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Performances inclusion for stable interval systems

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***Abstract*— This paper presents the performances inclusion on time and frequency domains of SISO stable interval systems. We demonstrate that an interval transfer function included in another interval transfer function will have its performances also included in those of the second one. While the results may be intuitive, the paper provides an analytical demonstration by using interval arithmetic and related tools. These results are of great interest for robust performances analysis and for controller design in parametric uncertain systems.**

I. Introduction

The first analysis of interval systems goes back to the works of *Kharitonov* [1][2]. It states that the robust stability of a real (resp. complex) interval polynomial can be deduced from the Hurwitz stability of four (resp. eight) vertex polynomials. Since this primary work, extensions of the robust stability analysis have been emerging. For instance, *Barlett et al.* [3] extend the Kharitonov's theorem to the edge theorem to state the stability of polytope of polynomials. Further, *Wang et al.* [4] present the multivariable edge theorem. In [5], *Jaulin et al.* present the δ-stability condition while in [6] *Dahleh et al.* adapt the Popov criterion, both for interval systems. Finally, the extension of the Kharitonov theorem for nonlinear systems is presented by *Chapellat et al.* [7]. Many works also report some elementary tools that may be useful to complete the robust stability: the algorithm to solve a set inversion problem in interval functions [8], the H_∞ norm [9] and the envelop of the Nyquist plots [10] all for interval systems.

More than the stability, a rigorous aim of the robust control design is to maintain the performances in presence of model uncertainty. Hence, the robust perfomances analysis for interval systems can be seen as the natural continuation of the robust stability analysis. The first works considering the robust performances for these systems date back to 1992 when *Dahleh et al* [11] synthesize a controller using thirty two point systems. Point systems mean systems whose parameters are point, not intervals. Much later, *Okuyama and Takemori* [12] provide a sufficient condition such that the roots of the characteristic polynomial are contained in a given circular area that can be linked to specified performances. In [13], *Bondia and Pico* employ a geometric approach to give a condition for the robustness of a closed-loop transfer to satisfy the specified time domain performances. The performances analysis in the latter approach were limited to first and second order reference models. More recently [14], robust temporal performances for n^{th}-order transfer functions were addressed. However, the considered numerator was limited to 0-degree. In this paper, we study the robust performances of generalized interval transfer functions. The study is performed both for the time and the frequency domains by using interval arithmetic and related algebraic tools. We especially prove that two interval systems with inclusion relation also have inclusion relation on their (temporal and frequential) performances. For as much as we treat the performances, we assume that the analyzed interval systems are stable. The results are more general than the previous works [12][13][14] since generalized structure of interval systems are considered and both time and frequency domains are treated. These results are useful for an *a posteriori* robustness analysis, or for the design of robust controllers when having systems with parametric uncertainties.

The paper is organized as follows. In section-II, we remind the real and complex interval arithmetic. We also introduce the notations used for interval systems. In section-III, we present the first fundamental result of the paper, which concerns the performances inclusion in the frequency domain. In section-IV, the second fundamental result is presented: the performances inclusion in the time domain. Finally, we present an illustrative example in section-V.

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II. Preliminary

A. Real interval arithmetic and functions

The real interval arithmetic and functions summarized here are based on [5][15][16].

A real (closed) interval $[x]$ is a connected, closed subset of $\mathbb{R}$. It is characterized by a lower bound x^- and an upper bound x^+ such as for all $x \in [x]$, we have $x^- \leq x \leq x^+$. We say that interval $[x] = [x^-, x^+]$ is degenerate iff $x^- = x^+$. By convention, a degenerate interval $[a, a]$ is identified by the real (point) number a. In the sequel, the designation *point* is alternately used to signify *degenerate interval*. The set of real intervals is denoted by $\mathbb{IR}$.

Given two intervals $[x] = [x^-, x^+]$ and $[y] = [y^-, y^+]$, so:

$$[x] + [y] = \left[x^- + y^-, x^+ + y^+\right] \tag{1}$$

$$[x] - [y] = \left[x^- - y^+, x^+ - y^-\right] \tag{2}$$

If $[y]=[x]$, we have: $[x]-[x]=[x^--x^+,x^+-x^-]\neq 0$, except if $x^-=x^+$.

