ times} \cdots \times n$, and $x \times \ldots_{p\ times} \ldots \times x$ is a pD tensor of dimension $n \times \cdots_{p\ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself. Such a polynomial can also be expressed in the summation form

$$f_k(x,t) = a_{0\ k}(t) + \sum_i a_{1\ ki}(t)x_i(t) + \sum_{ij} a_{2\ kij}(t)x_i(t)x_j(t) + \ldots$$
$$+ \sum_{i_1 \ldots i_p} a_{p\ ki_1 \ldots i_p}(t)x_{i_1}(t) \ldots x_{i_p}(t), \quad k, i, j, i_1 \ldots\ i_p = 1, \ldots, n.$$

The estimation problem is to find the optimal estimate $\hat{x}(t)$ of the system state $x(t)$, based on the observation process $Y(t) = \{y(s), t_0 \leq s \leq t\}$, that minimizes the conditional expectation of the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \mid F_t^Y]$$

at every time moment t. Here, $E[z(t) \mid F_t^Y]$ means the conditional expectation of a stochastic process $z(t) = (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As known [25], this optimal estimate is given by the conditional expectation

$$\hat{x}(t) = m(t) = E(x(t) \mid F_t^Y)$$

of the system state $x(t)$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As usual, the matrix function

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y]$$

is the estimation error variance matrix.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differential of the conditional expectation $E(x(t) \mid F_t^Y)$ and its variance $P(t)$ (cited after [25]) and given in the following section.

III. Optimal Filter for Incompletely Measured Polynomial States over Linear Observations

The optimal filtering equations could be obtained using the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ in case of the linear drift term $A_0(t) + A(t)x(t)$ in the observation equation (see [25])

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + E(x(t)[A(t)(x(t) - m(t))]^T \mid F_t^Y) \times$$
$$(B(t)B^T(t))^{-1}(dy(t) - (A_0(t) + A(t)m(t)), \tag{4}$$

where $f(x,t)$ is the polynomial drift term in the state equation. The equation (4) should be complemented with the initial condition $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$.

Trying to compose a closed system of the filtering equations, the equation (4) should be complemented with the equation for the error variance $P(t)$. For this purpose, the formula for the Ito differential of the variance $P(t) = E((x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y)$ in case of the linear drift term $A_0(t) + A(t)x(t)$ in the observation equation could be used (cited again after [25]):

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y)$$
$$+ E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$- (E(x(t)(x(t) - m(t))^T \mid F_t^Y)A^T(t) \times$$
$$(B(t)B^T(t))^{-1}A(t)E((x(t) - m(t))x^T(t)) \mid F_t^Y))dt.$$

Using the variance formula $P(t) = E((x(t) - m(t))x^T(t)) \mid F_t^Y)$, the last equation can be represented as

$$dP(t) = (E((x(t) - m(t))(f(x,t))^T \mid F_t^Y)$$
$$+ E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$- P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt. \tag{5}$$

The equation (5) should be complemented with the initial condition $P(t_0) = E[(x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid F_{t_0}^Y]$.

The equations (4) and (5) for the optimal estimate $m(t)$ and the error variance $P(t)$ form a non-closed system of the filtering equations for the nonlinear state (1) over linear observations (2). The non-closeness means that the system (4),(5) includes terms depending on x, such as $E(f(x,t) \mid F_t^Y)$, and $E((x(t)-m(t))f^T(x,t)) \mid F_t^Y)$, which are not expressed yet as functions of the system variables, $m(t)$ and $P(t)$.

As shown in [34], [35], a closed system of the filtering equations for a system state (1) with polynomial drift over linear observations can be obtained in case of Gaussian white noises in the state and observation equations. In the considered case of Poisson white noises, the following transformations are introduced.

