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Terminal Sliding Mode Control of Z-axis MEMS Gyroscope with Observer Based Rotation Rate Estimation

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Abstract - **This paper presents a novel methodology for approximation of the unknown time-varying rotation rate using sliding mode observer as well as a robust control scheme for improving the performance of the MEMS gyroscope despite the coupling between vibratory gyroscope modes and inherent model uncertainties. Terminal sliding mode control (TSMC) is invoked to develop tracking control of the drive and sense modes based on the uncertain model of vibratory gyroscope and subsequently the swiftness of TSMC scheme in comparison with conventional sliding mode control (SMC) is demonstrated. The robust terms of proposed sliding mode observer are designed such that the unknown functions including Coriolis acceleration and quadrature error terms are tracked and then the unknown rotation rate and stiffness coupling are constructed. The asymptotic stability and robustness of the proposed control and observer are proved using second method of Lyapunov. Finally, effectiveness of the proposed observer based control for approximation of the unknown time-varying rotation rate is demonstrated through simulations.**

Keywords—**Robustness, Terminal Sliding Mode Control, Sliding Mode Observer, MEMS Gyroscope**

I. INTRODUCTION

Gyroscopes are the inertial sensors which measure the rotational rate of an object. Microelectromechanical technologies have provided possibility of modeling and fabricating of gyroscopes with small size, low cost and low power consumption [1], [2]. These advantages offer wide application spectrum of MEMS gyroscopes in the aerospace industry, military, automotive such as high performance navigation and guidance systems, ride stabilization, roll-over detection and prevention, and next generation airbag and brake systems and electronics markets such as image stabilization in digital cameras and camcorders, virtual reality products, inertial pointing devices, and computer gaming industry [3], [4].

A vibratory Z-axis MEMS gyroscope which is sensitive to the angular rate about the Z-axis perpendicular to the plane of silicon substrate, developed by Berkeley Sensor and Actuator Center [5]. The fundamental architecture of a vibratory MEMS gyroscope is comprised of a drive-mode oscillator that generates and maintains a constant linear or angular momentum. Drive-mode oscillator is coupled to a sense-mode Coriolis accelerometer that measures the sinusoidal Coriolis force induced due to the combination of the drive vibration and an unknown angular rate input. In other words, when the gyroscope is exposed to an unknown rotational rate, the Coriolis acceleration causes that the energy transfers from drive-mode to sense-mode providing the information of the unknown rotation rate.

Since majority of the micro-machines gyroscopes utilize vibrating mechanical elements to sense angular rate, inherent fabrication imperfections along with environmental variations make frequency mismatch between two vibration modes, unknown disturbances, and parameter variations which significantly limit the performance, stability, and robustness of the vibratory MEMS gyroscope [6]. As a consequence, to overcome these difficulties, introducing a robust control system is necessary for the MEMS gyroscope ensuring its desired performance.

The most challenging control issue includes minimization of coupling between the actuation and sensing modes along with the unknown time-varying angular rate measurement. In the literature, several control methodologies have been proposed to enhance performance and robustness of MEMS gyroscope. Most of these designs are based on constraining the oscillation degree-of-freedom of the proof mass to lie only in the drive direction. In these designs, a part of Coriolis force induced in the proof mass is transferred from driving mode to sensing mode while proof mass is not allowed to oscillate in the sense direction [7]-[13]. Park and Horowitz *et al.* [7], [8] proposed two different adaptive controllers for a MEMS gyroscope controlling the entire operation of the device while the angular rate was assumed constant. Dong *et al.* [9] designed an adaptive controller with time-varying rotational rate according what happens in reality, but the parameters of the controller make it difficult to implement. Batur *et al.* [10], [11] introduced sliding mode control to MEMS gyroscopes meeting constant angular rate. These approaches utilize respectively demodulation and adaption technique for angular rate estimation.

This paper proposes a novel methodology for approximation of the unknown time-varying rotation rate by using a sliding mode observer along with a robust control system based on terminal sliding mode control (TSMC) for minimizing the coupling between two operational modes of MEMS gyroscopes while time-varying rotation rate and the stiffness coupling between gyroscope modes arising from mechanical imperfections are unknown.

Section 2 describes the model of a Z-axis MEMS gyroscope utilized in the paper. Section 3 outlines a terminal sliding mode control law for the MEMS gyroscope based on uncertain model. A sliding mode observer based control scheme for the unknown time-varying rotation rate approximation is developed in Section 4. Simulation results and comparative discussions are presented in Section 5 and some conclusions are made in Section 6.

II. Dynamic Model of Z-axis MEMS Gyroscope

A typical vibratory MEMS gyroscope is comprised of a proof mass, a suspension system, and electrostatic actuations and sensing mechanisms for forcing an oscillatory motion and sensing the position and velocity of the proof mass as well as a rigid frame which is rotated about the rotation axis [14]. Dynamics of the MEMS gyroscope is derived with respect to two coordinate systems: the inertial frame fixed in an inertial space, and the gyroscope frame fixed to the rotation platform.