The multiplication and subdivision are:

$$[x].[y]=\left[\min\left(x^-y^-,x^-y^+,x^+y^-,x^+y^+\right),\ \max\left(x^-y^-,x^-y^+,x^+y^-,x^+y^+\right)\right] \tag{3}$$

and

$$[x]\,/\,[y]=[x].[1/y^+,1/y^-],\quad 0\notin\ [y] \tag{4}$$

An interval $[x]$ is said to be included in an interval $[y]$, i.e. $[x]\subset[y]$, iff $[x]\cap[y]=[x]$. We have $[x]>[y]$ if $x^->y^+$. The real interval $[x]$ is positive if $x^->0$. The interval numbers $[0,0]=0$ and $[1,1]=1$ serve as additive and multiplicative identities, respectively. Interval arithmetic is associative and commutative with respect to addition and multiplication. The distributive law for interval does not hold in general. However, the following relation, called subdistributivity, holds: $[x]([y]+[z])\subseteq[x][y]+[x][z]$. Furthermore, if $[x]+[y]=[x]+[z]$, the cancellation law for addition holds, and $[y]=[z]$. The same property holds for multiplication: if $[x][y]=[x][z]$ and $0\notin[x]$, thus $[y]=[z]$. We have the following property:

$$[x]\subseteq[y]\quad\Leftrightarrow\quad a[x]\subseteq a[y] \tag{5}$$

where a is a real point number.

If f is a function $f:\mathbb{R}\rightarrow\mathbb{R}$, then its interval counterpart $[f]$ satisfies:

$$[f]([x])=[\{f(x):x\in[x]\}] \tag{6}$$

The interval function $[f]$ is called inclusion function because $f([x])\subseteq[f]([x])$, for all $[x]\in\mathbb{IR}$. An inclusion function $[f]$ is thin if for any degenerate interval $[x]=x$, $[f](x)=f(x)$. It is minimal if for any $[x]$, $[f]([x])$ is the smallest interval that contains $f([x])$. The minimal inclusion function for f is unique and denoted by $[f]^*([x])$. An easy way to compute an inclusion function for f is to replace in the expression of f all x by $[x]$ and all operations on points by their interval counterpart. Thus, one obtains the natural inclusion function.

B. Complex interval arithmetic: the rectangular form

Three forms exist to represent complex interval numbers: 1) the circular form [17][18] which uses a complex point number and a radius, 2) the polar form [19] which is an extension of the polar form of complex point number, 3) and the rectangular form [18][20][21] whose the real and imaginary parts are interval. It is obvious that when transforming a Laplace transfer function into frequential transfer function (using $s=j\omega$), we obtain the rectangular form. Therefore, in the sequel we give a preliminary of the latter and afterwards we employ it.

A complex interval number $[I]$ is characterized by an ordered pair of interval real number $([A],[B])$, such as $[I]=[A]+[B]j=[a^-,a^+]+[b^-,b^+]j$. The negative of I is $-[I]=-[A]-[B]j$ and its conjuguate is $[I]^*=[A]-[B]j$.

Let $[I]=[A_i]+[B_i]j=\left[a_i^-,a_i^+\right]+\left[b_i^-,b_i^+\right]j$ and $[J]=[A_j]+[B_j]j=\left[a_j^-,a_j^+\right]+\left[b_j^-,b_j^+\right]j$ two complex interval numbers, so:

$$[I]+[J]=([A_i]+[A_j])+([B_i]+[B_j])j \tag{7}$$

$$[I]-[J]=([A_i]-[A_j])+([B_i]-[B_j])j \tag{8}$$

$$[I]\cdot[J]=([A_i][A_j]-[B_i][B_j])+([A_i][B_j]+[B_i][A_j])j \tag{9}$$

The following definitions are provided for complex interval numbers [20].

$$[I]+[I]^*=2\cdot[A_i]+0j \tag{10}$$

$$[I]\cdot[I]^*=\left([A_i]^2+[B_i]^2\right)+0j \tag{11}$$

$$[I]\,/\,[J]=\left(\frac{[A_i][A_j]+[B_i][B_j]}{[A_j]^2+[B_j]^2}\right)+\left(\frac{[B_i][A_j]-[A_i][B_j]}{[A_j]^2+[B_j]^2}\right)j \tag{12}$$

The division defined above yields a complex interval that is generally far too pessimist. Other definitions of division were therefore introduced, see [22][23]. Because their properties, which are more interesting than the definition in this paper, are equivalent we will use the above definition.

If we have two complex intervals $[I]$ and $[J]$, so $[I]\subseteq[J]$ means $[A_i]\subseteq[A_j]$ and $[B_i]\subseteq[B_j]$.