First, note that the matrix A can always be assumed a matrix of complete rank, m, which is equal to the dimension of the linearly independent observations $y(t) \in R^m$; if not so, excessive linearly dependent observations, corresponding to linearly dependent rows of the matrix A, must be removed from consideration. In doing so, the number of Poisson processes in the observation equations can also be reduced to m, the dimension of independent observations, by summarizing and re-numerating the Poisson processes in each observation equation (2). Therefore, the matrix B can always be assumed a square matrix of dimension $m \times m$, such that $B(t)B^T(t)$ is a positive definite matrix (see Section 2 for this condition). Next, the new matrices $\bar{A}(t)$ and $\bar{B}(t)$ are defined as follows. The matrix $\bar{A}(t) \in R^{n\times n}$ is obtained from $A(t) \in R^{m\times n}$ by adding $n-m$ linearly independent rows such that the resulting matrix $\bar{A}(t)$ is invertible. The matrix $\bar{B}(t) \in R^{n\times n}$ is made from the matrix $B(t) \in R^{m\times m}$ by placing $B(t)$ in the upper left corner of $\bar{B}(t)$, defining the other $n-m$ diagonal entries of $\bar{B}(t)$ equal to infinity, and setting to zero all other entries of $\bar{B}(t)$ outside the main diagonal or outside the submatrix $B(t)$. In other words, $\bar{B}(t) = diag[B(t), \beta I_{(n-m)\times(n-m)}]$, where $\beta = \infty$, and $I_{(n-m)\times(n-m)}$ is the identity matrix of dimension $(n-m)\times(n-m)$. Thus, the new observation equation is given by

$$d\bar{y}(t) = (\bar{A}_0(t) + \bar{A}(t)x(t))dt + \bar{B}(t)dW_2(t), \qquad (6)$$

where $\bar{y}(t) \in R^n$, $\bar{A}_0(t) = [A_0^T(t), 0_{n-m}]^T \in R^n$, and 0_{n-m} is a vector of $n-m$ zeros.

The key point of the introduced transformation is that the new observation process $\bar{y}(t)$ is physically equivalent to the old one $y(t)$, since the fictitious last $n-m$ components of $\bar{y}(t)$ consist of pure noise in view of infinite intensities of Poisson white noises in the corresponding $n-m$ equations, and the first m components of $\bar{y}(t)$ coincide with $y(t)$. In addition, the entire observation matrix $\bar{A}(t)$ is invertible, and the matrix $(\bar{B}(t)\bar{B}^T(t))^{-1} \in R^{n\times n}$ exists and equals to the $n\times n$ – dimensional square matrix, whose upper left corner is occupied by the submatrix $(B(t)B^T(t))^{-1} \in R^{m\times m}$ and all other entries are zeros.

In terms of the new observation equation (6), the filtering equations (4) and (5) take the form

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + P(t)\bar{A}^T(t)(\bar{B}(t)\bar{B}^T(t))^{-1}$$
$$(d\bar{y}(t) - (\bar{A}_0(t) + \bar{A}(t)m(t))dt), \qquad (7)$$
$$dP(t) = (E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) +$$
$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$-P(t)\bar{A}^T(t)(\bar{B}(t)\bar{B}^T(t))^{-1}\bar{A}(t)P(t))dt. \qquad (8)$$

with the initial conditions $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$ and $P(t_0) = E[(x(t_0)-m(t_0)(x(t_0)-m(t_0)^T \mid F_{t_0}^Y]$.

Since the new observation matrix $\bar{A}(t)$ is invertible for any $t \geq t_0$, the random variable $x(t)-m(t)$ is conditionally Poisson with respect to the new observation process $\bar{y}(t)$, and therefore with respect to the original observation process $y(t)$, for any $t \geq t_0$ (see [25]). Hence, the following considerations are applicable to the filtering equations (4),(5).