With the definition of r_B, v_B, and a_B as the position, velocity, and acceleration vectors with respect to the rotating gyroscope frame, A as the linear acceleration of the gyroscope frame, and Ω as the angular velocity vector of the gyroscope frame; the expression for the equation of the motion of the proof mass reduces to

$$\vec{F}_{ext} = m[A + a_B + \dot{\Omega} \times r_B + \Omega \times (\Omega \times r_B) + 2\Omega \times v_B] \tag{1}$$

where $\vec{F}_{ext}$ is a total applied force to the proof mass, which includes spring, damping and control forces.

In a Z-axis gyroscope, by supposing the stiffness of spring in z direction much larger than that in x, y directions, motion of proof mass is constrained to only along the xy plane as shown in Fig.1 [7].

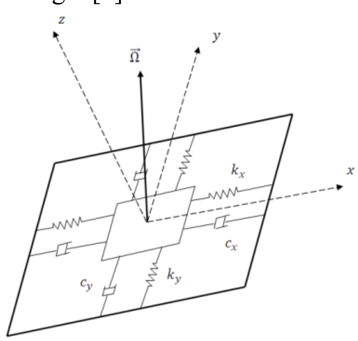

Fig.1. A simplified model of a z-axis MEMS Gyroscope

Decomposing the motion into the two principle oscillation directions and assuming that the linear accelerations are negligible, the two equations of motion along the drive and sense axes can be expressed as

$$\begin{aligned} m\ddot{x} + c_x\dot{x} + \left(k_x - m(\Omega_y^2 + \Omega_z^2)\right)x + m(\Omega_x\Omega_y - \dot{\Omega}_z)y &= u_x + 2m\Omega_z\dot{y} \\ m\ddot{y} + c_y\dot{y} + \left(k_y - m(\Omega_x^2 + \Omega_z^2)\right)y + m(\Omega_x\Omega_y + \dot{\Omega}_z)x &= u_y - 2m\Omega_z\dot{x} \end{aligned} \tag{2}$$

where x and y are the coordinates of the proof mass with respect to the gyroscope frame, m is the proof mass, c_x, c_y are damping coefficients, k_x, k_y are spring coefficients, Ω_i while $i = x, y, z$ are the angular velocity components along each axis of the gyroscope frame and u_x, u_y are control forces. The two last terms in equation (2), $2m\Omega_z\dot{y}, 2m\Omega_z\dot{x}$ are the Coriolis forces and are the terms which are used to construct the unknown time-varying angular rate Ω_z. Under typical assumptions $\Omega_x^2 \approx \Omega_y^2 \approx \Omega_z^2 \approx \Omega_x\Omega_y \approx 0$, only the component of the angular rate Ω_z causes a dynamic coupling between the x and y axes [7].

Taking into account fabrication imperfections occurring always and causing dynamic coupling between two modes and substituting $\omega_i = \sqrt{\frac{k_i}{m}}$, $c_i = 2m\xi_i\omega_i$ while $i = x, y$, the dynamic equation (2) are modified as follows

$$\begin{aligned} \ddot{x} + 2\xi_x\omega_x\dot{x} + \omega_x^2 x + \omega_{xy} y &= b_x u_x + 2\Omega_z\dot{y} \\ \ddot{y} + 2\xi_y\omega_y\dot{y} + \omega_y^2 y + \omega_{xy} x &= b_y u_y - 2\Omega_z\dot{x} \end{aligned} \tag{3}$$

where ω_x, ω_y are natural frequencies of drive and sense modes, ξ_x, ξ_y are damping coefficients, $\omega_{xy}x, \omega_{xy}y$ are constant unknown quadrature error terms caused by stiffness couplings between two axes due to fabrication imperfections, and b_x, b_y are the constants that account for sensor, actuator, and amplifier gains [12].

Moreover, by defining $q = [x \quad y]^T$ and $U = [u_x \quad u_y]^T$, the dynamics of Z-axis MEMS gyroscope is rewritten in vector form as

$$\ddot{q} + (D + 2\Omega)\dot{q} + (K_a + K_b)q = BU \tag{4}$$

where $D = \begin{bmatrix} 2\xi_x\omega_x & 0 \\ 0 & 2\xi_y\omega_y \end{bmatrix}$, $\Omega = \begin{bmatrix} 0 & -\Omega_z \\ \Omega_z & 0 \end{bmatrix}$, $K_a = \begin{bmatrix} \omega_x^2 & 0 \\ 0 & \omega_y^2 \end{bmatrix}$, $K_b = \begin{bmatrix} 0 & \omega_{xy} \\ \omega_{xy} & 0 \end{bmatrix}$, and $B = \begin{bmatrix} b_x & 0 \\ 0 & b_y \end{bmatrix}$. In this paper, the bounded structural uncertainties of the system are assumed to be in the following form

$$\omega^{lower} < \omega_{xy} < \omega^{upper} \quad , \quad \Omega^{lower} < \Omega_z < \Omega^{upper}$$

III. Terminal Sliding Mode Control

This section proposes a robust sliding mode controller for the MEMS gyroscope described by (4). The objective of control problem is to force drive and sense modes to oscillate at specified amplitudes and high frequencies (much more than time-varying rotation rate frequency) despite the fact that the motions in the x and y directions are coupled and the Coriolis acceleration and quadrature error terms are unknown. It is important to note that contrary to conventional drive-mode control approaches that maintain the proof mass to oscillate only in the x direction for measuring unknown rotation rate in sense direction [7]-[13], here there is no constraint on motion of proof mass.

