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Forward/Backward Adaptation Law for Nonlinearly Parameterized Systems

Zhiyong Chen

Abstract— **Linear parameterizations is a central assumption in adaptive estimation and control strategies and it in turn becomes a bottleneck for prevalent applications of adaptive control in many nonlinearly parameterized systems encountered in practice. In literature, there have been some attempts to breakthrough this bottleneck by investigating the characteristics of nonlinearities in artistic arguments. However, it is still open for an implementable strategy that is powerful for nonlinearly parameterized systems as the certainty equivalence principle for linearly parameterized systems. This paper aims to contribute a novel attempt to this open problem by proposing an adaptation algorithm which does not explicitly rely on the expression of the nonlinearities and allows blind tuning for satisfactory performance. The algorithm is supported by rigorous analysis on asymptotic stability and parameter convergence as well as numerical simulation.**

Index Terms— **Nonlinear systems; Nonlinear parametrization; Adaptive control**

I. INTRODUCTION

Adaptive control has evolved to a level of considerable maturity over the past 50 years. Many genuine industrial applications have been implemented and strong supporting theory has been developed [1]. A standard adaptive methodology is set for systems containing a number of constant unknown parameters to estimate the parameters online and hence achieve stability and performance. Specifically, a perfect knowledge controller is designed assuming the parameters be known and then an adaptation function is added by replacing the parameters by their corresponding estimates. This procedure is summarized as a certainty equivalence principle [2]. The success of an adaptive controller relies on an effective adaptation law to achieve the stability and performance of the closed-loop system, as well as the parameter convergence to real values whenever possible.

The existence of a successful adaptation law relies heavily on the linear parametrization assumption which requires the unknown parameters appear linearly in their dynamics, (see, e.g., the classical approach [3], the nonlinear design [2], and most of the recent adaptive control strategies [4]–[7]). However, this assumption has been a bottleneck for prevalent applications of adaptive control in many nonlinearly parameterized systems encountered in industries. The popular nonlinearly parameterized examples studied in literature cover physical systems, chemical industry processes and biotechnology, such as friction dynamics [8], magnetic bearing dynamics [9], fermentation processes [10], bioreactor processes [11], etc. Therefore, it is inevitable for control engineers to seek new methodologies for dealing with nonlinearly parameterized systems.

Since the mid-1990s, there have been some attempts to breakthrough the bottleneck caused by the linear parametrization assumption. The efforts were mainly on investigating the characteristics of nonlinearities through various artistic arguments. Robust control approach is one of the early attempts which is capable of handling bounded uncertainties including unknown constant parameters. The underlying principle of robust control is to use sufficiently high gain terms to dominate the uncertainties which can appear in dynamics, linearly or nonlinearly. An adaptation law can be further added to estimate the bound of uncertainties if it is unknown. Such a robust adaptive combined framework for nonlinearly parameterized systems has been widely used in the recent development of nonlinear control. For example, a global adaptive control of nonlinearly parameterized system was achieved by output feedback in [12], a robust adaptive control technique was incorporated in the backstepping control design with flat zones to tackle nonlinear parametrization in [13], a non-smooth feedback framework was established in [14], and an adaptive output regulation controller was proposed for systems containing nonlinearly parameterized uncertainties in [15]. Because of the uncertainty domination nature of robust approach, the robust adaptive approach, on one hand, exhibits its capability in dealing with complex nonlinearities, on the other hand, has its inherent disadvantages. One is that a system's equilibrium point is assumed not perturbed by uncertainties for an effective domination; the other is that no parameter estimation is needed and little can be said about the estimation convergence property. Therefore, it motives a more technically hard problem of handling nonlinearly parameterized systems using adaptive control solely. Along this direction, the existing attempts have led to several research lines as summarized below.

One of the major research lines for adaptive control of nonlinearly parameterized systems is on the convex/concave property of nonlinear functions. Direct adaptive controllers have been first reported in the literature for convexly parameterized systems in [16] (in Russian, which was brought to the attention of the western world in [17].) The nonconvexly parameterized systems were first studied in [18] where a globally stable adaptive controller was derived for asymptotic regulation to within a desired precision ϵ using a min-max algorithm. The algorithm was further used in [19] for global stability with further extensions in [20]. Also, a re-parametrization technique was used to convexify a nonconvexly parameterized system in [21]. The algorithm in

Z. Chen is with the School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW 2308, Australia, Tel: +61 2 4921 6352, Fax: +61 2 4921 6993, Email: zhiyong.chen@newcastle.edu.au. The work was substantially supported by the Australian Research Council under grant No. DP0878724.

