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Observer-based Output Feedback Control of Discrete-time Lur'e Systems with Sector-bounded Slope-restricted Nonlinearities

Kwang Ki Kevin Kim and Richard D. Braatz, *IEEE Fellow*

***Abstract*— Many well studied classes of dynamical systems such as actuator-constrained linear systems and dynamic artificial neural networks can be written as discrete-time Lur'e systems with sector-bounded and/or slope-restricted nonlinearities. Two types of observer-based output feedback control design methods are presented and analyzed with regard to robustness to model uncertainties and insensitivity to output disturbances. The controller designs are formulated in terms of linear matrix inequalities (LMIs) that are solvable with standard software. The design equations are illustrated in numerical examples.**

I. Introduction

Lyapunov methods and LMI-based computational algorithms provide simple but powerful ways to analyze nonlinear dynamical systems and design stabilizing controllers [1], [2]. Many well-known stability results were developed for a benchmark problem known as *the Lur'e problem* [3], [4], [5]. The Popov and Circle criteria are sufficient frequency-domain conditions for stability of the feedback interconnection of a continuous linear time-invariant system with a sector-bounded nonlinearity [6]. The discrete-time counterparts are known as the Tsypkin and Jury-Lee criteria [7], [8]. Such systems consist of the interconnection of a linear time-invariant system in feedback with a nonlinear operator:

$$\begin{aligned} x_{k+1} &= Ax_k + B_p p_k, \\ q_k &= C_q x_k + D_{qp} p_k, \quad p_k = -\phi(q_k, k), \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times n_p}$, $C_q \in \mathbb{R}^{n_q\times n}$, $D_{qp} \in \mathbb{R}^{n_q\times n_p}$, and the nonlinear operator $\phi \in \Phi$ where Φ is a set of static functions that satisfy $\phi(0, k) \equiv 0$ for all $k \in \mathbb{Z}_+$ and have some specified input-output characteristics.

Several observer designs for certain classes of nonlinear systems have been suggested. Many approaches use Luenberger-type observers with gain L for systems in which the nonlinear feedback interconnections are exactly known. Results include convergence analysis for a given Luenberger-type observer [9], analytical results on eigenvalue assignments based on the multi-valued comparison lemma (Lemma 3.4 [6]) [10], insights into observer design for Lipschitz systems using analysis of eigenstructural sufficient conditions on the stability matrix $(A - LC_y)$ [11], design of reduced-order observers for Lipschitz nonlinear systems [12], solutions of an H_∞-optimization problem that satisfies the standard regularity assumptions and a parameterization of all stabilizing observers for Lipschitz nonlinearities [13]. In the presence of model uncertainties, however, the estimation of the states may not be sufficiently accurate in such Luenberger observers. High-gain observers [6] have been proposed to allow a separation principle where a state feedback controller and state observer are designed separately, and then an output feedback control is applied with the estimate $\hat{x}$ of the state x in the presence of uncertainty in the nonlinearities. A different type of observer structure has been suggested with an output-injection term into the nonlinear mappings [14], and observers for Lur'e systems have been designed with multi-valued maximal monotone mappings in the feedback path by rendering a suitable operator passive [15].

The input-to-state stability (ISS) of an observer is relevant in the certainty-equivalent output feedback control for nonlinear systems. ISS has been successfully employed in the stability analysis and control synthesis of nonlinear systems [16], [17]. The discrete-time counterpart of ISS has also been investigated [18]. A discrete-time separation principle with local detectability was obtained in [19], [20] and a robust separation principle was obtained in the presence of uncertainty in the nonlinearities [18]. If a discrete-time (locally) detectable system can be stabilized by a state feedback law then it can also be (locally) stabilized by a feedback law that depends on the output of a (weak) detector (Theorem 1, [19]). The connection of detectability and the ISS condition for global stabilization was investigated in (Theorem 2.3, [20]) and similar results for an adaptive control system were reported in [21]. Roughly speaking, if a state observer is convergent to the state exponentially, which implies that the error dynamics satisfies an ISS property, then a certainty-equivalent output feedback control that replaces the state x by its estimate $\hat{x}$ in a stabilizing state feedback control stabilizes the overall system.

Two observer-based design methods are proposed for discrete-time systems: (a) two-step separation of controller-observer design and (b) one-step linearization of constraints with the variable reduction (i.e., Finsler's) lemma. The two-step separation of controller-observer design satisifies a separation property in a suitable sense and is robust to model uncertainties and insensitive to output disturbances. In other words, the state observer is sufficiently robust to be insensitive to uncertainty up to a certain degree. The one-step linearization of constraints on the observer and controller gains does not require any separation property and sufficient LMI conditions are proposed to obtain both the control feedback gain and the observer gain simultaneously.

This paper is organized as follows. Section II summarizes some results from convex analysis and state feedback control for the system (1). Section III derives two types of observer-based controllers for Lur'e-type systems and shows their robustness to model uncertainty and insensitivity to output disturbances. The resulting LMI problems are solved for a numerical example in Section IV using off-the-shelf software [22], [23], in which the desired control design specifications are achieved. Section V concludes this paper.