Lemma 2.1: Let a be a real point number. If $[I]\subseteq[J]$, then $a[I]\subseteq a[J]$.

Proof: Since $a[A_i]\subseteq a[A_j]$ and $a[B_i]\subseteq a[B_j]$ (according to Eq. 5), therefore $a[I]\subseteq a[J]$. ■

C. Functions of (rectangular) complex interval

Let $[I]$, $[J]$, $[K]$ and $[L]$ be complex intervals such that $[I]\subseteq[K]$ and $[J]\subseteq[L]$.

Lemma 2.2: $[I]\otimes[J]\subseteq[K]\otimes[L]$ for $\otimes\in\{+,-,\cdot,/\}$.

Proof: See [20]. ■

Theorem 2.1 (Containment theorem): Let $f(x_1,x_2,\ldots,x_n)$ be a rational expression in the point variables $x_1,x_2,\ldots,x_n$. If $[X_1]\subseteq[Y_1]$, $[X_2]\subseteq[Y_2]$, …, $[X_n]\subseteq[Y_n]$ are complex interval variables, then all inclusion functions $[f]$ counterpart of f verify

$$[f]([X_1],[X_2],\ldots,[X_n])\subseteq[f]([Y_1],[Y_2],\ldots,[Y_n])$$

Proof: See [20]. ■

Remark 2.1: Theorem 2.1 holds when $[X_i]$ and $[Y_i]$ are complex intervals. Therefore, it also holds for real intervals.

Corollary 2.1: Let $f(x_1,x_2,\ldots,x_n,t_1,t_2,\ldots,t_m)$ be a rational expression of the point variables $x_1,x_2,\ldots,x_n,t_1,t_2,\ldots,t_m$. If $[X_1]\subseteq[Y_1]$, $[X_2]\subseteq[Y_2]$, …, $[X_n]\subseteq[Y_n]$, then

$$[f]([X_1],[X_2],\ldots,[X_n],t_1,t_2,\ldots,t_m) \subseteq [f]([Y_1],[Y_2],\ldots,[Y_n],t_1,t_2,\ldots,t_m)$$

Proof: Rewriting t_i $(i\in\{1,\ldots,m\})$ by $[T_i]$ such as $[T_i]=[t_i,t_i]=t_i$ is a degenerate real interval number and knowing that $[T_i]\subseteq[T_i]$, we apply the Theorem 2.1 and derive the results in Corollary 2.1. ■

Notations: for a compact notation, we will use: $[X] = \begin{pmatrix}[X_1] & [X_2] & \ldots & [X_n]\end{pmatrix}$ and $[Y] = \begin{pmatrix}[Y_1] & [Y_2] & \ldots & [Y_n]\end{pmatrix}$. Therefore, $[X]\subseteq[Y]$ means that the inclusion relation holds for each element.

Lemma 2.3: If $[X]\subseteq[Y]$ and k an independent variable, then $\sum\limits_{k=\alpha}^{\beta}[f]([X],k)\subseteq\sum\limits_{k=\alpha}^{\beta}[f]([Y],k)$, for any $\alpha\le\beta$ and $\alpha,\beta\in\mathbb{Z}$.

Proof: Corollary 2.1 expresses that $[f]([X],k)\subseteq[f]([Y],k)$. Since $[f]([X],k)\subseteq[f]([X],k)$ and $[f]([Y],k)\subseteq[f]([Y],k)$, we can apply Lemma 2.2 to prove that $[f]([X],\alpha)+\ldots+[f]([X],\beta)\subseteq[f]([Y],\alpha)+\ldots+[f]([Y],\beta)$.

■

Theorem 2.2: If $[X]\subseteq[Y]$, t an independent variable and f a Rieman integrable function of t, then: $\int\limits_{t_o}^{t_f}[f]([X],t)\,dt\subseteq\int\limits_{t_o}^{t_f}[f]([Y],t)\,dt$

Proof: Replace k in Lemma 2.3 by $t_o+k\frac{t_f-t_o}{N}$ such as $N=\beta-\alpha+1>0$ and multiply the left and the right terms by the positive point number $\frac{t_f-t_o}{N}$, so the following inclusion still holds according to Lemma 2.2: $\frac{t_f-t_o}{N}\sum\limits_{k=1}^{N}[f]\left([X],t_o+k\frac{t_f-t_o}{N}\right)\subseteq\frac{t_f-t_o}{N}\sum\limits_{k=1}^{N}[f]\left([Y],t_o+k\frac{t_f-t_o}{N}\right)$ If the original function f is Riemann integrable along t, therefore the previous Riemann sum leads to the result of Theorem 2.2 when $N\to\infty$. ■