[22] represents another point of view on dealing with non-convexly parameterized models. Along this line, a recent result is a class of polynomial adaptive controllers based on a piecewise linearly approximated model for nonlinearly parameterized systems [23]. Also, nonlinearly parameterized systems are studied with the examination on the monotonicity property of nonlinear functions in, e.g., [24], [25]. Another major research line is for a class of systems with fractional parametrization whose unknown parameters appear linearly in both numerator and denominator. Early results in this line focused on scalar systems as in [10], [11], [26]. The multi-variable systems were studied in [27]. More extensions can be found in recent literature. For instance, a class of adaptive repetitive control was proposed in [28] for nonlinearly parameterized systems subject to periodic/repeated disturbances and uncertainties. In [29] a new adaptive control was proposed by introducing a biasing vector function into parameter estimate, which can be used for a class of strict-feedback nonlinearly parameterized systems. Finally, intelligent computation is another methodology for handling nonlinearly parameterized uncertainties. In particular, complex uncertainties of nonlinear structures might be simplified based on the universal approximation theorem including nonlinearly parameterized fuzzy approximation [30], [31] or neural network approximation [32].

In the aforementioned attempts to nonlinearly parameterized systems, the intelligent computation method does not rely on the explicit expression of nonlinear functions which are approximated by a fuzzy logic or a neural network, but the other methods usually do. To the best knowledge of the author, there exists no algorithm for general nonlinearities. In this paper, we aim to propose a novel methodology that does not rely on the explicit expression of nonlinearities or an approximation based intelligent computation. As its outstanding features, the new adaptation law is of simple structure, is independent of the diversity of nonlinearities, and allows blind tuning of parameters. Such a design philosophy reminds us typical PID (proportional-integral-derivative) controllers. Admittedly, PID controllers are extremely prevalent in industrial applications. This is due to their simple structure which allows blind tuning of parameters to achieve satisfactory regulation performance. For the same reason, we expect that the adaptation law developed in this paper has its potential applications as that previously achieved by PID controllers. The present adaptation law is based on an idea of repeatedly switching the evolution direction of the estimate parameter. Without knowing the precise position of the real parameter, the estimated parameter blindly switches its searching direction for a fixed interval. As a result, it moves forward to be closer to its real value in one interval but reverses backward in the next one, and repeats. An adaptation algorithm is properly designed to make the forward rate effectively faster than the backward one, such that eventually the parameter asymptotically approaches its real value. As its name means, this approach is called a forward/backward adaptation law. The establishment of a forward/backward adaptation law is the main contribution of this paper, which is supported by rigorous analysis on asymptotic stability and parameter convergence and numerical simulation.

II. A Forward/Backward Adaptation Law

We consider a general nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), d(t), t) + g(x(t), d(t), t)[\gamma(r(t), \theta) + u(t)] \\ y(t) &= h(x(t), t) \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input, $r \in \mathbb{R}^p$ a measurable signal, $\theta \in \mathbb{R}$ an unknown parameter, $d \in \mathbb{R}^l$ a disturbance vector, and $y \in \mathbb{R}^\ell$ a performance output. It is assumed that a preliminary controller has been designed such that the system $\dot{x} = f(x, d, t)$ is globally asymptotically stable (GAS) when the uncertain term $\gamma(r, \theta)$ is not taken into account. In this paper, the main objective is to propose an adaptive compensator to deal with the uncertain term $\gamma(r, \theta)$. If it is successful, a general stabilization problem for systems containing an unknown term can be solved separately: first for the ideal case with the term neglected and then for the original case by incorporating an adaptive compensator.

We first give a brief revisit to the conventional certainty equivalence principle used in adaptive control for linearly parameterized systems. In particular, the controller is given as follows,

$$u(t) = -\gamma(r(t), \hat{\theta}(t)) \tag{2.2}$$

where $\hat{\theta} \in \mathbb{R}$ is the estimated value for θ. The conventional speed-gradient based adaptation law relies on two assumptions. First, the term $\gamma(r, \theta)$ is linearly parameterized, i.e., $\gamma(r, \theta) = a(r)b(\theta)$ for some functions $a \in \mathbb{R}^{m\times q}$ and $b \in \mathbb{R}^q$ with $q \geq 1$. Secondly, the asymptotically stable system $\dot{x} = f(x, d, t)$ admits an explicit Lyapunov function $V(x, t)$ such that $dV(x, t)/dt \leq -\rho(x)$ for a class $\mathcal{K}_\infty$ function ρ. As a result, the controller (2.2) and the adaptation law for $\hat{\vartheta} := b(\hat{\theta})$ can be designed in the following form:

$$u = -a(r)\hat{\vartheta}, \quad \dot{\hat{\vartheta}} = \Lambda \left[\frac{\partial V(x,t)}{\partial x} g(x,t) a(y)\right]^\intercal$$

where the Λ is any positive definite diagonal matrix for tuning the update rate (see, e.g., [2].)

There are three inherent technical difficulties or disadvantages for the conventional speed-gradient adaptive control design including: (i) most importantly, the uncertain term $\gamma(r, \theta)$ must be linearly parameterized; (ii) the existence of an explicit Lyapunov function $V(x, t)$ is assumed for the nominal system $\dot{x} = f(x, d, t)$; (iii) when $q > 1$, the uncertain term is over-parameterized and a higher order adaptation law is needed for ϑ instead of θ. To overcome these disadvantages, we propose a novel adaptation law in this paper, which is given below followed by explanation, stability analysis, and simulation in the remaining sections.

For the system (2.1) and the controller (2.2), we propose a forward/backward adaptation law as follows.

Forward/Backward Adaptation (FBA) Law:

$$\begin{aligned}&\dot{\hat{\theta}}(t) = \epsilon(-1)^k \chi_k,\ k := \lfloor t/\tau \rfloor\\ &\chi_o = 0;\ \chi_k := \sup_{s\in[k\tau-\alpha\tau,k\tau)} \|y(s)\|,\ k \geq 1;\\ &\tau > 0,\ \epsilon > 0,\ 1/4 > \alpha > 0 \end{aligned} \tag{2.3}$$

where $\lfloor \cdot \rfloor$ represents the floor function.

The main feature of the FBA law (2.3) is that it does not explicitly depend on the expression of the system vector field or a Lyapunov function. It has a uniform structure for all systems with three parameters τ, ϵ, α to be tuned. Its main mechanism can be explained as follows. In an FBA law, the update direction of $\hat{\theta}$ switches between positive and negative for every interval τ. In other words, $\hat{\theta}$ moves forward to be closer to its real value θ in one interval but reverses backward in the next one, and repeats. To drive $\hat{\theta}$ to eventually approach its real value, we expect the forward rate effectively faster than the backward one. To make it possible, the rate is determined by the performance output $\|y\|$ during the preceding interval. For instance, in a forward interval (vice verse in a backward interval), the parameter gets closer to its real value and a smaller estimation error results and is reflected in the performance output y, which hence gives a smaller backward rate in the following interval. To reflect the influence of the estimation error on the performance output, we make the following assumption.

Assumption 1: Consider the closed-loop system composed of (2.1), (2.2), and (2.3), there exists a sufficiently large τ, such that, for $t \geq \tau$ and $\tilde{\theta} \geq 0$,

$$|\hat{\theta}(s) - \theta| \geq \tilde{\theta},\ \forall s \in [t - 2\alpha\tau, t] \ \Rightarrow \sup_{s\in[t-\alpha\tau,t)} \|y(s)\| \geq \beta(\tilde{\theta}) \tag{2.4}$$

$$|\hat{\theta}(s) - \theta| \leq \tilde{\theta},\ \forall s \in [t - 2\alpha\tau, t] \ \Rightarrow \sup_{s\in[t-\alpha\tau,t)} \|y(s)\| \leq \beta(\tilde{\theta}) \tag{2.5}$$

for a strictly increasing function β satisfying $0 < l \leq \dot{\beta}(\tilde{\theta}) \leq L < \infty,\ \forall \tilde{\theta} \geq 0$. ∎