II. Mathematical Preliminaries

A. Notations and Definitions

The notation used in this paper is standard: $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers, respectively; $\|\cdot\|$ is the Euclidean norm for vectors, or the corresponding induced matrix norm for matrices; $\mathbf{0}$ and $\mathbf{I}$ denote

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K. Kim is with University of Illinois at Urbana-Champaign, Urbana, IL, 61801, USA, kkim78@illinois.edu and Massachusetts Institute of Technology, Cambridge, MA, 02142, USA, kwangki@mit.edu

R. D. Braatz is with Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA braatz@mit.edu.

the null matrix whose components are all zeros and the identity matrix of compatible dimension, respectively; ℓ_∞^n is the set of all measurable essentially bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^n$ with ℓ_∞^n-norm defined by $\|f\|_{\ell_\infty^n} \triangleq \max_{1\le i\le n}\left\{\sup_{k\ge 0}|f_i(k)|\right\} < \infty$, where the subscript i denotes the ith element of a vector.

Recall the definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions from the nonlinear system stability literature. A function $\alpha_0 : \mathbb{R}^n \to \mathbb{R}_+$ is said to be proper or radially unbounded if $\alpha_0(\sigma)$ goes to ∞ as $\|\sigma\| \to \infty$. A function $\alpha_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if α_1 is continuous, strictly increasing, and $\alpha_1(0) = 0$. A function α_0 is of class $\mathcal{K}_\infty$ if furthermore α_1 is proper. A function $\alpha_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed $k \ge 0$, $\alpha_2(\cdot, k)$ is a class $\mathcal{K}$-function and for each fixed $\sigma \ge 0$ the function $\alpha_2(\sigma, \cdot)$ is non-increasing and $\alpha_2(\sigma, k) \to 0$ as $k \to \infty$.

Definition 1: (Some classes of nonlinear operators) A nonlinear mapping $\phi : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \mathbb{R}^{n_q}$ is said to be an element of $\Phi_{\rm sb}^{|\alpha|}$ if the inequality $\left[\alpha_i^{-1}\phi_i(\sigma,k)+\sigma\right]\left[\alpha_i^{-1}\phi_i(\sigma,k)-\sigma\right] \le 0$ holds for all $\sigma \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$, and $i = 1,\ldots,n_q$, where the subscript i denotes the ith element of the vector. A nonlinear mapping $\phi : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \mathbb{R}^{n_p}$ is said to be an element of $\bar{\Phi}_{\rm sb}^{\alpha}$ if the inequality $\|\phi(\sigma,k)\| \le \alpha\|\sigma\|$ holds for all $\sigma \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$. A nonlinear mapping $\phi : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \mathbb{R}^{n_p}$ is said to be an element of $\bar{\Phi}_{\rm sr}^{\mu}$ if $\|\phi(\sigma,k)-\phi(\hat{\sigma},k)\| \le \mu\|\sigma-\hat{\sigma}\|$ holds for all $\sigma \neq \hat{\sigma} \in \mathbb{R}^{n_q}$, $k \in \mathbb{Z}_+$.

B. Variable Reduction Lemma

In LMI-based robust control theory, it is common to transform a set of nonconvex inequalities to an LMI that is either equivalent or is a conservative approximation, or to eliminate some decision variables in the original inequalities such that the reduced LMI is convex in the remaining variables. In the elimination process, the eliminated variables that satisfy the original non-convex inequalities can be reconstructed from the solution of the reduced LMI. *Finsler's lemma* (also known as the *the variable reduction lemma*) is a well-known result for the elimination of parameters.

Lemma 1: (Finsler's lemma [1]) The following statements are equivalent:
(a) $\zeta^{\rm T}S\zeta > 0$ for all $\zeta \neq 0$ such that $Rx = 0$;
(b) $(R^\perp)^{\rm T}SR^\perp > 0$ for $RR^\perp = 0$;
(c) $S + \rho R^{\rm T}R > 0$ for some scalar ρ;
(d) $S + XR + R^{\rm T}X^{\rm T} > 0$ for some unstructured matrix X.
Furthermore, an extension of the equivalence of (b) and (d) is that for given matrices U and R there exists X such that the LMI $S + UXR + R^{\rm T}X^{\rm T}U^{\rm T} > 0$ holds for some S if and only if S satisfies the two LMIs $U^\perp S(U^\perp)^{\rm T} > 0$ and $(R^\perp)^{\rm T}SR^\perp > 0$.

C. Controlled Discrete-Time Lur'e Systems

The main contribution of this paper is to propose designs for state observers and dynamic output feedback controllers for some classes of Lur'e systems with multi-valued nonlinear mappings in the (negative) feedback interconnection so that the observation error dynamics are global (or local) asymptotically (or exponentially) stable in the presence of the internal and/or external disturbances. The focus is on the controlled discrete-time Lur'e systems:

$$\begin{aligned} x_{k+1} &= Ax_k + B_p p_k + \chi(x_k,u_k,k),\\ y_k &= C_y x_k,\ q_k = C_q x_k,\ p_k = -\phi(q_k,k), \end{aligned} \tag{2}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^{n_y}$ denote the state and the measurement going into the state observer, respectively, $q_k \in \mathbb{R}^{n_q}$ and $p_k \in \mathbb{R}^{n_p}$ are the variables entering and exiting the nonlinearity, respectively, and $u_k \in \mathbb{R}^{n_u}$ is the control input at the sampling time $k \in \mathbb{Z}_+$. In addition, the nonlinear function $\chi : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{Z}_+ \to \mathbb{R}^n$ is Lipschitz in the first argument and the nonlinear operator $\phi \in \Phi$, where Φ is a set of static functions that satisfy $\phi(0,k) \equiv 0$ for all $k \in \mathbb{Z}_+$ and have some specified input-output characteristics given in Definition 1.