If the function $f(x,t)$ is polynomial function of the state x with time-dependent coefficients, the expression of the terms $E(f(x,t) \mid F_t^Y)$ in (7) and $E((x(t)-m(t))f^T(x,t)) \mid F_t^Y)$ in (9) would also include only polynomial terms of x. Then, those polynomial terms can be represented as functions of $m(t)$ and $P(t)$ using the following property of a Poisson random variable $x(t)-m(t)$: all its odd conditional moments can be represented as functions of the variance $P(t)$. For example, $m_1 = E[(x(t)-m(t)) \mid Y(t)]= 0$, $m_2 = E[(x(t)-m(t))^2 \mid Y(t)]$, $m_3 = E[(x(t)-m(t))^3 \mid Y(t)]$ are equal to P, $m_4 = E[(x(t)-m(t))^4 \mid Y(t)]$=$3P^2+P$, ... etc. After representing all polynomial terms in (7) and (9), that are generated upon expressing $E(f(x,t) \mid F_t^Y)$, and $E((x(t)-m(t))f^T(x,t)) \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$, a closed form of the filtering equations would be obtained.

Finally, in view of definition of the matrices $\bar{A}(t)$ and $\bar{B}(t)$ and the new observation process $\bar{y}(t)$, the filtering equations (7),(9) can be written again in terms of the original observation equation (2) using $y(t)$, $A(t)$, and $B(t)$

$$dm(t) = E(f(x,t) \mid F_t^Y)dt + P(t)A^T(t)(B(t)B^T(t))^{-1} \times$$
$$(dy(t) - (A_0(t) + A(t)m(t))dt), \qquad (10)$$
$$dP(t) = (E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$$
$$+E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) + b(t)b^T(t)$$
$$-P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt, \qquad (11)$$

with the initial conditions $m(t_0) = E(x(t_0) \mid F_{t_0}^Y)$ and $P(t_0) = E[(x(t_0)-m(t_0))(x(t_0)-m(t_0))^T \mid F_{t_0}^Y]$.

Furthermore, a closed form of the filtering equations is obtained from (10) and (11) for a third-order function $f(x,t)$ in the equation (1), as follows. It should be noted, however, that application of the same procedure would result in designing a closed system of the filtering equations for any polynomial function $f(x,t)$ in (1).

A. Optimal Filter for Third-Order Polynomial State

Let the function

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + a_3(t)xxx^T \qquad (12)$$

be a third-order polynomial, where x is an n-dimensional vector, $a_0(t)$ is an n-dimensional vector, $a_1(t)$ is a $n\times n$-dimensional matrix, $a_2(t)$ is a 3D tensor of dimension $n\times n\times n$, $a_3(t)$ is a 4D tensor of dimension $n\times n\times n\times n$.

In this case, the representations for $E(f(x,t) \mid F_t^Y)$ and $E((x(t)-m(t))(f(x,t))^T \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ are derived as follows

$$E(f(x,t) \mid F_t^Y) = a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + 3a_3(t)m(t)P(t) + a_3(t)m(t)m(t)m^T(t) + a_3(t)P(t) * 1, \tag{13}$$

$$E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y) + E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) = a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + a_2(t)P(t) * 1 + (a_2(t)(2m(t)P(t) + P(t) * 1))^T + a_3(t)[(P(t) * (1 * 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) * 1^T] + (a_3(t)[(P(t) * (1 * 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) * 1^T])^T. \tag{14}$$

Here, the vector 1 is an n-dimensional vector with all its components equal to 1, and the vector $a_3P(t) * 1 \in R^n$ and matrices $a_3(t)P(t) * 1 * 1^T \in R^{n\times n}$ and $a_3(t)m(t)P(t) * 1^T \in R^{n\times n}$ are defined as

$$(a_3(t)P(t) * 1)_i = \sum_{j,k,l} a_{3\ ijkl}(t)P_{jk}(t)1_l, \quad i = 1,\ldots,n,$$

$$(a_3(t)P(t) * 1 * 1^T)_{ij} = \sum_{h,k,l} a_{3\ ihkl}(t)P_{hk}(t)1_l1_j, \quad i,j = 1,\ldots,n,$$

$$(a_3(t)m(t)P(t) * 1^T)_{ij} = \sum_{h,k,l} a_{3\ ihkl}(t)m_h(t)P_{kl}(t)1_j, \quad i,j = 1,\ldots,n.$$