Remark 2.1: The assumption plays the role that the deviation of $\hat{\theta}$ from the true value θ is reflected in the performance output y, which usually can be chosen as $y = x$. The strength of the assumption is explained as follows. If $\hat{\theta}$ is away from θ by at least $\tilde{\theta}$ during the interval $[t - 2\alpha\tau, t]$, then, allowing a sufficiently long time $\alpha\tau$ for settling, the superior value of $\|y\|$ is larger than $\beta(\tilde{\theta})$ during $[t - \alpha\tau, t)$. A similar explaination is given for (2.5) when $\hat{\theta}$ is close to θ with a difference less than $\tilde{\theta}$. In particular, we have $\beta(0) = 0$. In fact, when $\tilde{\theta} = 0$, i.e., $\hat{\theta} = \theta$ during some time interval, it easily implies that $\|y\| = 0$ for the same interval. More importantly, it makes the FBA law practically implementable that we do not need explicitly examine this assumption. The FBA law can be implemented with blind tuning for the parameters τ, ϵ, and α for any system satisfying the assumption. ∎

Successful applications of an FBA law are supported by the precise analysis on the asymptotic stability and parameter convergence. To facilitate the analysis, we define some concepts below. In the FBA law (2.3), it is noted that, for each k, $\dot{\hat{\theta}}$ is fixed over the interval $[k\tau,\ (k+1)\tau]$. So, the trajectory of $\hat{\theta}(t)$ is continuous and piecewise linear (with a sawtooth waveform). We call each straight line $\hat{\theta}(t),\ t \in [k\tau,\ (k+1)\tau)$ as a *segment* k, denote as

$$\mathrm{S}_k := \{\hat{\theta}(t) \mid t \in [k\tau,\ (k+1)\tau)\}.$$

The union of more than one adjacent segments can be denoted as $\mathrm{S}_{k,i} := \mathrm{S}_k \cup \cdots \cup \mathrm{S}_{k+i}$ for any integer $i \geq 0$. For completeness, we have $\mathrm{S}_{k,0} = \mathrm{S}_k$ and $\mathrm{S}_{k,\infty} = \mathrm{S}_k \cup \mathrm{S}_{k+1} \cup \cdots$. The two *ends* of S_k are denoted as

$$b_k := \hat{\theta}(k\tau),\ e_k := \hat{\theta}((k+1)\tau)$$

respectively. Then,

$$\mathrm{S}_{k,i}^- := b_k,\ \mathrm{S}_{k,i}^+ := e_{k+i}$$

are the two ends of the segments $\mathrm{S}_{k,i}$. Obviously, we have $e_k = b_{k+1}$ because the trajectory $\theta(t)$ is continuous.

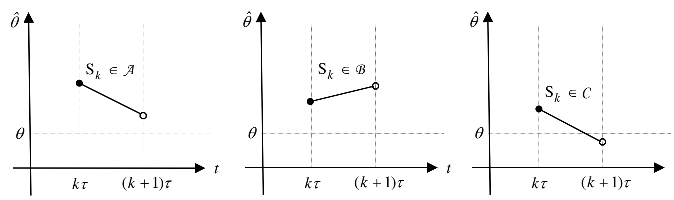

Fig. 1. Segments of classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

A segment S_k is said to be a *class* $\mathcal{A}$ *segment*, denoted as $\mathrm{S}_k \in \mathcal{A}$, if and only if $\theta \notin \mathrm{S}_k$ and $|b_k|_\theta \geq |e_k|_\theta$ [1]. Similarly, we can define *class* $\mathcal{B}$ *segment* and *class* $\mathcal{C}$ *segment* (see Fig. 1). Specifically, we say $\mathrm{S}_k \in \mathcal{B}$, if and only if $\theta \notin \mathrm{S}_k$ and $|b_k|_\theta < |e_k|_\theta$; and $\mathrm{S}_k \in \mathcal{C}$ if and only if $\theta \in \mathrm{S}_k$. Geometrically, a class $\mathcal{A}$ segment gets closer to the true value of θ but a class $\mathcal{B}$ segment gets farther from θ. Neither of them gets through θ, while a class $\mathcal{C}$ segment does. For a segment $\mathrm{S}_{k,i}$, we denote $\mathrm{S}_k \in \mathcal{X}_1, \cdots, \mathrm{S}_{k+i} \in \mathcal{X}_{i+1}$ in a compact form of $\mathrm{S}_{k,i} \in \mathcal{X}_1 \ldots \mathcal{X}_{i+1}$ where $\mathcal{X}_1, \cdots, \mathcal{X}_{i+1} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$.

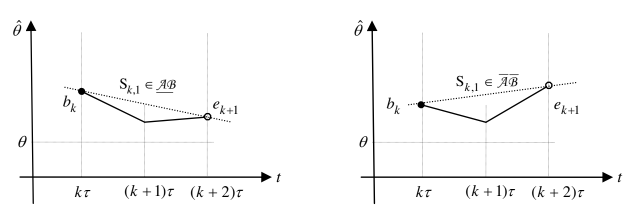

Fig. 2. Segments of classes $\underline{\mathcal{AB}}$ and $\overline{\mathcal{AB}}$.