D. State Feedback Controllers

The following result is a sufficient condition for the stability of the Lur'e system with $\phi \in \Phi_{\rm sb}^{|\alpha|}$ or $\phi \in \bar{\Phi}_{\rm sb}^{\alpha}$ that will be used to design controllers.

Lemma 2: The system (1) with the memoryless nonlinearity $\phi \in \Phi_{\rm sb}^{|\alpha|}$ is globally asymptotically stable (g.a.s.) if there exists a positive-definite matrix $Q = Q^{\rm T}$ and a diagonal positive-definite matrix T such that the LMI

$$\begin{bmatrix} -Q & * & * & * \\ \mathbf{0} & -T & * & * \\ AQ & -B_pT & -Q & * \\ C_qQ & \mathbf{0} & \mathbf{0} & -S_\alpha T \end{bmatrix} < 0, \tag{3}$$

is feasible, where $S_\alpha = \mathrm{diag}\{1/\alpha_1^2,\cdots,1/\alpha_{n_p}^2\}$. Similarly, the system (1) with the memoryless nonlinearity $\phi \in \bar{\Phi}_{\rm sb}^{\alpha}$ is g.a.s. if there exists $Q = Q^{\rm T} > 0$ such that the LMI (3) with $S_\alpha = \gamma\mathbf{I}$, $\gamma \equiv 1/\alpha^2$, and $T = \mathbf{I}$ is feasible.

Now consider the system (2) with a control affine term $\chi(x_k,u_k,k) = B_u u_k$ such that the pair (A, B_u) is controllable. Then our design objective is to determine a linear state feedback control law $u_k = K_s x_k$, where K_s is the control gain matrix of compatible dimension. Applying this feedback control law to the system (2) results in the closed-loop system:

$$x_{k+1} = (A + B_u K_s)x_k - B_p\phi(q_k,k). \tag{4}$$

The system (2) is said to be stabilized by the state feedback control law $u_k = K_s x_k$ if the closed-loop system is stable.

Lemma 3: The closed-loop system (4) with $\phi \in \Phi_{\rm sb}^{|\alpha|}$ and $S_\alpha = \mathrm{diag}\{1/\alpha_1^2,\cdots,1/\alpha_{n_p}^2\}$ is globally asymptotically stabilized by the state feedback control $u_k = K_s x_k$ with $K_s = WQ^{-1}$ if the LMI

$$\begin{bmatrix} -Q & * & * & * \\ \mathbf{0} & -T & * & * \\ AQ + B_uW & -B_pT & -Q & * \\ C_qQ & \mathbf{0} & \mathbf{0} & -S_\alpha T \end{bmatrix} < 0 \tag{5}$$

is feasible for $Q = Q^{\rm T} > 0$, a diagonal matrix $T > 0$, and W. A similar sufficient stability condition can be derived for the closed-loop system (4) for a Lur'e system with $\phi \in \bar{\Phi}_{\rm sb}^{\alpha}$, which is a concatenated conic-sector condition. If the LMI (5) with $S_\alpha = \gamma\mathbf{I}$, $\gamma \equiv 1/\alpha^2$, and $T = \mathbf{I}$ is feasible, then the closed-loop system (4) is stabilized by the state feedback control law $u_k = K_s x_k$.

III. Main Results

A. Robust Observer and Controller Design I

This section proposes a two-step separation of controller-observer design for discrete-time Lur'e systems, and investigates its robustness to perturbations in the state observer side and a separation property to ensure its certainty-equivalence. The proposed design is extended to the case when the nonlinear term is not exactly known but its estimate is used in the state observer side.

1) LMI conditions for observer design: This section derives an observer with estimation error dynamics that is globally exponential stable (g.e.s.). For the system dynamics (2), consider the estimator with measurement output injection [14]:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k - L_1\tilde{y}_k - B_p\phi(\hat{q}_k - L_2\tilde{y}_k,k) + \chi(\hat{x}_k,u_k,k),\\ \hat{y}_k &= C_y\hat{x}_k, \qquad \hat{q}_k = C_q\hat{x}_k, \end{aligned} \tag{6}$$

where $\tilde{y} \triangleq y - \hat{y}$ is the output observer error. Then the dynamics of the state estimation error $e \triangleq x - \hat{x}$ are described by

$$\begin{aligned} e_{k+1} &= (A + L_1 C_y) e_k - B_p \hat{\phi}(z_k, k; q_k) + \chi_\delta(e_k, u_k, k), \\ z_k &= (C_q + L_2 C_y) e_k, \end{aligned} \tag{7}$$

where $\hat{\phi}(z_k, k; q_k) \triangleq \phi(q_k, k) - \phi(\hat{q}_k - L_2 \tilde{y}_k, k) = \phi(q_k, k) - \phi(q_k - z_k, k)$ and $\chi_\delta(e_k, u_k, k) \triangleq \chi(x_k, u_k, k) - \chi(\hat{x}_k, u_k, k)$

Assumption 1: Suppose that $\chi(x, u, k)$ is continuously differentiable and (globally) Lipschitz in x with a Lipschitz constant γ_χ, uniformly in (u, k), i.e.,

$$\|\chi(x_1, u, k) - \chi(x_2, u, k)\| \le \gamma_\chi \|x_1 - x_2\|, \tag{8}$$

for any $(u, k) \in \mathbb{R}^{n_u} \times \mathbb{Z}_+$, where (u, k) is a shorthand for the concatenated vector $(u^{\mathrm{T}}, k)^{\mathrm{T}}$.

The next lemma shows that the state estimation error dynamics belong to a class of Lur'e systems.

