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Stability of Switched Linear Discrete-Time Descriptor Systems: A Commutation Condition

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Abstract— **In this paper, we study stability of switched linear discrete-time descriptor systems. Under the assumption that all subsystems are stable and there is no impulse occurring at the switching instants, we establish a new pairwise commutation condition under which the switched system is stable. We also show that when the proposed commutation condition holds, there exists a common quadratic Lyapunov function (CQLF) for the subsystems. These results are natural and important extensions to the existing results for switched systems in the state space representation.**

Index Terms—**Switched linear discrete-time descriptor systems, stability, pairwise commutation, impulse-free arbitrary switching, common quadratic Lyapunov functions (CQLFs), matrix inequalities.**

I. Introduction

This paper analyzes stability properties for switched systems composed of a family of linear discrete-time descriptor subsystems. It is known that descriptor systems (also called as singular systems or implicit systems) have high abilities in representing dynamical systems [1], [2], since they can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even improper part of the system in the same form. Due to the existence of the static part (or the algebraic constraint in other words), most descriptor systems have impulsive modes, which makes the analysis and design problems quite difficult, compared with the state space representation. So far, there have been many references on descriptor systems, focusing on stability analysis and stabilization [3], $\mathcal{H}_\infty$ control [4], etc.

On the other hand, there has been increasing interest recently in stability analysis and design for switched systems; see the survey papers [5], [6], [7] and the references cited therein. It is commonly recognized [5] that there are three basic problems for stability analysis and design of switched systems: (i) find conditions for stability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching laws; and (iii) construct a stabilizing switching law. Specifically, Problem (i) deals with the case that all subsystems are stable. This problem seems trivial, but it is important since we can find many examples where all subsystems are stable but improper switchings can make the whole system unstable [8]. In addition, if we know that a switched system is stable under arbitrary switching, then we can consider higher control specifications for the system.

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There have been several results for Problem (i) with the state space representation. For example, Ref. [9] showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. Ref. [10] extended this result from the commutation condition to a Lie-algebraic condition. Ref. [11], [12] and [13] extended the consideration to the case of $\mathcal{L}_2$ gain analysis and the case where both continuous-time and discrete-time subsystems exist, respectively. In our previous papers [14], [15], we extended the existing result of [9] to switched linear descriptor systems. In that context, we showed that in the case where all descriptor subsystems are stable, if the descriptor matrix and all subsystem matrices are commutative pairwise, then the switched system is stable under arbitrary switching. The recent papers [16] and [17] established sufficient conditions for stability of switched linear descriptor systems under arbitrary switching in the name of common Lyapunov functions. It is noted that [16] did not deal with the variable jump occurring at switching instants, while [17] proposed an additional condition involving consistency projectors for that purpose.

The present paper is motivated by the observation that the commutation condition proposed in [14], [15] is still conservative, although it is an extension to the existing commutation condition in [9]. The reason is that the commutation condition in [14], [15] is required to hold among the descriptor matrix and the subsystem matrices. However, it is known that the dynamics of a linear descriptor system is determined by the pair of descriptor matrix and system matrix. Thus, it is natural that the commutation condition be expressed by the pair or the combination of these matrices. Based on this observation, this paper proposes a new commutation condition for stability analysis of switched linear descriptor systems, which is a natural extension to the commutation conditions in [9], [14], [15]. The relation between the proposed commutation condition and the existence of common quadratic Lyapunov functions (CQLFs) is also clarified in a constructive way.

This paper is organized as follows. In Section II, we give some preliminaries of linear descriptor systems and formulate the stability analysis problem under consideration. Section III states and proves the commutation condition for the switched descriptor systems under impulse-free arbitrary switching. The condition includes the existing commutation conditions [9], [14], [15] as special cases. Section IV proves that if the commutation conditions holds, then there exists a CQLF for the subsystems. This is also a natural extension to the existing result in the literature. Finally, Section V concludes the paper.

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II. Preliminaries & Problem Formulation

A. Preliminaries

we first introduce some definitions [4] and a preliminary result for linear discrete-time descriptor systems.