Theorem 2.3: If $[X]\subseteq[Y]$, t_i $(i\in\{1,\ldots,m\})$ are independent variables and f a Rieman integrable function of t_i, then: $\int\int\ldots\int[f]([X],t_1,t_2,\ldots,t_m)\,dt_1dt_2\ldots dt_m\subseteq$

$\int\int\ldots\int[f]([Y],t_1,t_2,\ldots,t_m)\,dt_1dt_2\ldots dt_m$

Proof: To prove Theorem 2.3, it suffices to use the Theorem 2.2, by noting that $[f_{X1}]=\int[f]([X],t_1,t_2,\ldots,t_m)\,dt_1\subseteq[f_{Y1}]=\int[f]([Y],t_1,t_2,\ldots,t_m)\,dt_1$ and repeating the demonstration until $f_{Xm}\subseteq f_{Ym}$, where $[f_{Xi}]=\int[f_{Xi-1}]\,dt_{i-1}$ and $[f_{Yi}]=\int[f_{Yi-1}]\,dt_{i-1}$ $(i\in\{1,\ldots,m\})$. ■

D. Modulus and argument of (rectangular) complex intervals

Let $[I]=[A]+[B]\,j$ be a rectangle complex interval.

Definition 2.1: We define by $[\rho]([I])=\left[\sqrt{[A]^2+[B]^2}\right]$ the inclusion modulus of $[I]$. This modulus corresponds to any interval containing $\sqrt{[A]^2+[B]^2}$ and is positive real interval. The minimal inclusion modulus denoted by $[\rho]^*([I])$, which is unique, corresponds to the smallest interval that contains $\sqrt{[A]^2+[B]^2}$.

Definition 2.2: We define by $[\varphi]([I])=\left[atan\left(\frac{[B]}{[A]}\right)\right]$ the inclusion argument of $[I]$. This argument corresponds to any interval containing $atan\left(\frac{[B]}{[A]}\right)$ and is real interval. The minimal inslusion argument denoted by $[\varphi]^*([I])$, which is unique, corresponds to the smallest interval that contains $atan\left(\frac{[B]}{[A]}\right)$

Let $[I]=[A_i]+[B_i]\,j$ and $[J]=[A_j]+[B_j]\,j$ be two rectangle complex interval numbers .

Lemma 2.4: If $[I]\subseteq[J]$, then $[\rho]([I])\subseteq[\rho]([J])$

Proof: Since $[A_i]\subseteq[A_j]$ and $[B_i]\subseteq[B_j]$, therefore $[A_i]^2+[B_i]^2\subseteq[A_j]^2+[B_j]^2$ according to Remark 2.1. The root square function being increasing monotonic on $\mathbb{R}^+$, the following function property is held for any subset: $\rho([I])=\sqrt{[A_i]^2+[B_i]^2}\subseteq\rho([J])=\sqrt{[A_j]^2+[B_j]^2}$. As the two related inclusion functions verify $[\rho]([I])\subseteq[\rho]([J])$, we demonstrate Lemma 2.4.

■

Lemma 2.5: If $[I]\subseteq[J]$, then $[\varphi]([I])\subseteq[\varphi]([J])$

Proof: Since $[A_i]\subseteq[A_j]$ and $[B_i]\subseteq[B_j]$, therefore $\frac{[B_i]}{[A_i]}\subseteq\frac{[B_j]}{[A_j]}$ according to Remark 2.1. The arctangent function being increasing monotonic on $\mathbb{R}$, the following function property is held for any subset: $\varphi([I])=atan\left(\frac{[B_i]}{[A_i]}\right)\subseteq\varphi([J])=atan\left(\frac{[B_j]}{[A_j]}\right)$. Once again, the related inclusion functions verify $[\varphi]([I])\subseteq[\varphi]([J])$, and therefore Lemma 2.5 is proven. ■

E. Interval systems

All along the paper, we shall be interested by transfer functions, and the state-space, the differential or the Rosenbrock representations shall not be considered. Furthermore, stable interval transfer functions are considered. The reason is that we study robust performances, which implicitely implies the stability.