Lemma 4: The nonlinear function $\hat{\phi}$ has the following properties:

(i) For any q_k, $\hat{\phi}$ vanishes at $z_k \equiv 0$, i.e., $\hat{\phi}(0, k; q_k) \equiv 0$ for all $q(k) \in \mathbb{R}^{n_q}$ and $k \in \mathbb{Z}_+$;

(ii) $\phi \in \bar{\Phi}^{\mu}_{\mathrm{sr}}$ implies that $\hat{\phi} \in \bar{\Phi}^{\mu}_{\mathrm{sb}}$.

From Assumption 1 and Lemma 4, the following theorem provides a sufficient condition for the g.e.s. of the error dynamics (7).

Theorem 1: If there exist matrices L_2, P, and Y such that the LMIs $P = P^{\mathrm{T}} > 0$ and

$$\begin{bmatrix} -\lambda P & * & * & * & * & * \\ \mathbf{0} & -\mathbf{I} & * & * & * & * \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & * & * & * \\ PA + YC_y & -PB_p & P & -P & * & * \\ C_q + L_2 C_y & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{\mu^2}\mathbf{I} & * \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{\gamma_\chi^2}\mathbf{I} \end{bmatrix} < 0, \tag{9}$$

are feasible for some $\lambda \in (0, 1)$, then the estimation error dynamics (7) with $\phi \in \bar{\Phi}^{\alpha}_{\mathrm{sb}} \cap \bar{\Phi}^{\mu}_{\mathrm{sr}}$ is g.e.s. for observer gains $L_1 = P^{-1} Y_1$ and L_2.

2) Stability analysis of a robust observer: Here the robust stability of the state estimation error dynamics is analyzed. Two cases of limited information are considered: (a) partial information of the nonlinear function and (b) output disturbances. In the proof of Theorem 1, a sufficient condition for the error dynamics to be g.e.s. was derived from the LMI condition (3) and vice versa. The nonlinear mapping $\chi(\cdot, \cdot, \cdot)$ is assumed known and the robust stability analysis focuses on the two cases of limited information (a) and (b). To simplify the expressions, here and below assume without loss of generality that $\chi = \chi(y_k, u_k, k)$ such that $\chi_\delta(\cdot, \cdot, k) \equiv 0$ for all $k \in \mathbb{Z}_+$.

a) Unknown nonlinear feedback interconnection: Assume that $\chi(\cdot, \cdot)$ is known, but $\phi \in \Phi$ is unknown so that its approximation ϕ_0 is used in the observer. Consider the estimator for the unmeasurable state:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k - L_1 \tilde{y}_k - B_p \phi_0 \left(\hat{q}_k - L_2 \tilde{y}_k, k\right) + \chi(y_k, u_k, k), \\ \hat{y}_k &= C_y \hat{x}_k, \qquad \hat{q}_k = C_q \hat{x}_k, \end{aligned} \tag{10}$$

then the error dynamics are

$$e_{k+1} = (A + L_1 C_y) e_k - B_p \tilde{\phi}(z_k, k; q_k), \tag{11}$$

where $\tilde{\phi}(z_k, k; q_k) := \phi(q_k, k) - \phi_0 \left(\hat{q}_k - L_2 \tilde{y}_k, k\right)$.

The next two lemmas show that the error dynamics (10) with the nonlinear approximation ϕ_0 can be represented as a Lur'e system.

Lemma 5: If a nonlinear approximation $\phi_0 : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \Omega$ of a nonlinear function $\phi : \mathbb{R}^{n_q} \times \mathbb{Z}_+ \to \Omega$, where $\Omega \subset \mathbb{R}^{n_p}$ is a known bounded set, is a nonlinear mapping satisfying

$$\|\phi(\sigma_1, k) - \phi_0(\sigma_2, k)\| \le \eta_0 \|\sigma_1 - \sigma_2\| \tag{12}$$

for all σ_1, $\sigma_2 \in \mathbb{R}^{n_q}$ and $k \in \mathbb{Z}_+$, then the nonlinear function $\tilde{\phi}(\cdot, \cdot; \cdot)$ in (10) satisfies $\|\tilde{\phi}(z_k, k; q_k)\| \le \eta_0 \|z_k\|$ for all $z_k \in \mathcal{Z} \subset \mathbb{R}^{n_q}$, $q_k \in \mathcal{Q} \subset \mathbb{R}^{n_q}$, and $k \in \mathbb{Z}_+$, i.e., $\tilde{\phi} \in \bar{\Phi}^{\eta_0}_{\mathrm{sb}}$.

Lemma 6: If a nonlinear approximation ϕ_0 of $\phi \in \bar{\Phi}^{\mu}_{\mathrm{sr}}$ is an nonlinear mapping satisfying

$$\|\phi(\sigma, k) - \phi_0(\sigma, k)\| \le \xi_0 \|\sigma\| \quad \forall \sigma \ \forall k \in \mathbb{Z}_+. \tag{13}$$

for all $\sigma \in \mathbb{R}^{n_q}$ and $k \in \mathbb{Z}_+$, then the nonlinear function $\tilde{\phi}(\cdot, \cdot; \cdot)$ in (10) satisfies $\|\tilde{\phi}(z_k, k; q_k)\| \le (\mu + \xi_0) \|z_k\|$ for all $z_k \in \mathcal{Z} \subset \mathbb{R}^{n_q}$, $q_k \in \mathcal{Q} \subset \mathbb{R}^{n_q}$, and $k \in \mathbb{Z}_+$, i.e., $\tilde{\phi} \in \bar{\Phi}^{\mu + \xi_0}_{\mathrm{sb}}$.