Definition 1: Consider the linear discrete-time descriptor system

$$Ex(k+1) = Ax(k) \tag{2.1}$$

where $x \in \mathcal{R}^n$ is the descriptor variable, the nonnegative integer k denotes the discrete-time instant, and $E, A \in \mathcal{R}^{n\times n}$ are constant matrices. The matrix E may be singular and we denote its rank by $r = \mathrm{rank}\, E \leq n$. The system (2.1) has a unique solution for any initial condition and is called *regular*, if $|zE - A| \not\equiv 0$. The finite eigenvalues of the matrix pair (E, A), that is, the solutions of $|zE - A| = 0$, and the corresponding (generalized) eigenvectors define exponential modes of (2.1). If the finite eigenvalues lie in the open unit disc of the complex plane, the solution *decays exponentially*. The infinite eigenvalues of (E, A) with the eigenvectors satisfying the relations $Ex_1 = 0$ determines static modes. The infinite eigenvalues of (E, A) with generalized eigenvectors x_k satisfying the relations $Ex_1 = 0$ and $Ex_k = x_{k-1}$ ($k \geq 2$) create *impulsive modes*. The system (2.1) has no impulsive mode if and only if rank E = deg $|zE - A|$. The system (2.1) is said to be *stable* if it is regular and has only decaying exponential modes and static modes (without impulsive modes). ∎

Lemma 1 (Weiertrass Form)[2]: *If the descriptor system (2.1) is regular, then there exist two nonsingular matrices M and N such that*

$$MEN = \begin{bmatrix} I_d & 0 \\ 0 & J \end{bmatrix}, \quad MAN = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-d} \end{bmatrix} \tag{2.2}$$

where $d = deg\ |zE - A|$, J is composed of Jordan blocks for the finite eigenvalues. If the system (2.1) is regular and there is no impulsive mode, then (2.2) holds with $d = r$ and $J = 0$. If the system (2.1) is stable, then (2.2) holds with $d = r$, $J = 0$ and furthermore Λ being (Schur) stable. ∎

B. Problem Formulation

Without losing generality, we consider the switched system composed of two discrete-time descriptor subsystems described by

$$\begin{cases} Ex(k+1) = A_1 x(k) \\ Ex(k+1) = A_2 x(k) \end{cases} \tag{2.3}$$

where $x \in \mathcal{R}^n$ is the descriptor variable, and $E, A_1, A_2 \in \mathcal{R}^{n\times n}$ are constant matrices. The matrix E may be singular and we denote its rank by $r = \mathrm{rank}\, E \leq n$.

In this paper, we consider Problem (i) for the switched system (2.3) under the assumption that the two subsystems in (2.3) are both stable. As for the stability analysis of switched linear systems in state space representation, such an analysis problem is well posed (or practical) since a switched descriptor system can be unstable even if all the descriptor subsystems are stable and there is no variable (state) jump at the switching instants. Furthermore, switchings between two descriptor subsystems can even result in impulse signals, even if the two subsystems do not have impulsive modes themselves. This happens when the variable vector $x(k_s)$, where k_s is a switching instant, does not satisfy the algebraic equation required in the subsequent subsystem. In order to exclude this possibility, Ref. [17] proposed an additional condition using the name of consistency projectors (for switched continuous-time descriptor systems). Here, in order to establish a commutation condition, we focus our attention on the case that there is no impulse occurring with the variable (state) vector at every switching instant, and call such kind of switching *impulse-free*. A more detailed discussion will be made in the next section how the switching should be done so that no impulse occurs.

Definition 2: Given a switching law, the switched system (2.3) is said to be *stable* if there is no impulse occurring, and starting from any initial value the system's trajectories converge to the origin exponentially. If there exists a switching law under which the switched system is stable, the switched system (2.3) is said to be *stabilizable* (under appropriate switching). ∎

We are now ready to state the analysis problem considered in the present paper.

Stability Analysis Problem Under Impulse-Free Arbitrary Switching: "Assume that the two descriptor subsystems in (2.3) are stable. Establish the commutation condition under which the switched system is stable under impulse-free arbitrary switching."