In conventional sliding mode control, variable control systems are designed to drive and then constrain the system stable to lie within a neighborhood of the switching function. The sliding mode control design approach consists of two components. The first involves the design of a switching function so that the sliding motion satisfies design specifications. The second is concerned with the selection of a control law which will make the switching function attractive to the system state [15]-[17]. The basic idea in terminal sliding mode scheme is making the convergence rate of control law exponentially fast when the state is near equilibrium.

The dynamics of a Z-axis MEMS gyroscope with bounded uncertainties concentrating in term $h(q, \dot{q})$ and estimating as $\hat{h}$, is rewritten in the following form

$$\ddot{q} = -D\dot{q} - K_a q + h(q, \dot{q}) + BU \tag{5}$$

where the estimation error on $h(q,\dot{q})$ which includes unknown Coriolis acceleration and quadrature error terms is assumed to be bounded by some known function $H(q,\dot{q})$ as

$$\left|h-\hat{h}\right| \le H \tag{6}$$

In order to maintain the proof mass to track a smooth desired trajectory $Q_d = [q_d \quad \dot{q}_d \quad \ddot{q}_d]^T$ which $q_d = [x_d \quad y_d]^T$ includes desired proof mass oscillations in the actuation and sensing directions at given frequencies and amplitudes, the terminal sliding manifold is defined as [18], [19]

$$S(q,t) = \dot{\tilde{q}} + \Lambda\tilde{q} + C\tilde{q}^{\frac{\alpha}{\beta}} \tag{7}$$

where $S = [s_x \quad s_y]^T$, $\tilde{q} = q - q_d$ is defined as tracking error, and Λ, C are positive definite constant matrixes to be respectively selected i.e. $\Lambda = dig\{\lambda_x, \lambda_y\}$ and $C = dig\{c_x, c_y\}$ and α, β are the positive odd integers to be chosen such that $\beta > \alpha$. It should be noted that the finite time convergence dynamic in terminal sliding mode control depending on design parameters C, α, β, in contrast with conventional sliding mode control, implies a swifter tracking capability in the Z-axis MEMS gyroscope control problem. To ensure that the state of the system approaches the terminal sliding surface, first derivative of the sliding surface should be converged to zero as follow

$$\begin{aligned}\dot{S} &= \ddot{\tilde{q}} + \Lambda\dot{\tilde{q}} + C\frac{\alpha}{\beta}\dot{\tilde{q}}\tilde{q}^{\frac{\alpha-\beta}{\beta}} \\ &= BU - D\dot{q} - K_a q + h(q,\dot{q}) - \ddot{q}_d + \Lambda\dot{\tilde{q}} + C\frac{\alpha}{\beta}\dot{\tilde{q}}\tilde{q}^{\frac{\alpha-\beta}{\beta}} = 0\end{aligned} \tag{8}$$

The best approximation of continuous equivalent control law that would achieve $\dot{S} = 0$ is

$$\hat{U} = B^{-1}[D\dot{q} + K_a q - \hat{h} + \ddot{q}_d - \Lambda\dot{\tilde{q}} - C\frac{\alpha}{\beta}\dot{\tilde{q}}\tilde{q}^{\frac{\alpha-\beta}{\beta}}] \tag{9}$$

In order to satisfy sliding condition [15] despite uncertainty on the dynamics of the MEMS gyroscope, a discontinuous term across the terminal sliding surface is added to $\hat{U}$. Consequently terminal sliding mode control law is proposed as

$$U = \hat{U} - B^{-1}E\, sign(S) \tag{10}$$

where $sign(S) = [sign(s_x) \quad sign(s_y)]^T$ and E is a positive definite constant matrix i.e. $E = dig\{\eta_x, \eta_y\}$ depending on the upper bounds of unknown Coriolis acceleration and quadrature error terms in both direction *x, y* and reaching times and its diagonal elements are selected under conditions as follow

$$\begin{aligned}\eta_x &= H_x + \rho_x = F_x + G_x + \rho_x \\ \eta_y &= H_y + \rho_y = F_y + G_y + \rho_y\end{aligned} \tag{11}$$

It is essential to recall that F_i, G_i while $i = x, y$ are respectively upper known bounds of estimation error on Coriolis acceleration and quadrature error terms in drive and sense directions according to (6) which are calculated simply by knowing upper and lower bounds of unknown quantities Ω_z, ω_{xy} and ρ_x, ρ_y are two strictly positive constants [15].

Proposition 1: Consider the Z-axis MEMS gyroscope (4) while time-varying rotation rate $\mathbf{\Omega_z}$ and stiffness coupling ω_{xy} are unknown. The robust sliding mode control law given by (10) under conditions (11) ensures that terminal sliding manifold S converges to zero in finite time and consequently forces both coupled drive and sense mode to track desired trajectories $q_d, \dot{q}_d$ oscillating at specified amplitudes and frequencies.