Combining Theorem 1 with Lemmas 5 and 6 results in sufficient LMI conditions for the stability of the error dynamics (11) when the nonlinear approximation ϕ_0 of ϕ satisfies the relation (12) or (13).

b) Effect of Output Disturbance on the Observer: Consider the state estimator in the presence of an output disturbance d:

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k - L_1(\tilde{y}_k + d_k) - B_p \phi(\hat{q}_k - L_2(\tilde{y}_k + d_k), k) \\ &\quad + \chi(y_k, u_k, k), \\ \hat{y}_k &= C_y \hat{x}_k, \qquad \hat{q}_k = C_q \hat{x}_k. \end{aligned} \tag{14}$$

Then the state estimation error dynamics are

$$e_{k+1} = (A + L_1 C_y) e_k - B_p \hat{\phi}(z_k, k; q_k) + \delta_k, \tag{15}$$

where $\delta_k \triangleq -B_p \left(\phi(\hat{q}_k - L_2 \tilde{y}_k) - \phi(\hat{q}_k - L_2 \tilde{y}_k - L_2 d_k)\right) + L_1 d_k$, $\hat{\phi}(\cdot, \cdot; \cdot)$ are the same as in (7), and the augmented disturbance δ_k satisfies

$$\|\delta_k\| \le c_1 \|d_k\|, \quad \forall k \in \mathbb{Z}_+, \tag{16}$$

where $c_1 \triangleq \mu \sigma_{\max}(L_2) + \sigma_{\max}(L_1)$.

Recall that a system whose equilibrium point is g.e.s. in the absence of the disturbance is input-to-state stable. If the estimation error dynamics (15) with $\delta_k \equiv 0$ is g.e.s and the measured-output disturbance satisfies $d \in \ell^{n_y}_\infty$, i.e., there exists a constant Δ_d such that $\|d\|_{\ell^{n_y}_\infty} := \Delta_d < \infty$, then there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function ρ such that, for any initial state e_0, the solution e_k of the system (15) satisfies

$$\begin{aligned} \|e_k\| &\le \beta\left(\|e_0\|, k\right) + \rho\left(\|\delta_{[0,k]}\|_{\ell^{n_y}_\infty}\right) \\ &\le \beta\left(\|e_0\|, k\right) + \rho\left(c_1 \Delta_d\right). \end{aligned} \tag{17}$$

Application of the definition of the ISS and a sufficient LMI condition (9) in Theorem 1 for the g.e.s. of the error dynamics (15) in the absence of δ_k guarantees the robustness of the state observer to output disturbances with a non-zero stability margin, which corresponds to the class $\mathcal{K}$ function ρ. In particular, if the output disturbance is an asymptotically vanishing perturbation, i.e., $\lim_{k\to\infty} \|d_k\| = 0$ or satisfies the linear growth bound $\|d_k\| \le \gamma_d \|e_k\|$ for all $k \in \mathbb{Z}_+$ with small $\gamma_d \ll 1$ then the origin of the system (15) is g.a.s.

An example of an output disturbance is an output quantizer [24]. In quantized measurement and control of continuous-time nonlinear systems, this ISS property of the error dynamics appears to be fundamental for incorporating an observer in certainty-equivalent output feedback control. In fact, ISS with respect to an output disturbance is a standing assumption in the results on quantized feedback control. Even for discrete-time nonlinear systems, the time scales (or samplings) in the plant, controllers, and observers may be different such that quantization-like effects are everywhere in the system. In such cases, it is important to ensure robustness against the quantization effect.

3) Optimal observer design: Consider the observer design objective of maximizing the decay rate.

Corollary 1: (Maximization of the decay rate of the estimation error dynamics) Observer matrices that maximize a lower bound on the decay rate of the estimation error dynamics (7) are obtained by solving the generalized eigenvalue problem (GEVP):

$$\min\ \lambda \quad \text{s.t.} \quad P > 0, \quad (9), \tag{18}$$

in the decision variables P, Y, L_2, and λ, where $L_1 = P^{-1}Y$.

A optimal value of $\lambda \in (0, 1)$ in (18) implies that the designed estimation error dynamics are g.e.s., with a smaller value of λ indicating a faster rate of exponential convergence.

4) Certainty-equivalence Control: The observer that ensures ISS estimation error dynamics guarantees the certainty-equivalence property of the closed-loop system, at least in a local sense.

Theorem 2: (Observer-based output feedback controller design I) Consider the system (2) with $\phi \in \bar{\Phi}_{sb}^{\alpha} \cap \bar{\Phi}_{sr}^{\mu}$ and $\chi(\cdot, u_k, k) = B_u u_k = B_u K_s \hat{x}_k$, and its state observer (6). If there exists a feasible solution to the EVP

$$\min_{Q,W}\ \gamma$$

$$\text{s.t. } Q > 0, \begin{bmatrix} -Q & * & * & * \\ \mathbf{0} & -\mathbf{I} & * & * \\ AQ + B_u W & -B_p & -Q & * \\ C_q Q & \mathbf{0} & \mathbf{0} & -\gamma \mathbf{I} \end{bmatrix} < 0, \tag{19}$$

and a feasible solution to the GEVP

$$\min_{P,Y,L_2}\ \lambda \tag{20}$$

$$\text{s.t. } P > 0, \begin{bmatrix} -\lambda P & * & * & * \\ \mathbf{0} & -\mathbf{I} & * & * \\ PA + YC_y & -PB_p & -P & * \\ C_q + L_2 C_y & \mathbf{0} & \mathbf{0} & -\frac{1}{\mu^2}\mathbf{I} \end{bmatrix} < 0,$$

then the overall system (2) with feedback gain $K_s = WQ^{-1}$ and estimator gains $L_1 = P^{-1}Y$ and L_2 is g.a.s. with $\gamma = 1/\alpha^2$.