Remark 1: There is a tacit assumption in the switched system (2.3) that the descriptor matrix E is the same in all the subsystems. Theoretically, this assumption is restrictive at present. However, as also discussed in [14], [15], the above problem settings and the results later can be applied to switching control problems for single linear descriptor systems. This is the main motivation that we presently consider the same descriptor matrix E in the switched system. For example, if for a single descriptor system $Ex(k+1) = Ax(k) + Bu(k)$ where $u(k)$ is the control input, we have designed two stabilizing descriptor variable feedbacks $u(k) = K_1 x(k)$, $u(k) = K_2 x(k)$, and furthermore the switched system composed of the descriptor subsystems characterized by $(E, A + BK_1)$ and $(E, A + BK_2)$ are stable under impulse-free arbitrary switching, then we can switch between the two controllers arbitrarily provided that no impulse occurs, and thus can consider higher control specifications. This kind of requirement is very important when we want more flexibility for multiple control specifications in real applications. ∎

III. New Commutation Condition

A. New Commutation Condition for Stability Under Impulse-Free Arbitrary Switching

Since (E, A_1) is stable, according to Lemma 1, there exist two nonsingular matrices M, N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MA_1N = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{3.1}$$

where Λ_1 is a (Schur) stable matrix. Then, we partition the matrices M, N into

$$N = \begin{bmatrix} N_L & N_R \end{bmatrix} , \qquad M = \begin{bmatrix} M_U \\ M_L \end{bmatrix} \tag{3.2}$$

where $N_L \in \mathcal{R}^{n\times r}$, $N_R \in \mathcal{R}^{n\times(n-r)}$, $M_U \in \mathcal{R}^{r\times n}$, and $M_L \in \mathcal{R}^{(n-r)\times n}$, and establish the first main result as follows.

Theorem 1: *If the two descriptor systems in (2.3) are stable, and furthermore the subsystems are commutative pairwise in the sense of*

$$A_1 N_L M_U A_2 = A_2 N_L M_U A_1 , \tag{3.3}$$

then the switched system (2.3) is stable under impulse-free arbitrary switching.

Proof: Using the nonsingular matrices M and N, we write the transformation of A_2 as

$$M A_2 N = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} . \tag{3.4}$$

According to (3.1) and (3.4), under the same nonsingular transformation $\bar{x} = N^{-1}x$, the two descriptor subsystems in (2.3) take the form of

$$\begin{aligned} \bar{x}_1(k+1) &= \Lambda_1 \bar{x}_1(k) \\ 0 &= \bar{x}_2(k) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \bar{x}_1(k+1) &= \bar{A}_{11}\bar{x}_1(k) + \bar{A}_{12}\bar{x}_2(k) \\ 0 &= \bar{A}_{21}\bar{x}_1(k) + \bar{A}_{22}\bar{x}_2(k) \end{aligned} \tag{3.6}$$

respectively, where $\bar{x} = N^{-1}x = [\bar{x}_1^\top \;\; \bar{x}_2^\top]^\top$, $\bar{x}_1 \in \mathcal{R}^r$, $\bar{x}_2 \in \mathcal{R}^{n-r}$. Since (E, A_2) is stable, we obtain from (3.6) that $\bar{A}_{22}$ is nonsingular,

$$\begin{aligned} \bar{x}_1(k+1) &= \Lambda_2 \bar{x}_1(k) , \quad \Lambda_2 = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} \\ \bar{x}_2(k) &= -\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1(k) . \end{aligned} \tag{3.7}$$

and furthermore Λ_2 is (Schur) stable. Note that the stability of the switched system (2.3) is equivalent to that of the switched system composed of (3.5) and (3.7).