Definition 2.3: We define by interval system denoted by $[G](s)$, s being the Laplace variable, a system whose the parameters are intervals:

$$[G](s)=\frac{[b_m]s^m+\ldots+[b_1]s^1+[b_0]}{[a_n]s^n+\ldots+[a_1]s^1+[a_0]}=\frac{\sum\limits_{l=0}^{m}[b_l]s^l}{\sum\limits_{k=0}^{n}[a_k]s^k}$$

The parameters $[a_k]$ and $[b_l]$ are considered to be constant real interval in order to assume linear time invariant (LTI) systems. The notations $[G](s)$ shall be used if the intervals $[a_k]$ and $[b_l]$ are known. Instead, the notation $[G]([a_k],[b_l],s)$ is used when they are unknown and to be sought for.

The notion of inclusion of systems should be defined. Consider two interval systems having the same polynomials degrees m and n:

$$[G_1(s)] = \frac{\sum_{l=0}^{m} [b_{1l}]\cdot(j\omega)^l}{\sum_{k=0}^{n} [a_{1k}]\cdot(j\omega)^k} \quad , \quad [G_2(s)] = \frac{\sum_{l=0}^{m} [b_{2l}]\cdot(j\omega)^l}{\sum_{k=0}^{n} [a_{2k}]\cdot(j\omega)^k} \tag{13}$$

Definition 2.4: $[G_1](s) \subseteq [G_2](s)$ means that for any $s \in [0, \infty)$, we have $[G_1] \subseteq [G_2]$.

Lemma 2.6: If $[b_{1l}] \subseteq [b_{2l}]$ and $[a_{1k}] \subseteq [a_{2k}]$, $\forall k, l$, then $[G_1](s) \subseteq [G_2](s)$.

Proof: Consider a rational function $G(b_1, b_2, \ldots, b_m, s) = \frac{\sum_{l=0}^{m} b_l s^l}{\sum_{k=0}^{n} a_k s^k}$ of the point variables $b_1, b_2, \ldots, b_m, s$. Thus, by using Corollary 2.1, the proof is straightforward. ■

III. Frequential performances inclusion

Performances of systems, of loop transfers and of closed-loop can be defined in the frequency domain. For instance, in the H_∞ control design techniques, upper bounds of the modulus are used to define the maximal settling times, statical errors and overshoots. In this section, we give the inclusion relation in the frequency domain (modulus and argument) of interval systems. The results are fundamental for the robust analysis and control design.

A. Inclusion theorem for the modulus (magnitude or gain)

First, compute the frequential transfer of an interval system. For that, we consider the interval system $[G](s)$ as defined in Definition 2.3. The frequential transfer is obtained by using $s = j\omega$:

$$[G](j\omega) = \frac{\sum_{l=0}^{m} [b_l]\cdot(j\omega)^l}{\sum_{k=0}^{n} [a_k]\cdot(j\omega)^k} \tag{14}$$

which can be rewritten:

$$[G](j\omega) = \frac{[A_{num}] + [B_{num}]\, j}{[A_{den}] + [B_{den}]\, j} \tag{15}$$

such as the real and imaginary parts of the numerator are:

$$[A_{num}] = \begin{cases} \sum_{l=0}^{\frac{m}{2}} \left[b_{(2l)}\right](-1)^l(\omega)^{2l} & if \quad m : even \\ \\ \sum_{l=0}^{\frac{m-1}{2}} \left[b_{(2l)}\right](-1)^l(\omega)^{2l} & elseif \end{cases} \tag{16}$$

and

$$[B_{num}] = \begin{cases} \sum_{l=0}^{\frac{m}{2}-1} \left[b_{(2l+1)}\right](-1)^l(\omega)^{2l+1} & if \quad m : even \\ \\ \sum_{l=0}^{\frac{m-1}{2}} \left[b_{(2l+1)}\right](-1)^l(\omega)^{2l+1} & elseif \end{cases} \tag{17}$$

The real part $[A_{den}]$ and imaginary part $[B_{den}]$ of the denominator have the same structure than Eq. 16 and Eq. 17 respectively. It suffices to replace coefficients $b_{(2l)}$ (resp. $b_{(2l+1)}$) by $a_{(2k)}$ (resp. $a_{(2k+1)}$), l by k and m by n.

Lemma 3.1: Consider the two interval systems defined by Eq. 13.