Proof: The stability and robustness analysis of the proposed terminal sliding mode control law in presence of unknown time-varying angular rate and stiffness coupling as the uncertainties is accomplished by choosing a Lyapunov function as

$$V(S) = \frac{1}{2}S^T S \tag{12}$$

Differentiating V with respect to time yields

$$\dot{V}(S) = S^T\dot{S} = S^T(h - \hat{h} - E\, sign(S)) \tag{13}$$

Expanding (13) yields

$$\begin{aligned}\dot{V} &= s_x\left[h_x - \hat{h}_x - \eta_x\, sign(s_x)\right] \\ &\qquad + s_y\left[h_y - \hat{h}_y - \eta_y\, sign(s_y)\right] \\ &= \left(h_x - \hat{h}_x\right)s_x - \eta_x|s_x| + \left(h_y - \hat{h}_y\right)s_y - \eta_y|s_y|\end{aligned} \tag{14}$$

Decomposition unknown function $h(q,\dot{q})$ to unknown quadrature error terms $f(q)$ and Coriolis acceleration terms $g(\dot{q})$ yields

$$\begin{aligned}\dot{V} &= \left(f_x - \hat{f}_x\right)s_x + (g_x - \hat{g}_x)s_x - \eta_x|s_x| \\ &\quad + \left(f_y - \hat{f}_y\right)s_y + (g_y - \hat{g}_y)s_y - \eta_y|s_y|\end{aligned} \tag{15}$$

It is obvious by rewriting (6) as follow

$$\begin{aligned}&\left(f_i - \hat{f}_i\right) \le F_i \qquad i = x, y \\ &(g_i - \hat{g}_i) \le G_i \qquad i = x, y\end{aligned} \tag{16}$$

and substituting η_x, η_y according to (11) into (15) makes

$$\begin{aligned}\dot{V} &= \left(f_x - \hat{f}_x\right)s_x + (g_x - \hat{g}_x)s_x - \left(F_x + G_x + \rho_y\right)|s_x| \\ &\quad + \left(f_y - \hat{f}_y\right)s_y + (g_y - \hat{g}_y)s_y - \left(F_y + G_y + \rho_y\right)|s_y| < 0\end{aligned} \tag{17}$$

In other words, $\dot{V}$ is strictly negative outside the terminal sliding surface and consequently sliding condition is verified and stability and robustness of control law is ensured. As a result, all system trajectories in directions x and y are respectively constrained to the terminal sliding surfaces s_x and s_y.

Remark 1: There is a possible singularity in terminal sliding mode control when $\tilde{q} \to 0$. Thus, the selection of α, β is critical in the design of an appropriate TSMC. To avoid singularity, α, β have been proposed to be chosen such that $2\alpha > \beta$ [20].

Remark 2: The control law (10) only remains continuous prior to entering into the terminal sliding manifold $S = 0$. It makes implementation so hard and impractical due to discontinuity of sign function at zero. Moreover, this discontinuity causes unwanted chattering phenomena which may excite the high frequency unmodeled dynamics. Thus, for continuous approximation of switching control law and alleviating chattering on terminal sliding surface, a saturation function is applied rather than sign function as follow

$$sat\left(\frac{\cdot}{\varphi}\right) = \begin{cases} -1 & if\ \cdot < \varphi \\ \dfrac{\cdot}{\varphi} & if\ |\cdot| \le \varphi \\ 1 & if\ \cdot > \varphi \end{cases} \tag{18}$$

where φ is called boundary layer thickness and is a positive constant.

This section is terminated with the fact that the control law (10) utilizes full state for feedback which requires both position and velocity sensor in micron dimensions. Let note that the sliding mode observer proposed in subsequent section for rotation rate estimation enable to provide an acceptable estimation of velocity which can be used for feedback rather than its actual value [24].

IV. Sliding Mode Observer Based Rotation Rate Estimation

In this section a sliding mode observer is proposed for the Z-axis MEMS gyroscope. In contrast with the most approaches which estimate unknown angular velocity using demodulation of sense control input u_y while sense mode is enforced to zero [7]-[11], the main objective of the robust observation in this paper is to approximate the unknown vector function $h(q,\dot{q})$ including Coriolis acceleration and quadrature error terms in addition to providing an acceptable estimation of the state system unavailable in output. As a result, the unknown time-varying rotation rate and the stiffness coupling between gyroscope modes are explicitly able to be reconstructed by demodulating of proposed sliding mode observer outputs.

A. Sliding mode observer design

Sliding mode observers are very useful means which have been developed for many reasons like working with reduced observation error dynamics, possibility of obtaining a step by step design, a finite time convergence for all the observable states and robustness under bounded uncertainties of the systems [21]-[23]. It is important to recall that the proposed robust control law in previous section ensures that the unknown terms of vibratory gyroscope would be bounded all the time.

The dynamics of a Z-axis MEMS gyroscope (5) is rewritten in the following state space form

$$\begin{aligned}\dot{q}_1 &= q_2\\ \dot{q}_2 &= -Dq_2 - K_a q_1 + h(q_1,q_2) + BU\end{aligned} \tag{19}$$

where $q_1 = q = [x \quad y]^T = [x_1 \quad y_1]^T$ denotes just measurable state variables in system output versus unavailable state variables $q_2 = \dot{q} = [\dot{x} \quad \dot{y}]^T = [x_2 \quad y_2]^T$, $h(q_1,q_2) = [h_x(y_1,y_2) \quad h_y(x_1,x_2)]^T$ includes unknown Coriolis acceleration and quadrature error terms in both gyroscope directions, and U is the proposed terminal sliding mode control law which guarantees unknown vector function $h(q_1,q_2)$ to be bounded in all operational time of device.