For two adjacent segments $\mathrm{S}_k \in \mathcal{A}$ and $\mathrm{S}_{k+1} \in \mathcal{B}$, it is obvious that

$$|\mathrm{S}_{k,1}^-|_\theta = |b_k|_\theta \geq |e_k|_\theta = |b_{k+1}|_\theta < |e_{k+1}|_\theta = |\mathrm{S}_{k,1}^+|_\theta.$$

[1] Throughout this paper, we denote $|x|_\theta := |x - \theta|$ for any $x \in \mathbb{R}$.

However, it is not clear which end ($|\mathrm{S}_{k,1}^-|_\theta$ or $|\mathrm{S}_{k,1}^+|_\theta$) is closer to θ. As illustrated in Fig. 2, two more notations are introduced to compare $|\mathrm{S}_{k,1}^-|_\theta$ and $|\mathrm{S}_{k,1}^+|_\theta$: we say $\mathrm{S}_{k,1} \in \underline{\mathcal{AB}}$ if and only if $|\mathrm{S}_{k,1}^-|_\theta \geq |\mathrm{S}_{k,1}^+|_\theta$; and $\mathrm{S}_{k,1} \in \overline{\mathcal{AB}}$ if and only if $|\mathrm{S}_{k,1}^-|_\theta < |\mathrm{S}_{k,1}^+|_\theta$ (see Fig. 2).

III. Stability Analysis and Parameter Convergence

In this section, we will show that under Assumption 1 the estimated parameter asymptotically approaches its real value through forward-backward iterations under the proposed FBA law, and hence the closed-loop system composed of (2.1), (2.2), and (2.3) is asymptotically stable. The analysis will be conducted by examining the evolution tendency of the trajectory $\hat{\theta}(t)$. Some lemmas will be first given and then followed by the main theorem.

Lemma 3.1: *Under Assumption 1, the trajectory of the closed-loop system composed of (2.1), (2.2), and (2.3) satisfies $S_{k,3} \notin \overline{\mathcal{AB}}\overline{\mathcal{AB}}$ for any $k \geq 0$.* ∎

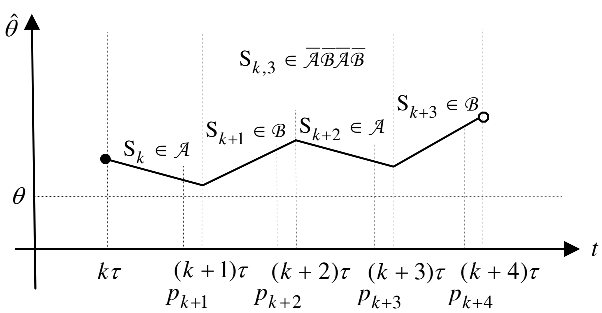

Fig. 3. A segment $\mathrm{S}_{k,3} \in \overline{\mathcal{AB}}\overline{\mathcal{AB}}$.

Proof: If it is not true, we assume $\mathrm{S}_{k,3} \in \overline{\mathcal{AB}}\overline{\mathcal{AB}}$ for some $k \geq 0$ (see Fig. 3). For convenience, we define a time $p_k := k\tau - 2\alpha\tau$.

On one hand, since $\mathrm{S}_{k+1} \in \mathcal{B}$ and $\mathrm{S}_{k+2} \in \mathcal{A}$, from Assumption 1 and the definition of χ_k in (2.3), we have $\chi_{k+2} \geq \beta(\hat{\theta}(p_{k+2}))$ and $\chi_{k+3} \leq \beta(\hat{\theta}(p_{k+3}))$, respectively. And hence,

$$\begin{aligned} L[\hat{\theta}(p_{k+3}) - \hat{\theta}(p_{k+2})] &\geq \beta(\hat{\theta}(p_{k+3})) - \beta(\hat{\theta}(p_{k+2})) \\ &\geq \chi_{k+3} - \chi_{k+2} > 0. \end{aligned} \quad (3.1)$$

The last inequality is from the assumption that $\mathrm{S}_{k+2,1} \in \overline{\mathcal{AB}}$.