If $[b_{1l}] \subseteq [b_{2l}]$ and $[a_{1k}] \subseteq [a_{2k}]$, then $[G_1](j\omega) \subseteq [G_2](j\omega)$.

Proof: We apply the real interval property as in Eq. 5 to $[b_{1l}] \subseteq [b_{2l}]$ and $[a_{1k}] \subseteq [a_{2k}]$ and we obtain:

$$\left[b_{1(2l)}\right](-1)^l(\omega)^{2l} \subseteq \left[b_{2(2l)}\right](-1)^l(\omega)^{2l}$$
$$\left[b_{1(2l+1)}\right](-1)^l(\omega)^{2l+1} \subseteq \left[b_{2(2l+1)}\right](-1)^l(\omega)^{2l+1}$$
$$\left[a_{1(2k)}\right](-1)^k(\omega)^{2k} \subseteq \left[b_{2(2k)}\right](-1)^k(\omega)^{2k}$$
$$\left[a_{1(2k+1)}\right](-1)^k(\omega)^{2k+1} \subseteq \left[a_{2(2k+1)}\right](-1)^k(\omega)^{2k+1}$$

Applying Remark 2.1 to the previous inclusions, we have:

$$\begin{array}{l} [A_{1num}] \subseteq [A_{2num}] \\ [B_{1num}] \subseteq [B_{2num}] \\ [A_{1den}] \subseteq [A_{2den}] \\ [B_{1den}] \subseteq [B_{2den}] \end{array}$$

where $[A_{inum}]$ and $[B_{inum}]$ are the real and imaginary parts respectively of the numerator of system i ($i \in \{1, 2\}$), and $[A_{iden}]$ and $[B_{iden}]$ their counterparts in the denominator. They are defined by Eq. 16 and Eq. 17.

As a result, we have $[A_{1num}] + [B_{1num}]\, j \subseteq [A_{2num}] + [B_{2num}]\, j$ and $[A_{1den}] + [B_{1den}]\, j \subseteq [A_{2den}] + [B_{2den}]\, j$. Finally, using Lemma 2.2 we have: $\frac{[A_{1num}]+[B_{1num}]j}{[A_{1den}]+[B_{1den}]j} \subseteq \frac{[A_{2num}]+[B_{2num}]j}{[A_{2den}]+[B_{2den}]j}$ and therefore $[G_1](j\omega) \subseteq [G_2](j\omega)$. ■

Theorem 3.1: Let $[G_1](s)$ and $[G_2](s)$ two interval systems as defined by Eq. 13.

$$if \begin{cases} [a_{1k}] \subseteq [a_{2k}], & \forall k = 1 \cdots n \\ and & \\ [b_{1l}] \subseteq [b_{2l}], & \forall l = 1 \cdots m \end{cases}$$

$$\Rightarrow \quad [\rho]([G_1](j\omega)) \subseteq [\rho]([G_2](j\omega))$$

Proof: From $[b_{1l}] \subseteq [b_{2l}]$ and $[a_{1k}] \subseteq [a_{2k}]$, we have $[G_1](j\omega) \subseteq [G_2](j\omega)$ according to Lemma 3.1. Using Lemma 2.4, we conclude that $[\rho]([G_1](j\omega)) \subseteq [\rho]([G_2](j\omega))$. ■

B. Inclusion theorem for the argument (phase)

Afterwards, let us decompose $[G](s)$ into the multiplication of many first orders systems. Assuming that it is always possible to find imaginary interval roots for any given interval polynomial, it is possible to find imaginary zeros and poles for $[G](s)$. The system defined in Definition 2.3 can be therefore rewritten as follows:

$$[G](s) = \frac{\prod_{l=0}^{m}(s+[Z_l])}{\prod_{k=0}^{n}(s+[P_k])} \tag{18}$$

where $[P_k]$ and $[Z_l]$ are the poles and zeros respectively. They are complex interval numbers.

Finally, the frequential transfer corresponding to Eq. 18 is:

$$[G](j\omega) = \frac{\prod_{l=0}^{m}(j\omega+[Z_l])}{\prod_{k=0}^{n}(j\omega+[P_k])} \tag{19}$$

Lemma 3.2: Consider the two interval systems defined by Eq. 13.

If $[b_{1l}] \subseteq [b_{2l}]$, and $[a_{1l}] \subseteq [a_{2l}]$, therefore $[G_1](j\omega) \subseteq [G_2](j\omega)$.