Let us consider classical sliding mode observer for the Z-axis MEMS gyroscope as follow [21]

$$\begin{aligned}\dot{\hat{q}}_1 &= \hat{q}_2 + L_1 sign(q_1 - \hat{q}_1)\\ \dot{\hat{q}}_2 &= -D(\hat{q}_2 + L_1 sign(q_1 - \hat{q}_1)) - K_a q_1 + BU\\ &\quad + L_2 sign(L_1 sign(q_1 - \hat{q}_1))\end{aligned} \tag{20}$$

where $\hat{q}_1, \hat{q}_2$ respectively represent the estimated value of actual state variables q_1, q_2 and L_1, L_2 are two positive definite constant matrixes representing observer gains to be respectively selected i.e. $L_1 = dig\{\lambda_1, \lambda_2\}$ and $L_2 = dig\{\lambda_3, \lambda_4\}$ and ultimately $sign(q_1 - \hat{q}_1) = [sign(x_1 - \hat{x}_1) \quad sign(y_1 - \hat{y}_1)]^T$.

By taking $e_i = q_i - \hat{q}_i$ while $i = 1,2$, the error observation dynamics are obtained from (19) and (20) as

$$\begin{aligned}\dot{e}_1 &= e_2 - L_1 sign(e_1)\\ \dot{e}_2 &= -D(e_2 - L_1 sign(e_1)) + h(q_1,q_2)\\ &\quad - L_2 sign(L_1 sign(e_1))\end{aligned} \tag{21}$$

By choosing observer gains such that satisfy the following inequalities

$$\begin{aligned}&\lambda_1 > |x_2 - \hat{x}_2|_{max} \qquad &&\lambda_2 > |y_2 - \hat{y}_2|_{max}\\ &\lambda_3 > |h_x(y_1,y_2)|_{max} &&\lambda_4 > |h_y(x_1,x_2)|_{max}\end{aligned} \tag{22}$$

Observation error trajectories (21) reach the sliding mode and then asymptotically converge to zero in finite time or in other words estimated state $\hat{q}_i$ while $i = 1,2$ converges to its actual value q_i. In sliding mode, the unknown term $h(q_1,q_2)$ including Coriolis accelerations and quadrature errors can be simply derived for reconstruction of the unknown time-varying rotation rate.

Proposition 2: Consider the state space representation of Z-axis MEMS gyroscope (19) while time-varying rotation rate Ω_z and stiffness coupling ω_{xy} are unknown.

1- The proposed sliding mode observer (20) under condition (22) converges to dynamics of uncertain Z-axis MEMS gyroscope (19) in finite time by just utilizing the measureable state variable q_1 in output.

2- Once the error observation trajectories reach the sliding mode, the unknown vector function $h(q_1,q_2)$ can be estimated as follow

$$\hat{h} = L_2 sign(L_1 sign(q_1 - \hat{q}_1)) \tag{23}$$

Proof: The convergence and robustness of the proposed observer (20) is also proved using second method of Lyapunov. The analysis criterion for the convergence of observation error on $q_1$is based on the following Lyapunov function

$$V_1 = \frac{1}{2} e_1^T e_1 \tag{24}$$

Differentiating V_1 with respect to time yields

$$\dot{V}_1 = e_1^T \dot{e}_1 = e_1^T (e_2 - L_1 sign(e_1)) \tag{25}$$

Obviously, choosing diagonal elements of L_1 according to (22) makes $\dot{V}_1 < 0$. It means by decreasing Lyapunov function with respect to time, $\hat{x}_1 \to x_1$ and $\hat{y}_1 \to y_1$ in finite times t_1, t_2 and remain equal to x_1 and y_1 for $t > t_1$ and $t > t_2$ respectively. Moreover, for $t > t_{12} = max\,(t_1,t_2)$, $\dot{e}_{1\approx} 0$, meaning

$$e_2 = L_1 sign(e_1) \tag{26}$$

Consequently for $t > t_{12}$ the observation error dynamics is now equal to

$$\begin{aligned}\dot{e}_1 &= 0\\ \dot{e}_2 &= h(q_1,q_2) - L_2 sign(e_2)\end{aligned} \tag{27}$$

Subsequently, the second Lyapunov function is defined for the convergence analysis of observation error on q_2 as follow

$$V_2 = \frac{1}{2}(e_1^T e_1 + e_2^T e_2) \tag{28}$$

Similarly, differentiating V_2 with respect to time yields

$$\dot{V}_2 = e_1^T \dot{e}_1 + e_2^T \dot{e}_2 = e_2^T\{h(q_1, q_2) - L_2 sign(e_2)\} \tag{29}$$

and choosing diagonal elements of L_2 according to (22) make $\dot{V}_2 < 0$. Thus, $\hat{x}_2 \to x_2$ and $\hat{y}_2 \to y_2$ in finite times t_3, t_4 by decreasing Lyapunov function with respect to time and remain equal to x_2 and y_2 for $t > t_3 > t_1$ and $t > t_4 > t_2$ respectively. Similar to what stated earlier, for $t > t_{34} = max\,(t_3, t_4)$, $\dot{e}_{2\approx}0$, meaning at t_{34} trajectories of observation error reach the sliding mode. Explicitly, an approximation of the unknown vector function $h(q_1, q_2)$ including Coriolis acceleration and quadrature error terms can be derived in sliding mode by just utilizing the measureable state variable q_1 as in (23).