Next, we proceed to prove Λ_1 and Λ_2 are commutative pairwise. Before that, we write several equations

$$\begin{gathered} M_U A_1 N_L = \Lambda_1 , \; M_U A_1 N_R = 0 , \; M_L A_1 N_L = 0 \\ M_U A_2 N_L = \bar{A}_{11} , \; M_U A_2 N_R = \bar{A}_{12} \\ M_L A_2 N_L = \bar{A}_{21} , \; M_L A_2 N_R = \bar{A}_{22} , \end{gathered} \tag{3.8}$$

which are easily derived from the transformations (3.1) and (3.4). Then,

$$\begin{aligned} \Lambda_1\Lambda_2 &= M_U A_1 N_L \left[M_U A_2 N_L - M_U A_2 N_R \bar{A}_{22}^{-1}\bar{A}_{21} \right] \\ &= M_U A_1 N_L M_U A_2 N_L - M_U A_1 N_L M_U A_2 N_R \bar{A}_{22}^{-1}\bar{A}_{21} \\ &= M_U A_2 N_L M_U A_1 N_L - M_U A_2 N_L M_U A_1 N_R \bar{A}_{22}^{-1}\bar{A}_{21} \\ &= M_U A_2 N_L M_U A_1 N_L , \end{aligned} \tag{3.9}$$

where the fact $M_U A_1 N_R = 0$ and the condition (3.3) are used to reach the final equation. Similarly,

$$\begin{aligned} \Lambda_2\Lambda_1 &= \left[M_U A_2 N_L - \bar{A}_{12}\bar{A}_{22}^{-1} M_L A_2 N_L \right] M_U A_1 N_L \\ &= M_U A_2 N_L M_U A_1 N_L - \bar{A}_{12}\bar{A}_{22}^{-1} M_L A_2 N_L M_U A_1 N_L \\ &= M_U A_2 N_L M_U A_1 N_L - \bar{A}_{12}\bar{A}_{22}^{-1} M_L A_1 N_L M_U A_2 N_L \\ &= M_U A_2 N_L M_U A_1 N_L , \end{aligned} \tag{3.10}$$

where the fact $M_L A_1 N_L = 0$ is used.

According to (3.9) and (3.10), $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Since both Λ_1 and Λ_2 are Schur stable, we obtain that the switched system composed of $\bar{x}_1(k+1) = \Lambda_1\bar{x}_1(k)$ and $\bar{x}_1(k+1) = \Lambda_2\bar{x}_1(k)$ is exponentially stable under arbitrary switching, and $\bar{x}_1(k)$ converges to zero exponentially.

Noticing that $\bar{x}_2(k) = 0$ in the first subsystem and $\bar{x}_2(k) = -\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1(k)$ in the second subsystem, $\bar{x}_2(k)$ also converges to zero exponentially under impulse-free arbitrary switching. This completes the proof. ∎

Remark 2: Using the same technique as in the proof of Theorem 1, we can easily establish and prove the result for the case where there are more than three descriptor subsystems involved. More precisely, for the switched system composed of $\mathcal{N}$ descriptor subsystems described by

$$E x(k+1) = A_i x(k) , \quad i = 1, \cdots, \mathcal{N} , \tag{3.11}$$

if all (E, A_i), $i = 1, \cdots, \mathcal{N}$, are stable, and furthermore the subsystems are commutative pairwise in the sense of satisfying

$$A_i N_L M_U A_j = A_j N_L M_U A_i , \quad \forall i \neq j , \tag{3.12}$$

then the switched system (3.11) is stable under impulse-free arbitrary switching. ∎

Remark 3: It is obtained from the transformation (3.1) that

$$M E N_L = \begin{bmatrix} I_r \\ 0 \end{bmatrix} , \quad M A_1 N_R = \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} , \tag{3.13}$$

and thus

$$M^{-1} = \begin{bmatrix} E N_L & A_1 N_R \end{bmatrix} . \tag{3.14}$$

Similarly, using (3.1) again,

$$M_U E N = \begin{bmatrix} I_r & 0 \end{bmatrix} , \quad M_L A_1 N = \begin{bmatrix} 0 & I_{n-r} \end{bmatrix} , \tag{3.15}$$

and thus

$$N^{-1} = \begin{bmatrix} M_U E \\ M_L A_1 \end{bmatrix} . \tag{3.16}$$

To substitute these inverse matrices into the first equation of (3.1), one obtains

$$\begin{aligned} E &= M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} \\ &= \begin{bmatrix} E N_L & A_1 N_R \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_U E \\ M_L A_1 \end{bmatrix} \\ &= E(N_L M_U)E . \end{aligned} \tag{3.17}$$