Proof: Every interval arithmetic operation applied to $[b_{1l}]$ and $[a_{1l}]$ may be associated with an interval arithmetic operation to $[b_{2l}]$ and $[a_{2l}]$. Especially, the computation of $[P_{2l}]$ and $[Z_{2l}]$ exactly uses the same computation than $[P_{1l}]$ and $[Z_{1l}]$. Therefore, we deduce that $[Z_{1l}] \subseteq [Z_{2l}]$ and $[P_{1l}] \subseteq [P_{2l}]$. Using the latter deduction, knowing that $j\omega = [j\omega, j\omega] \subseteq j\omega = [j\omega, j\omega]$ and applying Theorem 2.1 (or also Lemma 2.2) to Eq. 19, we demonstrate $[G_1](j\omega) \subseteq [G_2](j\omega)$. ∎

Theorem 3.2: Let $[G_1](s)$ and $[G_2](s)$ two interval systems as defined by Eq. 13.

$$if \begin{cases} [a_{1k}] \subseteq [a_{2k}], & \forall k = 1\cdots n \\ and & \\ [b_{1l}] \subseteq [b_{2l}], & \forall l = 1\cdots m \end{cases}$$

$$\Rightarrow \quad [\varphi]([G_1](j\omega)) \subseteq [\varphi]([G_2](j\omega))$$

Proof: From $[b_{1l}] \subseteq [b_{2l}]$ and $[a_{1k}] \subseteq [a_{2k}]$, we derive $[G_1](j\omega) \subseteq [G_2](j\omega)$ according to Lemma 3.2. So using Lemma 2.5, we conclude that $[\varphi]([G_1](j\omega)) \subseteq [\varphi]([G_2](j\omega))$. ∎

C. The frequential performances inclusion theorem

Bringing together Theorem 3.1 and Theorem 3.2, we have:

Theorem 3.3: Let $[G_1](s)$ and $[G_2](s)$ two interval systems as defined by Eq. 13.

$$if \begin{cases} [a_{1k}] \subseteq [a_{2k}], & \forall k = 1\cdots n \\ and & \\ [b_{1l}] \subseteq [b_{2l}], & \forall l = 1\cdots m \end{cases}$$

$$\Rightarrow \begin{cases} [\rho]([G_1](j\omega)) \subseteq \rho([G_2](j\omega)) \\ and \\ [\varphi]([G_1](j\omega)) \subseteq \varphi([G_2](j\omega)) \end{cases}$$

Theorem 3.3 constitutes the first main result of this paper. It states that for two systems such that their parameters are linked by the inclusion relation, their frequential performances will also be linked by the inlcusion relation. These frequential performances are given in the Bode, Nyquist or Black-Nichols diagram.

IV. Temporal performances inclusion

In this section, we shall present the time domain counterpart of the previous frequency domain analysis.

Reconsider the interval system $[G](s)$ as defined in Definition 2.3. So, its impulse response denoted by $[g](t)$ is given by:

$$[g](t) = \left[\mathcal{L}^{-1}\{[G](s)\}\right] = \left[\frac{1}{2\pi j}\int_{-\infty}^{+\infty}[G](s)e^{st}ds\right] \tag{20}$$

where $\mathcal{L}^{-1}\{[G](s)\}$ is the inverse Laplace transform of $[G](s)$.

Theorem 4.1: Let $[G_1](s)$ and $[G_2](s)$ two interval systems as defined by Eq. 13.

$$if \begin{cases} [a_{1k}] \subseteq [a_{2k}], & \forall k = 1\cdots n \\ and & \\ [b_{1l}] \subseteq [b_{2l}], & \forall l = 1\cdots m \end{cases}$$

$$\Rightarrow \quad [g_1](t) \subseteq [g_2](t)$$

Proof: From Lemma 2.6, we have $[G_1](s) \subseteq [G_2](s)$. As e^{st} is a real point number for any s and t, therefore: $[G_1](s)e^{st} \subseteq [G_2](s)e^{st}$ accordingly to Eq. 5. Using Theorem 2.2, we obtain

$\int_{-\infty}^{+\infty}[G_1](s)e^{st}ds \subseteq \int_{-\infty}^{+\infty}[G_2](s)e^{st}ds$

However, since

$\frac{1}{2\pi j} = [0,0] + \left[\frac{-1}{2\pi}, \frac{-1}{2\pi}\right]j \subseteq \frac{1}{2\pi j} = [0,0] + \left[\frac{-1}{2\pi}, \frac{-1}{2\pi}\right]j$

we deduce, by using Lemma 2.2, that

$\frac{1}{2\pi j}\int_{-\infty}^{+\infty}[G_1](s)e^{st}ds \subseteq \frac{1}{2\pi j}\int_{-\infty}^{+\infty}[G_2](s)e^{st}ds.$

Therefore: $[g_1](t) \subseteq [g_2](t)$. ∎

Theorem 4.1 constitutes the second main result of this paper. It states that for two systems such that their parameters are linked by the inclusion relation, their time domain performances will also be linked by the inlcusion relation. These time domain performances are often defined and/or given with the impulse and step responses.