B. Rotation rate estimation

The proposed sliding mode observer provides an appropriate estimate of the unknown part of gyroscope dynamics including modeling errors and structural uncertainties. According to (4), the unknown vector function $h(q_1, q_2)$ has the following structure

$$h(q_1, q_2) = K_b q + 2\Omega\dot{q} \tag{30}$$

Considering (30) shows that in both drive and sense modes, both Coriolis acceleration and quadrature error terms can be amplitude modulated signals centered at the resonant frequencies of the drive and sense axes by using proposed robust control law. Since $q, \dot{q}$ signals in both directions have a relative phase shift of 90°, the undesired quadrature errors from the useful Coriolis accelerations can be separated through the demodulation technique in both directions without any restrictions. Subsequently, the unknown time-varying rotation rate is estimated by filtering the induced demodulated signals.

Applying the sliding mode observer (20) and the terminal sliding mode control law (10) to the Z-axis MEMS gyroscope, the outputs of the drive and sense axis are forced to track the desired trajectories with ideal amplitudes and resonant frequencies much more than time-varying rotation rate frequency and subsequently the unmeasurable state variables and unknown dynamics of vibratory gyroscope are precisely estimated.

The desired trajectory of the drive axis is $x_d = Asin(\omega t)$. Since the proposed robust controller forces the tracking errors to zero, the position and velocity of the proof mass in x-direction would be $x = Asin(\omega t)$, and $\dot{x} = A\omega cos\,(\omega t)$. Substitution of $x, \dot{x}$ into (30) in sense direction, the structural uncertainties due to unknown time-varying rotation rate and the unknown quadrature error terms in sense direction takes the form

$$h_y = -A\omega_{xy} sin\,(\omega t) - 2A\omega\Omega_z cos\,(\omega t) \tag{31}$$

where $\omega \gg \omega_{rate}$ in the Z-axis MEMS gyroscope. It should be noted that the unknown time-varying rotation rate is considered as a sinusoidal signal [7], and $\Omega_z = \Omega_0 + \Omega_1 sin\,(\omega_{rate} t)$ where $\Omega_0, \Omega_1, \omega_{rate}$ are respectively bias, amplitude, and frequency of the rotation rate. Multiplying (31) by $cos\,(\omega t)$ yields

$$\begin{aligned} h_y cos(\omega t) = &-\frac{1}{2} A\omega_{xy}\, sin(2\omega t) \\ &- A\omega\Omega_z\, cos(2\omega t) - A\omega\Omega_z \end{aligned} \tag{32}$$

Since $\omega \gg \omega_{rate}$, the high frequency signals will be filtered out through a low-pass filter (LPF) and thus the time-varying rotation rate Ω_z can be reconstructed as

$$\Omega_z = F_{LPF}\left\{-\frac{h_y cos(\omega t)}{A\omega}\right\} \tag{33}$$

where $F_{LPF}\{.\}$ represent the function of the low-pass filter. The unknown signal h_y is estimated by proposed sliding mode observer according to (23) and as a result, the rotation rate can be estimated by

$$\hat{\Omega}_z = F_{LPF}\left\{-\frac{\lambda_4 sign(\lambda_2 sign(y - \hat{y}))cos(\omega t)}{A\omega}\right\} \tag{34}$$

V. Simulation Results

In this section, the proposed control and unknown rotation rate estimation scheme based on sliding mode observer is simulated on a model of Berkeley vibratory Z-axis MEMS gyroscope [2] with key parameters given in Table.1. Moreover, the actual time-varying rotation rate and coupling stiffness which are unknown, verified $\Omega_{min} < \Omega_z < \Omega_{max}$ and $\omega_{min} < \omega_{xy} < \omega_{max}$. The nominal values of these quantities are given as $\Omega_z = 0.1 + 0.04\, sin(\omega_{rate} t)$ while $\omega_{rate} = 100\pi\ {rad}/{s}$ and $\omega_{xy} = 6000\ rad^2/s^2$ assumed to be varied by 50% of the nominal values in simulations.

Table-1: The Z-axis MEMS Gyroscope parameters used in simulations

Parameter	Value	Unit
$\boldsymbol{m}$	2×10^{-9}	*kg*
$\boldsymbol{\omega_x}$	*81681.4*	*rad/s*
$\boldsymbol{\omega_y}$	*80864.6*	*rad/s*
$\boldsymbol{\xi_x}$	4.5455×10^{-5}	*N/A*
$\boldsymbol{\xi_y}$	3.125×10^{-4}	*N/A*
$\boldsymbol{b_x}$	4.169×10^{8}	kg^{-1}
$\boldsymbol{b_y}$	4.169×10^{8}	kg^{-1}

The key design parameters of the proposed observer based scheme for control and rotation rate estimation of the studied gyroscope are given in Table.2. It is important to note that in all simulation results, the boundary layer thicknesses in $sat\left(\frac{\cdot}{\varphi}\right)$ is selected equal to 0.5.