Substituting the expression (13) in (10) and the expression (14) in (11), the filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + 3a_3(t)m(t)P(t) + a_3(t)m(t)m(t)m^T(t) + a_3(t)P(t) * 1 + P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))dt],$$
$$m(t_0) = E(x(t_0) \mid F_t^Y)), \tag{15}$$

$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + a_2(t)P(t) * 1 + (a_2(t)(2m(t)P(t) + P(t) * 1))^T + a_3(t)[(P(t) * (1 * 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) * 1^T] + (a_3(t)[(P(t) * (1 * 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) * 1^T])^T + b(t)b^T(t))dt - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt. \tag{16}$$
$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid F_t^Y)).$$

By means of the preceding derivation, the following result is proved.

Theorem 1. The optimal finite-dimensional filter for the third-order state (1), where the third-order polynomial $f(x,t)$ is defined by (12), over the incomplete linear observations (2), is given by the equation (15) for the optimal estimate $m(t) = E(x(t) \mid F_t^Y)$ and the equation (16) for the estimation error variance $P(t) = E[(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y]$.

Thus, based on the general non-closed system of the filtering equations (7),(9), it is proved that the closed system of the filtering equations can be obtained for any polynomial state (1) over incomplete linear observations (2). Furthermore, the specific form (15),(16) of the closed system of the filtering equations corresponding to a third-order state is derived. In the next section, performance of the designed optimal filter for a third-order state over incomplete linear observations is verified against a conventional mean-square filter for stochastic polynomial systems with Gaussian noises, obtained in [35].

IV. Example

This section presents an example of designing the optimal filter for a third-order bi-dimensional state and for over scalar linear observations and comparing it to a conventional mean-square filter for stochastic polynomial systems with Gaussian noises [35].

Let the bi-dimensional real state $x(t)$ satisfy the third-order system

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10}, \tag{17}$$
$$\dot{x}_2(t) = 0.1x_2^3(t) + \psi_1(t), \quad x_2(0) = x_{20},$$

and the scalar observation process be given by the linear equation

$$y(t) = x_1(t) + \psi_2(t), \tag{18}$$

where $\psi_1(t)$ and $\psi_2(t)$ are white Poisson noises, which are the weak mean square derivatives of standard Poisson process (see [25]). The equations (17),(18) present the conventional form for the equations (1),(2), which is actually used in practice [36], [37].

The filtering system (17),(18) includes two state components $x(t) = [x_1(t), x_2(t)]^T \in R^2$ and only one observation channel $y(t) \in R$, measuring the state component $x_1(t)$. Note that the observation matrix $A = [1\ 0] \in R^{(1\times 2)}$ is non-square and, therefore, non-invertible. Moreover, the state nonlinear component $x_2(t)$ is unmeasured. The filtering problem is to find the optimal estimate for the third-order state (17), using incomplete linear observations (18) confused with independent randomly driven isolated disturbances modeled as Poisson white noises.

Let us show how to calculate the coefficients of the vector polynomial (3) for the system (17). Indeed, the matrix coefficient a_1 is a 2×2 - matrix, equal to $a_1 = [0\ 1 \mid 0\ 0]$, the 3D tensor coefficient a_2 consists of zeros only, since the quadratic or bilinear terms are absent in (17), and the 4D tensor coefficient a_3 has only one non-zero entry, $a_{3\ 2222} = 0.1$, whereas its other entries are zeros. Therefore, according to (15),(16), this only non-zero term should enter the equation for m_2, multiplied by $3m_2P_{22} + m_2^3 + P_{22}$, the equation for $P_{21} = P_{12}$, multiplied by $3m_2^2P_{21} + 3P_{22}P_{21} + P_{22} + 3m_2P_{22} = 3m_2^2P_{12} + 3P_{22}P_{12} + P_{22} + 3m_2P_{22}$, in view of symmetry of the variance matrix P, and the equation for P_{22}, multiplied by $2P_{22} + 6P_{22}^2 + 6m_2P_{22} + 6m_2^2P_{22}$.