In addition, since $M_U E N_L = I_r$, it leads to

$$(N_L M_U) E (N_L M_U) = N_L (M_U E N_L) M_U = N_L M_U \,. \tag{3.18}$$

Based on the above observation, $N_L M_U$ is actually a Pseudo inverse matrix of E. ∎

Remark 4: The commutation condition (3.3) does not depend on the choice of M, N in (3.1). Suppose we choose two different nonsingular matrices $\bar{M}$, $\bar{N}$ such that

$$\bar{M} E \bar{N} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{M} A_1 \bar{N} = \begin{bmatrix} \bar{\Lambda}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{3.19}$$

Since both Λ_1 and $\bar{\Lambda}_1$ include the dynamic part of the descriptor system (E, A_1), they have the same eigenvalues and thus there exists a nonsingular matrix T_1 satisfying $\bar{\Lambda}_1 = T_1^{-1} \Lambda_1 T_1$. Then, comparing (3.19) with (3.1), we obtain easily that

$$N_L = \bar{N}_L T_1^{-1} \,, \ M_U = T_1 \bar{M}_U \,. \tag{3.20}$$

Therefore, $N_L M_U = \bar{N}_L \bar{M}_U$ and the condition (3.3) takes the same form. ∎

In the end of this subsection, we give a remark discussing the "impulse-free" property at the switchings for the switched descriptor system. In real applications, we need to know in which space (area) the switchings do not result in impulses.

Remark 5: According to the proof of Theorem 1, under the same nonsingular variable transformation $\bar{x} = N^{-1} x$, the two descriptor subsystems in (2.3) are decomposed into a differential equation and an algebraic equation, as described in (3.5) and (3.6). It is clear that impulses occur when the two algebraic equations are not consistent. On the contrary, switchings in the variable (state) space where the two algebraic equations are both satisfied will not result in impulses. Since $\bar{A}_{22}$ is nonsingular, $\bar{A}_{21} \bar{x}_1 = 0$ is required in a necessary and sufficient manner so that the two algebraic equations are consistent. To summarize, the "impulse-free" space (area) is obtained as $\left\{ x \in \mathcal{R}^n \middle| \bar{A}_{21} \begin{bmatrix} I_r & 0 \end{bmatrix} N^{-1} x = 0 \right\}$, which provides an easy-to-check condition for "impulse-free" switching. ∎

B. Comparison with Existing Commutation Conditions

In this subsection, we consider the relation of Theorem 1 with the existing commutation conditions in [9], [15].

Lemma 2:[9] *Consider the switched system composed of*

$$x(k+1) = A_1 x(k) \,, \quad x(k+1) = A_2 x(k) \tag{3.21}$$

and assume that A_1 and A_2 are (Schur) stable matrices such that $A_1 A_2 = A_2 A_1$. Then,

1) *the switched system is exponentially stable under arbitrary switching;*
2) *there exists a common quadratic Lyapunov function $V(x) = x^\top P x$ for the subsystems.*

In the case that E is nonsingular, the commutation condition in the above lemma is written as

$$\begin{aligned} &(E^{-1} A_1)(E^{-1} A_2) = (E^{-1} A_2)(E^{-1} A_1) \\ &\iff A_1 E^{-1} A_2 = A_2 E^{-1} A_1 \,. \end{aligned} \tag{3.22}$$

In our discussion, when E is nonsingular, we obtain $r = n$ and simply choose $N = E^{-1}, M = I_n$ or $N = I_n, M = E^{-1}$ in the transformation (3.1). Then, the commutation condition (3.3) is the same as (3.22). This implies that the commutation condition together with the stability result in this paper is an extension of Ref. [9] to switched linear descriptor systems.

Next, we proceed to compare Theorem 1 with the commutation condition proposed in [14], [15].