Table-2: Observer based controller parameters used in simulations

Controller				Observer	
Paramet	**Value**	**Parameter**	**Value**	**Parameter**	**Value**
$\boldsymbol{\lambda_x}$	2×10^4	$\boldsymbol{c_x}$	5×10^2	$\boldsymbol{\lambda_1}$	1×10^6
$\boldsymbol{\lambda_y}$	2×10^4	$\boldsymbol{c_y}$	5×10^2	$\boldsymbol{\lambda_2}$	1×10^6
$\boldsymbol{\rho_x}$	1×10^4	$\boldsymbol{\alpha}$	5	$\boldsymbol{\lambda_3}$	1×10^6
$\boldsymbol{\rho_y}$	1×10^4	$\boldsymbol{\beta}$	9	$\boldsymbol{\lambda_4}$	1×10^6

At first, we will illustrate the efficiency of the proposed robust control scheme for considered vibratory gyroscope. The desired position for the drive mode is assumed to be $x_d = Asin(\omega t)$, where $A = 10^{-6}$ and $\omega = 26800\pi\ rad/s$ and the desired position for sense mode in the easiest case can be $y_d = 0$. The initial value of the proof mass in both directions are assumed to be $x(0) = -1\ \mu m$, $y(0) = 0.5\mu m$ and $\dot{x}(0) = \dot{y}(0) = 0$. A rapid tracking control in both directions can be achieved by using the proposed terminal sliding mode control.

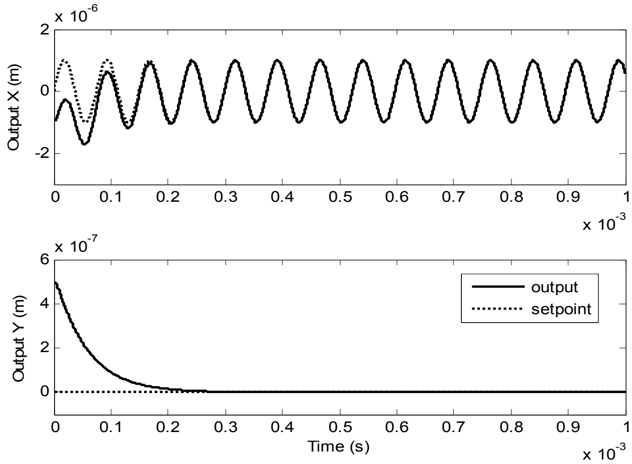

Fig.2. The gyroscope outputs in both drive and sense axis under terminal sliding mode control

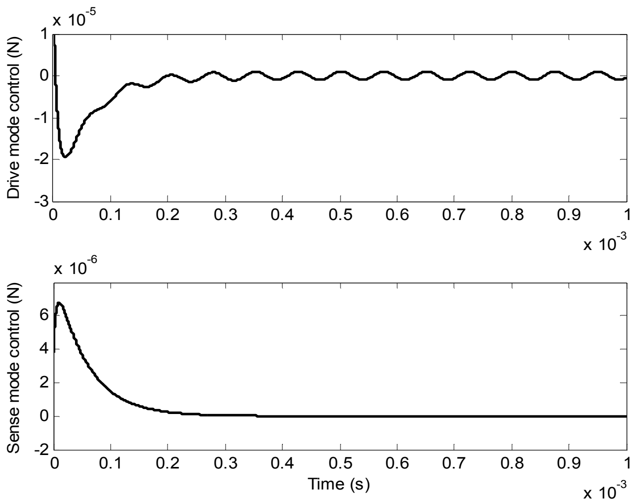

Fig.3. Terminal sliding mode control inputs in both drive and sense axis

The gyroscope outputs and terminal sliding mode control inputs in both drive and sense axis for desired trajectories tracking are respectively depicted in Fig.2 and Fig.3. The corresponding steady state accuracies on the drive and sense axes under terminal sliding mode control being $|x - x_d| \leq 1.5 \times 10^{-17}m$ and $|y - y_d| \leq 1 \times 10^{-12}$. Furthermore, Fig.6 shows the phase trajectories of tracking errors in both drive and sense modes under terminal sliding mode control.

Subsequently, a comparable investigation is accomplished between the proposed terminal sliding mode and conventional sliding mode control laws applied to uncertain vibratory gyroscope (4) and the results are shown in Fig.4 where the tracking errors corresponding to terminal sliding mode control more severely decreases in contrast with the conventional sliding mode control due to the proposed finite time convergence algorithm.