An optimal solution K_s^* of the EVP (19) maximizes an upper bound on α of the system (2) while achieving g.a.s. and optimal solutions L_1^* and L_2^* of the GEVP (20) maximize a lower bound on the decay rate of the estimation error dynamics (7). The Lyapunov matrices Q and P in Theorem 2 are independent, i.e., the existence of a common Lyapunov function is not required. The value of the local slope μ in Step 2 should be set so that $\mu \geq \alpha$, to avoid conservatism in the definition of the nonlinearities. The upper bound on the local slope is the same as the maximum sector bound for many typical memoryless nonlinearities (e.g., hyperbolic tangent, saturation, and dead-zone nonlinearities), in which case μ in Step 2 should be set equal to the α computed in Step 1.

B. Robust Observer and Controller Design II

This section proposes one-step linearization of design constraints with the variable reduction lemma (Lemma 1). The purpose of this section is to propose a state feedback control and state observer design whose design parameters are obtained by solving LMIs, instead of solving BMIs.

1) LMI conditions for design: Instead of the design method in Theorem 2 consisting of an EVP followed by a GEVP, consider the objective of designing for a fixed decay rate for the closed-loop system with $\alpha = \mu$. The closed-loop system with the observer-based feedback can be written as

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B_u K_s & B_u K_s \\ \mathbf{0} & A + L_1 C_y \end{bmatrix}}_{A_{\text{cl}}} \begin{bmatrix} x_k \\ e_k \end{bmatrix} - \underbrace{\begin{bmatrix} B_p & \mathbf{0} \\ \mathbf{0} & B_p \end{bmatrix}}_{B_{p,\text{cl}}} \underbrace{\begin{bmatrix} \phi(q_k, k) \\ \hat{\phi}(z_k, k; q_k) \end{bmatrix}}_{\phi_{\text{cl}}},$$

$$\begin{bmatrix} q_k \\ z_k \end{bmatrix} = \underbrace{\begin{bmatrix} C_q & \mathbf{0} \\ \mathbf{0} & C_q + L_2 C_y \end{bmatrix}}_{C_{q,\text{cl}}} \begin{bmatrix} x_k \\ e_k \end{bmatrix}. \tag{21}$$

The closed-loop system, which is the feedback interconnection of the system whose transfer function is $G_{\text{cl}}(s) \triangleq C_{q,\text{cl}}(s\mathbf{I} - A_{\text{cl}})^{-1} B_{p,\text{cl}}$ and the nonlinearity within the set $\phi_{\text{cl}} \in \bar{\Phi}_{sb}^{\mu}$ is g.e.s. if the inequality

$$\begin{bmatrix} -\lambda X & * & * & * \\ \mathbf{0} & -\mathbf{I} & * & * \\ XA_{\text{cl}} & -XB_{p,\text{cl}} & -X & * \\ C_{q,\text{cl}} & \mathbf{0} & \mathbf{0} & -\frac{1}{\mu^2}\mathbf{I} \end{bmatrix} < 0 \tag{22}$$

is feasible for some $X = X^{\text{T}} > 0$ and $\lambda \in (0, 1)$. This is not a convex feasibility problem due to bilinear product terms of the decision matrix variables. Since BMI problems are in general nonconvex and hence difficult to solve, there has been much interest in identifying special cases in which the BMI problem can be reduced to an LMI feasibility problem. The following result is that the feasibility of the BMI (22) is implied, with some conservatism, by the feasibility of two LMIs.

Theorem 3: The BMI (22) is feasible for L_1, K_s, L_2, and a block-diagonal matrix $X = \text{diag}\{X_1, X_2\} = X^{\text{T}} > 0$ if and only if the two LMIs

$$\bar{B}_u^{\perp} \Pi_\lambda (\bar{B}_u^{\perp})^{\text{T}} < 0 \quad \text{and} \quad (\bar{E}^{\text{T}})^{\perp} \Pi_\lambda ((\bar{E}^{\text{T}})^{\perp})^{\text{T}} < 0, \tag{24}$$

are feasible for $Y_1 \triangleq X_1^{-1}$, $W_1 \triangleq K_s X_1^{-1}$, X_2, $W_2 \triangleq X_2 L_1$, and L_2 with a given $\lambda \in (0, 1)$, where the matrix Π_λ is given in (23) and $\bar{B}_u^{\text{T}} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & B_u^{\text{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, $\bar{E} \triangleq \begin{bmatrix} 0 & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Remark 1: All of the results for analysis and synthesis of a robust observer obtained from Section III-A can be trivially extended to the system (21).

IV. Numerical Examples

The proposed observer-based design methods are demonstrated for a numerical example. Consider the system (2) with the nonlinear mapping $\phi \in \bar{\Phi}_{sb}^{\alpha} \cap \bar{\Phi}_{sr}^{\mu}$, system matrices

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -0.2703 & -0.0124 & 0.2703 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.1075 & 0 & 0.0743 & 0 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 0 \\ 0.0216 \\ 0 \\ 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 & 0 \\ 0.2703 & 0 \\ 0 & 0 \\ -0.1075 & 0.0332 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and $\chi(\cdot, u_k, k) = B_u u_k$.