Lemma 3:[15] *If the two descriptor systems in (2.3) are stable, and furthermore the descriptor matrix E and the two system matrices A_1, A_2 are commutative pairwise, i.e.,*

$$E A_1 = A_1 E \,, \ E A_2 = A_2 E \,, \ A_1 A_2 = A_2 A_1 \,, \tag{3.23}$$

then the switched system (2.3) is stable under impulse-free arbitrary switching. ∎

The next theorem shows that the commutation condition (3.3) is an extension to (3.23).

Theorem 2: *If the commutation condition (3.23) holds, there exist nonsingular matrices M, N in (3.1) such that the condition (3.3) is satisfied.*

Proof: As shown in the proof of Theorem 1 in [15], when the commutation condition (3.23) holds, there exist nonsingular matrices M, N such that

$$\begin{aligned} MEN &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \\ MA_1 N &= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, MA_2 N = \begin{bmatrix} \Lambda_2 & 0 \\ 0 & X \end{bmatrix} \end{aligned} \tag{3.24}$$

where Λ_1 and Λ_2 are (Schur) stable matrices satisfying $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$, and $X \in \mathcal{R}^{(n-r)\times(n-r)}$ is nonsingular. Here, without causing confusion, we used the same notations as before.

After some simple calculation, we obtain

$$\begin{aligned} &M A_1 N_L M_U A_2 N \\ &= (M A_1 N)(N^{-1} N_L)(M_U M^{-1})(M A_2 N) \\ &= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \end{bmatrix} \begin{bmatrix} \Lambda_2 & 0 \\ 0 & X \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_1 \Lambda_2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &M A_2 N_L M_U A_1 N \\ &= (M A_2 N)(N^{-1} N_L)(M_U M^{-1})(M A_1 N) \\ &= \begin{bmatrix} \Lambda_2 & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_2 \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.26}$$

Since $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$ and N, M are nonsingular, we obtain $A_1 N_L M_U A_2 = A_2 N_L M_U A_1$, which is exactly the commutation condition (3.3) in Theorem 1. ∎

IV. Relation with Existence of CQLF

Part 2) of Lemma 2 in the previous section states that when the commutation condition $A_1 A_2 = A_2 A_1$ holds for switched linear systems in the state space representation, there exists a CQLF for the subsystems. The next theorem extends this important part for Theorem 1. More precisely, it shows that when the commutation condition (3.3) holds, there exists a CQLF for the subsystems in (2.3).

Theorem 3: *If the two descriptor subsystems in (2.3) are stable, and furthermore the commutation condition (3.3) holds, then there are nonsingular symmetric matrices $P_i \in \mathcal{R}^{n\times n}$, $i = 1, 2$, such that*

$$E^\top P_i E \geq 0 \tag{4.1}$$

$$A_i^\top P_i A_i - E^\top P_i E < 0 \tag{4.2}$$

and furthermore

$$E^\top P_1 E = E^\top P_2 E\,. \tag{4.3}$$

It is known [18] that (4.1)-(4.2) guarantees $V_i(x) = x^\top E^\top P_i E x$ is a Lyapunov function for stability of the i-th subsystem. Thus, (4.1)-(4.3) shows there is a CQLF $x^\top E^\top P_i E x$ for the two subsystems.

Proof of Theorem 3: In the proof of Theorem 1, we have obtained that when the two descriptor systems in (2.3) are stable and (3.3) holds, the original switched system is equivalent to the switched system composed of (3.5) and (3.7). Since Λ_1 and Λ_2 are commutative, we obtain a common positive definite matrix P_Λ satisfying

$$\Lambda_1^\top P_\Lambda \Lambda_1 - P_\Lambda < 0\,, \quad \Lambda_2^\top P_\Lambda \Lambda_2 - P_\Lambda < 0\,. \tag{4.4}$$

Use the above matrix P_Λ to define

$$P_i = M^\top \begin{bmatrix} P_\Lambda & P_{12}^i \\ (P_{12}^i)^\top & P_{22}^i \end{bmatrix} M\,. \tag{4.5}$$

where the matrices P_{12}^i, P_{22}^i are determined later, only assuming P_{22}^i is symmetric presently. Then,

$$\begin{aligned} E^\top P_i E &= N^{-\top}(MEN)^\top (M^{-\top} P_i M^{-1})(MEN)N^{-1} \\ &= N^{-\top} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_\Lambda & P_{12}^i \\ (P_{12}^i)^\top & P_{22}^i \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} \\ &= N^{-\top} \begin{bmatrix} P_\Lambda & 0 \\ 0 & 0 \end{bmatrix} N^{-1} \geq 0\,, \end{aligned} \tag{4.6}$$

which shows (4.1) and (4.3) are true.