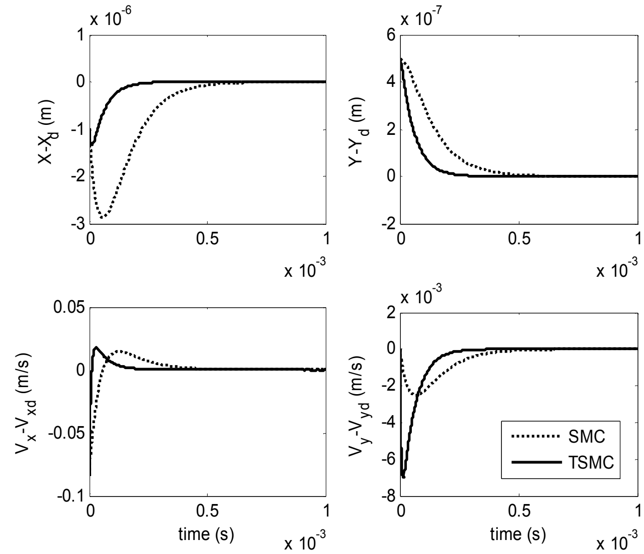

Fig.4. Tracking error comparison between TSMC and SMC

In the next step, the performance of the proposed sliding mode observer based rotation rate estimation scheme for vibratory MEMS gyroscope is demonstrated through simulations. The initial conditions for designed observer are selected all zero.

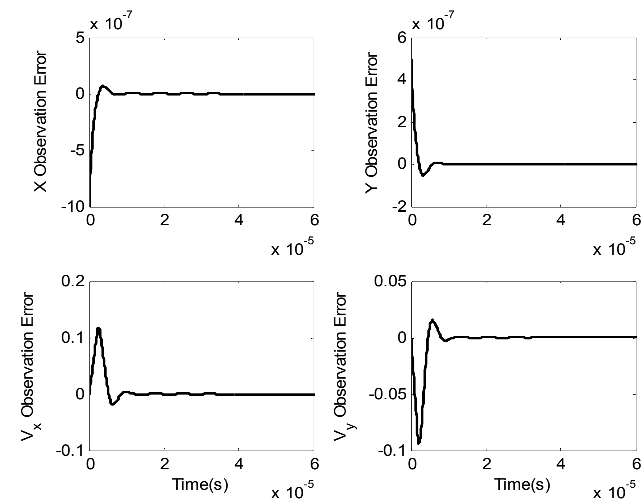

Fig.5. Observation errors of vibratory gyroscope in both drive and sense axis

Fig.5 explicitly shows that the estimated state variables of gyroscope reach their actual values in finite times while the rotation rate and stiffness coupling are unknown. Finally, the time-varying rotation rate estimations at three different frequency values $f_{rate} = 50H$, $f_{rate} = 100\ Hz$, and $f_{rate} = 200\ Hz$ $(\omega_{rate} = 2\pi f_{rate})$ are illustrated in Fig.6 without any changes in key parameters of the proposed observer based control scheme.

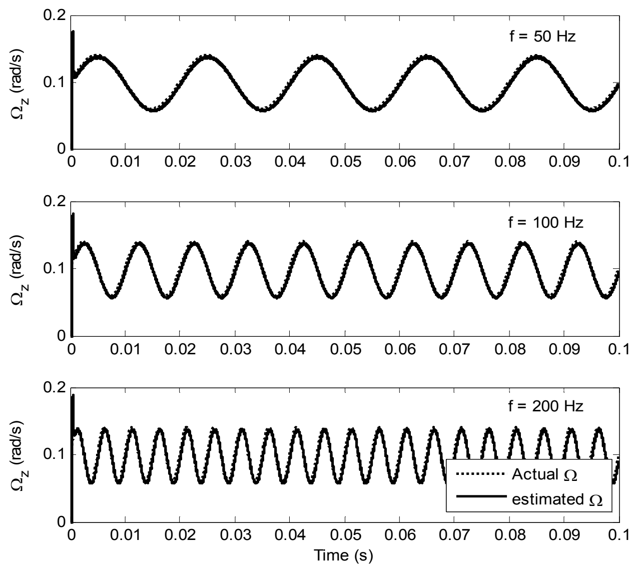

Fig.6. The rotation rate estimations at three different f_{rate}

It is important to note that the transfer function of the low- pass filter is chosen as $G_{LPF} = \frac{1}{(1+\tau s)^2}$ where the time constant τ is $6.7 \times 10^{-5}\, {rad}/{s}$.

VI. Conclusions

A novel observer based control scheme using sliding mode theory is applied for a Z-axis MEMS gyroscope while the time-varying rotation rate and stiffness coupling between both gyroscope modes are unknown. A terminal sliding mode control (TSMC) is proposed to minimize the coupling between two operational modes and to force both drive and sense mode to oscillate at specified amplitudes and frequencies despite unknown Coriolis acceleration and quadrature error terms. Subsequently, a sliding mode observer is proposed able to reconstruct the unknown time-varying rotation rate when the error observation trajectories reach the sliding mode. Moreover, the proposed robust control law ensures that the unknown terms of vibratory gyroscope will be bounded for all the time. The stability and robustness of the proposed controller and observer as well as their convergence in finite times are proved using second method of Lyapunov. Simulation results demonstrate high tracking performance and robustness of the control in both drive and sense axis along with acceptable estimation of the unknown time-varying rotation rate using sliding mode observer. Furthermore, numerical simulations show advantageous of desired trajectory tracking speed of the TSMC in contrast with conventional SMC.

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