On the other hand, it is not difficult to see that

$$\hat{\theta}(p_{k+3}) - \hat{\theta}(p_{k+2}) = -\epsilon\chi_{x+2}(1-2\alpha)\tau + \epsilon\chi_{k+1}2\alpha\tau. \quad (3.2)$$

As a result, (3.1) and (3.2) imply that

$$L\epsilon\tau[\chi_{k+1}2\alpha - \chi_{x+2}(1-2\alpha)] \geq \chi_{k+3} - \chi_{k+2} > 0, \quad (3.3)$$

and clearly $\chi_{k+1} - \chi_{k+2} > 0$.

Next, since $\mathrm{S}_k \in \mathcal{A}$ we have $\chi_{k+1} \leq \beta(\hat{\theta}(p_{k+1}))$, which, together with $\chi_{k+2} \geq \beta(\hat{\theta}(p_{k+2}))$, implies

$$\begin{aligned} L[\hat{\theta}(p_{k+1}) - \hat{\theta}(p_{k+2})] &\geq \beta(\hat{\theta}(p_{k+1})) - \beta(\hat{\theta}(p_{k+2})) \\ &\geq \chi_{k+1} - \chi_{k+2} > 0. \end{aligned} \quad (3.4)$$

Again, it is not difficult to see that

$$\hat{\theta}(p_{k+1}) - \hat{\theta}(p_{k+2}) = \epsilon\chi_k 2\alpha\tau - \epsilon\chi_{x+1}(1-2\alpha)\tau. \quad (3.5)$$

As a result, (3.4) and (3.5) imply that

$$L\epsilon\tau[\chi_k 2\alpha - \chi_{x+1}(1-2\alpha)] \geq \chi_{k+1} - \chi_{k+2} > 0, \quad (3.6)$$

and clearly $\chi_k > \chi_{k+1}$, which contradicts the assumption of $\mathrm{S}_{k,1} \in \overline{\mathcal{AB}}$. ∎

Lemma 3.2: *Suppose the trajectory of the closed-loop system composed of (2.1), (2.2), and (2.3) with $\epsilon\tau L\alpha < 1/(4a) - 1$ satisfies Assumption 1. For any $k \geq 0$, if $S_{k,3} \in \underline{\mathcal{AB}}\overline{\mathcal{AB}}$, then $|S_{k,3}^-|_\theta \geq |S_{k,3}^+|_\theta$.* ∎

Proof: By using the same arguments, we have (3.3) and (3.6) which imply that

$$\begin{aligned} &\chi_{k+1} + \chi_{k+3} \\ &\leq [1 + L\epsilon\tau 2\alpha]\chi_{k+1} + [1 - L\epsilon\tau(1-2\alpha)]\chi_{k+2} \\ &\leq [1 + L\epsilon\tau 2\alpha]\frac{2\alpha}{1-2\alpha}\chi_k + [1 - L\epsilon\tau(1-2\alpha)]\chi_{k+2} \\ &\leq \varepsilon(\chi_k + \chi_{k+2}) \end{aligned}$$

where

$$\varepsilon := \max\{[1 + L\epsilon\tau 2\alpha]2\alpha/(1-2\alpha),\ 1 - L\epsilon\tau(1-2\alpha)\} < 1.$$

As a result,

$$\begin{aligned} |\mathrm{S}_{k,3}^-|_\theta - |\mathrm{S}_{k,3}^+|_\theta &= \epsilon\tau(\chi_k + \chi_{k+2} - \chi_{k+1} - \chi_{k+3}) \\ &\geq \epsilon\tau(1-\varepsilon)(\chi_k + \chi_{k+2}) \geq 0. \end{aligned} \quad (3.7)$$

The proof is thus complete. ∎

Lemma 3.3: *Suppose the trajectory of the closed-loop system composed of (2.1), (2.2), and (2.3) satisfies Assumption 1. If there exists an integer $\ell > 0$, such that $S_{2k,1} \in \underline{\mathcal{AB}}$ for any $k \geq \ell$, then $\lim_{t\to\infty}\hat{\theta}(t) = \theta$.* ∎

Proof: From the definition of $\underline{\mathcal{AB}}$, the sequence $|b_{2k}|_\theta$ is monotonically decreasing along k. As a result, we have $\lim_{k\to\infty}|b_{2k}|_\theta = \rho \geq 0$. Now, it suffices to prove $\rho = 0$. For any small $\sigma > 0$, there exists an integer K, such that

$$\rho \leq |b_{2k}|_\theta < \rho + \sigma,\ \forall k \geq K.$$

Next, we will prove that

$$\chi_{2k+1} \leq \frac{2a\kappa\sigma}{(1-4a)\epsilon\tau},\ \kappa := \max\{1/(2al), 1\},\ \forall k \geq K. \quad (3.8)$$