To prove (4.2), we first obtain

$$\begin{aligned} & A_1^\top P_1 A_1 \\ &= N^{-\top}(MA_1N)^\top (M^{-\top} P_1 M^{-1})(MA_1N)N^{-1} \\ &= N^{-\top} \begin{bmatrix} \Lambda_1^\top & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} P_\Lambda & P_{12}^1 \\ (P_{12}^1)^\top & P_{22}^1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} N^{-1} \\ &= N^{-\top} \begin{bmatrix} \Lambda_1^\top P_\Lambda \Lambda_1 & \Lambda_1^\top P_{12}^1 \\ (P_{12}^1)^\top \Lambda_1 & P_{22}^1 \end{bmatrix} N^{-1}\,, \end{aligned} \tag{4.7}$$

and thus

$$\begin{aligned} & A_1^\top P_1 A_1 - E^\top P_1 E \\ &= N^{-\top} \begin{bmatrix} \Lambda_1^\top P_\Lambda \Lambda_1 - P_\Lambda & \Lambda_1^\top P_{12}^1 \\ (P_{12}^1)^\top \Lambda_1 & P_{22}^1 \end{bmatrix} N^{-1}\,. \end{aligned} \tag{4.8}$$

Since N is nonsingular and $\Lambda_1^\top P_\Lambda \Lambda_1 - P_\Lambda < 0$, we can simply choose $P_{12}^1 = 0$ and $P_{22}^1 = -\eta I$ with any positive scalar η to achieve $A_1^\top P_1 A_1 - E^\top P_1 E < 0$.

Next, we choose $P_{12}^2 = 0$ in P_2 and obtain by similar calculation that

$$\begin{aligned} & A_2^\top P_2 A_2 \\ &= N^{-\top}(MA_2N)^\top (M^{-\top} P_2 M^{-1})(MA_2N)N^{-1} \\ &= N^{-\top} \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}^\top \begin{bmatrix} P_\Lambda & 0 \\ 0 & P_{22}^2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} N^{-1}\,. \end{aligned} \tag{4.9}$$

Using the fact

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix} = \begin{bmatrix} \Lambda_2 & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \tag{4.10}$$

we obtain from (4.9) that

$$\begin{aligned} & A_2^\top P_2 A_2 \\ &= \bar{N}^{-\top} \begin{bmatrix} \Lambda_2^\top P_\Lambda \Lambda_2 & \Lambda_2^\top P_\Lambda \bar{A}_{12} \\ \bar{A}_{12}^\top P_\Lambda \Lambda_2 & \bar{A}_{22}^\top P_{22}^2 \bar{A}_{22} + \bar{A}_{12}^\top P_\Lambda \bar{A}_{12} \end{bmatrix} \bar{N}^{-1} \end{aligned} \tag{4.11}$$

where

$$\bar{N} = N \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix} \tag{4.12}$$

is also nonsingular. Together with the fact

$$\bar{N}^{-\top} \begin{bmatrix} P_\Lambda & 0 \\ 0 & 0 \end{bmatrix} \bar{N}^{-1} = N^{-\top} \begin{bmatrix} P_\Lambda & 0 \\ 0 & 0 \end{bmatrix} N^{-1}\,, \tag{4.13}$$

we reach

$$\begin{aligned} & A_2^\top P_2 A_2 - E^\top P_2 E \\ &= \bar{N}^{-\top} \begin{bmatrix} \Lambda_2^\top P_\Lambda \Lambda_2 - P_\Lambda & \Lambda_2^\top P_\Lambda \bar{A}_{12} \\ \bar{A}_{12}^\top P_\Lambda \Lambda_2 & \bar{A}_{22}^\top P_{22}^2 \bar{A}_{22} + \bar{A}_{12}^\top P_\Lambda \bar{A}_{12} \end{bmatrix} \bar{N}^{-1}. \end{aligned} \tag{4.14}$$