The two-step design method proposed in Section III-A is applied to achieve robust stability and performance of the closed-loop system. The design objective is to maximize $\alpha = \mu$ that quantifies the magnitude of the nonlinearity and to maximize the decay rate of the estimation error dynamics (7) such that the closed-loop system

$$
\Pi_\lambda \triangleq \begin{bmatrix}
-\lambda Y_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{pmatrix} Y_1 A^{\mathrm{T}} \\ +W_1^{\mathrm{T}} B_u^{\mathrm{T}} \end{pmatrix} & \mathbf{0} & Y_1 C_q^{\mathrm{T}} & \mathbf{0} \\
\mathbf{0} & -\lambda X_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{pmatrix} A^{\mathrm{T}} X_2 \\ +C_y^{\mathrm{T}} W_2^{\mathrm{T}} \end{pmatrix} & \mathbf{0} & \begin{pmatrix} C_q^{\mathrm{T}} \\ +C_y^{\mathrm{T}} L_2^{\mathrm{T}} \end{pmatrix} \\
\mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & -B_p^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & -B_p^{\mathrm{T}} X_2 & \mathbf{0} & \mathbf{0} \\
\begin{pmatrix} AY_1 \\ +B_u W_1 \end{pmatrix} & \mathbf{0} & -B_p & \mathbf{0} & -Y_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \begin{pmatrix} X_2 A \\ +W_2 C_y \end{pmatrix} & 0 & -X_2 B_p & \mathbf{0} & -X_2 & \mathbf{0} & \mathbf{0} \\
C_q Y_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{\mu^2}\mathbf{I} & \mathbf{0} \\
\mathbf{0} & \begin{pmatrix} C_q \\ +L_2 C_y \end{pmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{\mu^2}\mathbf{I}
\end{bmatrix}. \tag{23}
$$

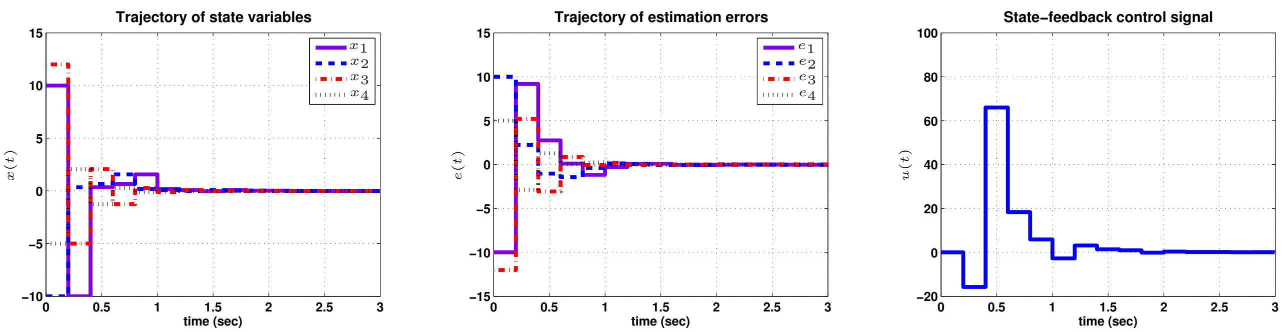


Fig. 1: Trajectories for a controlled Lur'e system with Design I in the presence of measurement disturbance and modeling error

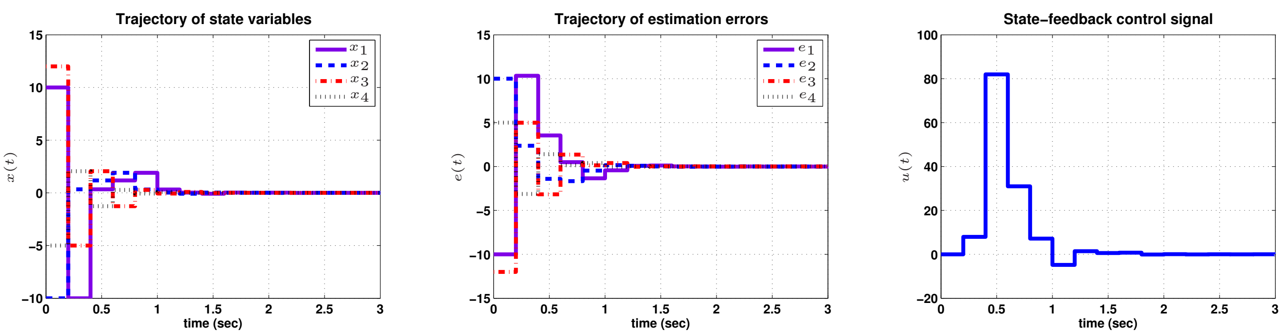


Fig. 2: Trajectories for a controlled Lur'e system with Design II in the presence of measurement disturbance and modeling error