As a result, the filtering equations (15),(16) take the following particular form for the system (17),(18)

$$\dot{m}_1(t) = m_2(t) + P_{11}(t)[y(t) - m_1(t)], \tag{19}$$

$$\dot{m}_2(t) = 0.1m_2^3(t) + 0.3P_{22}(t)m_2(t) + 0.1P_{22}(t) + P_{12}(t)[y(t) - m_1(t)],$$

with the initial condition $m(0) = E(x(0) \mid y(0)) = m_0$,

$$\dot{P}_{11}(t) = 2P_{12}(t) - P_{11}^2(t), \tag{20}$$

$$\dot{P}_{12}(t) = 1.1P_{22}(t) + 0.3m_2^2(t)P_{12}(t) + 0.3m_2(t)P_{22}(t) + 0.3P_{22}(t)P_{12}(t) - P_{11}(t)P_{12}(t),$$

$$\dot{P}_{22}(t) = 1 + 0.2P_{22}(t) + 0.6m_2^2(t)P_{22}(t) + 0.6m_2(t)P_{22}(t) + 0.6P_{22}^2(t) - P_{12}^2(t),$$

with the initial condition $P(0) = E((x(0) - m(0))(x(0) - m(0))^T \mid y(0)) = P_0$.

The estimates obtained upon solving the equations (19)–(20) are compared to the estimates satisfying the conventional mean-square polynomial filter equations for the third-order state (17) over the incomplete linear observations (18) (see [35]):

$$\dot{m}_{k1}(t) = m_{k2}(t) + P_{k11}(t)[y(t) - m_{k1}(t)], \tag{21}$$

$$\dot{m}_{k2}(t) = 0.1m_{k2}^3(t) + 0.3P_{k22}(t)m_{k2}(t) + P_{k12}(t)[y(t) - m_{k1}(t)],$$

with the initial condition $m(0) = E(x(0) \mid y(0)) = m_0$,

$$\dot{P}_{k11}(t) = 2P_{k12}(t) - P_{k11}^2(t), \tag{22}$$

$$\dot{P}_{k12}(t) = P_{k22}(t) + 0.3m_{k2}^2(t)P_{k12}(t) + 0.3P_{k22}(t)P_{k12}(t) - P_{k11}(t)P_{k12}(t),$$

$$\dot{P}_{k22}(t) = 1 + 0.6m_{k2}^2(t)P_{k22}(t) + 0.6P_{k22}^2(t) - P_{k12}^2(t),$$

Numerical simulation results are obtained solving the systems of filtering equations (19)–(20), and (21)–(22). The obtained values of the estimates $m_1(t)$, $m_2(t)$, $m_{k1}(t)$, and $m_{k2}(t)$ satisfying the equations (19), and (21), respectively, are compared to the real values of the state variables $x_1(t)$ and $x_2(t)$ in (17).

For each of the two filters (19)–(20) and (21)–(22), and the reference system (17)–(18), involved in simulation, the following initial values are assigned: $x_{10} = -2.5$, $x_{20} = -0.35$, $m_{10} = -14.6$, $m_{20} = -1.38$, $P_{110} = 20$, $P_{120} = 0.9$, $P_{220} = 0.06$. Realizations of white Poisson noises $\psi_1(t)$ and $\psi_2(t)$ in (21) are generated using the Simulink chart suggested in [38].

The following graphs are obtained: graphs of the errors between the reference state components $x_1(t)$ and $x_2(t)$, satisfying the equations (17), and the optimal filter estimate components $m_1(t)$ and $m_2(t)$, satisfying the equations (19), are shown in Figs. 1 and 2; graphs of the errors between the reference state components $x_1(t)$ and $x_2(t)$, satisfying the equations (17), and the conventional mean-square polynomial filter estimate components $m_{k1}(t)$ and $m_{k2}(t)$, satisfying the equations (21), are shown in Figs. 3 and 4. It can be observed that the estimation error given by the optimal filter rapidly reaches and then maintains near-zero values. This presents a definitive advantage of the designed optimal filter. On the contrary, the estimation error given by the conventional mean-square polynomial filter diverges to infinity at $T = 1.7842$.