Since $\bar{A}_{22}$ is nonsingular, we can always choose $P_{22}^2 = -\mu I$ with a large positive scalar μ such that $\bar{A}_{22}^\top P_{22}^2 \bar{A}_{22} + \bar{A}_{12}^\top P_\Lambda \bar{A}_{12}$ is negative definite enough. Combining with the fact that $\Lambda_2^\top P_\Lambda \Lambda_2 - P_\Lambda < 0$, we obtain that we can always choose P_{22}^2 such that $A_2^\top P_2 A_2 - E^\top P_2 E < 0$. This completes the whole proof. ∎

Remark 6: As also shown in [16], the proofs of Theorem 1 and Theorem 3 suggest that $V(\bar{x}) = \bar{x}_1^\top P_\Lambda \bar{x}_1$ is a common quadratic Lyapunov function for the systems (3.5) and (3.7). In fact, this is rationalized by the following equation.

$$\begin{aligned} & x^\top E^\top P_i E x \\ &= (N^{-1}x)^\top (MEN)^\top (M^{-\top} P_i M^{-1})(MEN)(N^{-1}x) \\ &= \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}^\top \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_\Lambda & 0 \\ 0 & P_{22}^i \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \\ &= \bar{x}_1^\top P_\Lambda \bar{x}_1 \end{aligned} \tag{4.15}$$

Therefore, although $E^\top P_i E$ is not positive definite and neither is $V_i(x) = x^\top E^\top P_i E x$, the function $V_i(x)$ plays the role of a CQLF for the two descriptor subsystems. ∎

Remark 7: The conditions (4.1)-(4.3) in Theorem 3 include a non-strict matrix inequality and an equation, which may not be easy to solve using the existing LMI Control Toolbox in Matlab [19]. As a matter of fact, the proof of Theorem 3 suggested an alternative method for solving it in the framework of strict LMIs: (a) decompose E as in (3.1) using nonsingular matrices M and N, and compute MA_2N; (b) solve the following simultaneous strict LMIs [20]

$$\begin{bmatrix} \Lambda_1^\top P_\Lambda \Lambda_1 - P_\Lambda & 0 \\ 0 & P_{22}^1 \end{bmatrix} < 0$$

$$\begin{bmatrix} \Lambda_2^\top P_\Lambda \Lambda_2 - P_\Lambda & \Lambda_2^\top P_\Lambda \bar{A}_{12} \\ \bar{A}_{12}^\top P_\Lambda \Lambda_2 & \bar{A}_{22}^\top P_{22}^2 \bar{A}_{22} + \bar{A}_{12}^\top P_\Lambda \bar{A}_{12} \end{bmatrix} < 0 \tag{4.16}$$

with respect to P_Λ and P_{22}^1, P_{22}^2, where $P_\Lambda > 0$ and P_{22}^1, P_{22}^2 are symmetric; (c) compute the original P_i with $P_i = M^\top \begin{bmatrix} P_\Lambda & 0 \\ 0 & P_{22}^i \end{bmatrix} M$, $i = 1, 2$. ∎

V. Concluding Remarks

In this paper, we have established a new commutation condition for stability of switched linear discrete-time descriptor systems under impulse-free arbitrary switching. We have also shown that when the proposed commutation condition holds, there exists a CQLF for the subsystems. These results are natural and important extensions to the existing results for switched systems in the state space representation.

It is easy to see that the proposed commutation condition can be used to deal with the case where there are both stable and unstable subsystems involved [21], and the case where there is no stable subsystem but there is a stable combination of the subsystems [22] taking the form of $(E, \lambda A_1 + (1-\lambda)A_2)$ [14], [15]. It is noted that careful consideration of excluding impulsive signals is still desired, especially in the case where the subsystems have different descriptor matrices.

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