(2) is stabilized by the control law $u_k = K_s \hat{x}_k$. The optimal solution for K_s^*, L_1^*, and L_2^* of the successive EVP and GEVP in Theorem 1 as obtained using a semi-definite programming (SDP) solver [22] is

$$
K_s^* = \begin{bmatrix} 18.8558 & 0.5750 & -8.1083 & 0.0000 \end{bmatrix},
$$
$$
L_1^* = \begin{bmatrix} -0.1158 & -0.6558 \\ 0.2666 & 0.0046 \\ 0.0321 & 0.2515 \\ -0.0857 & -0.0021 \end{bmatrix}, \; L_2^* = \begin{bmatrix} 1.7341 & 0.4482 \\ 0.4482 & -0.3401 \end{bmatrix},
$$

where the maximum upper bound on the sector and slope for the nonlinearity ϕ is $\alpha^* = \mu^* = 2.5281$ and the decay rate with $\lambda^* = 0.2959$ is achieved. To maximize insensitivity to output disturbances on the observer, L_1 and L_2 are computed that minimize the value c_1 in (16). This can be done by using the bisection method and solving EVPs at each step iteration. The computed value of c_1 for L_1^* and L_2^* is $c_1^* = 3.3915$. Figure 1 shows the time trajectories for the closed-loop system with $\phi(q) = \alpha^* \tanh(q)$, in the presence of the vanishing disturbance $d_k = 2.7\sin(k\pi)e^{-0.01k}$ in (14) and a modeling error with $\phi_0(q) = 0.5\alpha^* \tanh(q)$ in (10). The states and estimation errors for a nonzero initial state converge quickly, as expected from the value of $\lambda^* = 0.2959$, with an insensitivity to the output disturbance and model uncertainty.

Now the one-step design method proposed in Section III-B is applied to the same system. Similar to the two-step design method, for the control objective of maximizing α such that the closed-loop system is stabilized by the control law $u_k = K_s \hat{x}_k$, the optimal solution for K_s^*, L_1^*, and L_2^* in the EVP in Theorem 3 obtained by a semi-definite programming (SDP) solver [22] is

$$
K_s^* = \begin{bmatrix} 18.8558 & 0.5750 & -8.1083 & 0.0000 \end{bmatrix},
$$
$$
L_1^* = \begin{bmatrix} 0.0000 & -1.0000 \\ 0.2703 & 0.0124 \\ 0.0000 & 0.0301 \\ -0.1075 & 0.0000 \end{bmatrix}, \; L_2^* = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix},
$$

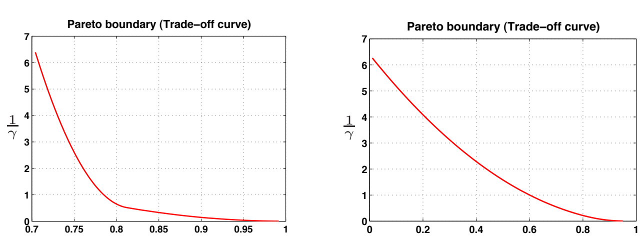
where the maximum upper bound on the sector and slope for the nonlinearity ϕ is $\alpha^* = \mu^* = 2.5281$ for $\lambda = 0.99$. The closed-loop trajectories for the closed-loop system (2) with $\phi(q) = \alpha^* \tanh(q)$, in the presence of the vanishing disturbance $d_k = 2.7 \sin(k\pi) e^{-0.01k}$ in (14) and a modeling error with $\phi_0(q) = 0.5\alpha^* \tanh(q)$ in (10), are shown in Fig. 2. To compare the potential effect of output disturbances with the previous design method, the value of c_1 for L_1^* and L_2^* was computed, which was $c_1^* = 3.5286$, somewhat higher than for the two-step design. This comparison suggests the closed-loop system for the one-step design method could be more sensitive to output disturbances than for the two-step design method, but this may not be true because sufficient conditions appear in the derivations of both design methods.

For this example, potential conservatism due to the use of a block-diagonalized Lyapunov matrix $X = \text{diag}\{X_1, X_2\}$ in Theorem 3 for the two-step method was not significant in terms of the achieved maximum upper bound on the sector and slope for the nonlinearity ϕ, although the computational algorithms are different. Theorem 1 considers the convergence rate of the estimation error dynamics in a separate design process from the computation of the control gain K_s, whereas all design variables are computed in an integrated manner in Theorem 3.

Fig. 3 shows the Pareto-optimality curves for the two design methods, which quantify the tradeoffs between insensitivity to disturbances and performance in terms of decay rate (rate of convergence). Note that the Pareto-optimality curves for the two different design methods have different meanings. Fig. 3a shows the tradeoff between the convergence (decay) rate of the estimation error dynamics and the upper bound on the sector-bounded (and/or slope-restricted) nonlinearities, where the estimation error dynamics are g.e.s. with λ independent of the controlled system. Contrary to this, Fig. 3b shows the tradeoff curve for the overall system, which is the concatenation of the controlled system and the estimation error dynamics. That is, the overall closed-loop system is g.e.s. with the decay rate $(1 - \lambda)$ and the upper bound $(1/\gamma^2)$ on the sector-bounded (and/or slope-restricted) nonlinearities. Both Pareto-optimality curve are monotonic and well-behaved, with the certainty-equivalence design having a sharper knee region for defining an optimal tradeoff (Fig. 3a).

V. Conclusions

Two LMI-based procedures are proposed for the design of observer-based output feedback controllers for a Lur'e-type system with conic-sector-bounded slope-restricted nonlinearities. Observer design methods are proposed for two different strategies: (a) based on an observer-controller separation and (b) based on simultaneous design derived from the Finsler's lemma. Both sets of LMIs are easily solved using existing solvers. Their robustness against model uncertainty and insensitivity to output disturbance were also investigated. Very similar controller and estimator designs and closed-loop responses were obtained in applications of the two methods to a numerical example.



(a) Certainty-equivalence control (b) Combined observer-controller

Fig. 3: Trade off between λ and γ

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