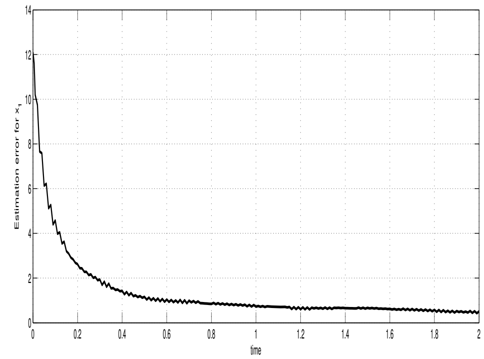

Fig. 1. Graph of the error between the real state $x_1(t)$, satisfying (17), and the optimal filter estimate $m_1(t)$, satisfying (19), in the entire simulation interval $[0,2]$.

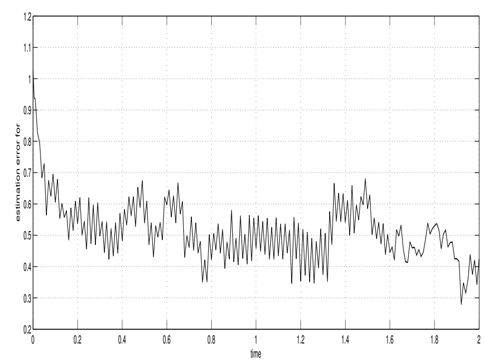

Fig. 2. Graph of the error between the real state $x_2(t)$, satisfying (17), and the optimal filter estimate $m_2(t)$, satisfying (19), in the entire simulation interval $[0,2]$.

Note that the optimal filtering error variance $P(t)$ does not converge to zero as time tends to the asymptotic time point, since the polynomial dynamics of third order is stronger than the quadratic Riccati terms in the right-hand side of the equations (20).

Thus, it can be concluded that the obtained optimal filter (19)–(20) for a bi-dimensional third-order state over incomplete linear observations yields definitely better estimates than the conventional filter for polynomial systems with Gaussian noises.

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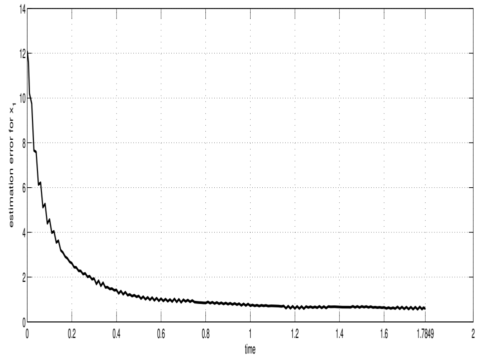

Fig. 3. Graph of the error between the real state $x_1(t)$, satisfying (17), and the estimate $m_{k1}(t)$, satisfying (21), in the simulation interval $[0, 1.7842]$.

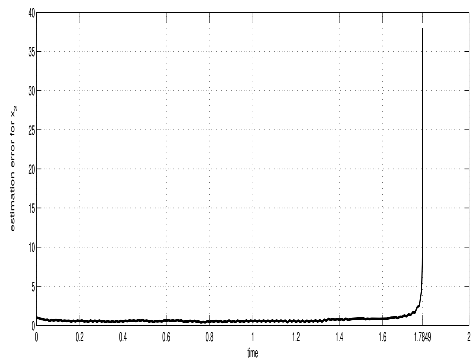

Fig. 4. Graph of the error between the real state $x_2(t)$, satisfying (17), and the estimate $m_{k2}(t)$, satisfying (21), in the simulation interval $[0, 1.7842]$.

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