V. Illustrative example

Consider the following two interval systems:

$$[G_1](s) = \frac{[20.82, 20.83]}{s^2 + [5, 5.2]s + [15.98, 16.56]} \tag{21}$$

and

$$[G_2](s) = \frac{[20.8, 20.85]}{s^2 + [4.8, 5.6]s + [15.37, 17.76]} \tag{22}$$

The system $[G_2]$ may represent a reference model whose the time-rseponse, the overshot and the static error are defined by bounds. The system $[G_1]$ represents a closed-loop system which includes a plant (whose parameters are uncertain and bounded by intervals) and a designed controller. Therefore, it is possible to analyze a posteriori if the controller inside the closed-loop $[G_1]$ ensures the performances.

According to Lemma 2.6, we have $[G_1](s) \subseteq [G_2](s)$. So, Theorem 3.3 and Theorem 4.1 predict that the controller will efficiently ensure the (frequential and temporal) performances of the closed-loop $[G_1]$.

If we plot the bode diagram of the two systems, we obtain the Fig. 1. As we can see in the figure, we have $[\rho]([G_1](j\omega)) \subseteq [\rho]([G_2](j\omega))$ and $[\varphi]([G_1](j\omega)) \subseteq [\varphi]([G_2](j\omega))$.

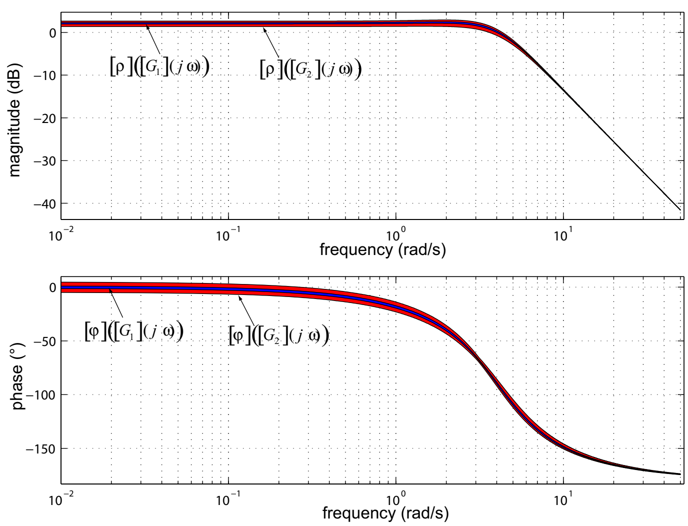

Fig. 1. Bode diagram of $[G_1](s)$ and $[G_2](s)$.

Fig. 2 pictures the step response of the two systems. Once again as predicted by the theory, the step response of $[G_1](s)$ is bounded by the one of $[G_2](s)$.

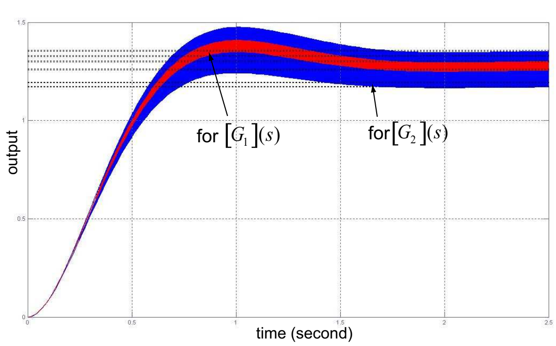

Fig. 2. Step response of $[G_1](s)$ and $[G_2](s)$.

VI. Conclusion

We analyzed the performances inclusion of interval systems. We have demonstrated that when interval systems are included each other, there is also an inclusion relation between their performances both in the frequency and in the time domains. The results can be used for stability and performances robustness analysis, or for the design of controller dedicated to parametric uncertain systems.

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