Otherwise, if (3.8) is not true, there exists a k such that

$$(1-4a)\chi_{2k+1} > \frac{2a\kappa\sigma}{\epsilon\tau}. \quad (3.9)$$

Also, we note that

$$\begin{aligned} \epsilon\tau(\chi_{2k} - \chi_{2k+1}) &= |b_{2k}|_\theta - |b_{2k+2}|_\theta \leq \sigma, \\ \epsilon\tau(\chi_{2k+2} - \chi_{2k+3}) &= |b_{2k+2}|_\theta - |b_{2k+4}|_\theta \leq \sigma. \end{aligned} \quad (3.10)$$

Since $\kappa \geq 1$, (3.9) and the first equation of (3.10) imply that

$$(1-2a)\chi_{2k+1} > 2a\chi_{2k+1} + \frac{2a\sigma}{\epsilon\tau} \geq 2a\chi_{2k} \quad (3.11)$$

and hence $\hat{\theta}(p_{2k+2}) > \hat{\theta}(p_{2k+1})$ where $p_k := k\tau - 2a\tau$. This further implies $\chi_{2k+2} > \chi_{2k+1}$ and $\hat{\theta}(p_{2k+2}) > \hat{\theta}(p_{2k+3})$.

Since $\beta(\hat{\theta}(p_{2k+2})) \le \chi_{2k+2}$ and $\beta(\hat{\theta}(p_{2k+3})) \ge \chi_{2k+3}$, we have

$$\begin{aligned} 0 < l(\hat{\theta}(p_{2k+2}) - \hat{\theta}(p_{2k+3})) &\le \beta(\hat{\theta}(p_{2k+2})) - \beta(\hat{\theta}(p_{2k+3})) \\ &\le \chi_{2k+2} - \chi_{2k+3} \le \sigma/(\epsilon\tau) \end{aligned}$$

where the second equation of (3.10) is used. On the other hand, we have

$$\begin{aligned} & \hat{\theta}(p_{2k+2}) - \hat{\theta}(p_{2k+3}) \\ = \ & (1-2a)\chi_{2k+2} - 2a\chi_{2k+1} \\ = \ & (1-2a)(\chi_{2k+2} - \chi_{2k+1}) + (1-4a)\chi_{2k+1} \\ \ge \ & (1-4a)\chi_{2k+1} > 2a\kappa\sigma/(\epsilon\tau). \end{aligned}$$

The above two inequalities imply that

$$\frac{\sigma}{\epsilon\tau l} \ge \hat{\theta}(p_{2k+2}) - \hat{\theta}(p_{2k+3}) > \frac{2a\kappa\sigma}{\epsilon\tau}$$

that is $\kappa < 1/(2al)$, which contradicts the definition of κ. Thus, (3.8) is proved.

From (3.8), we have $\lim_{k\to\infty} \chi_{2k+1} = 0$, which easily implies $\lim_{k\to\infty} \chi_{2k} = 0$. Then, it is ready to show $\rho = 0$. ■

Theorem 3.1: *Under Assumption 1, if ϵ is sufficiently small such that*

$$\epsilon\tau L < \min\{1,\ [1/(4\alpha) - 1]/\alpha\}, \tag{3.12}$$

then the system composed of (2.1), (2.2), and (2.3) is globally asymptotically stable in the sense of

$$\lim_{t\to\infty} x(t) = 0,\ \lim_{t\to\infty} \hat{\theta}(t) = \theta \tag{3.13}$$

for any initial states $x(0)$ and $\hat{\theta}(0)$. ▮

Proof: Without loss of generality, we assume $\hat{\theta}(0) \ne \theta$. We will first show that $\mathrm{S}_k \notin \mathcal{C}$ for any k. Otherwise, let S_k be the first segment of class $\mathcal{C}$. Therefore, $\mathrm{S}_{k-1} \in \mathcal{B}$ and hence, $\chi_k \le \beta(|b_k|_\theta)$. Now, we have

$$|b_k|_\theta + |b_{k+1}|_\theta = \epsilon\chi_k\tau$$

and

$$\begin{aligned} |b_{k+1}|_\theta &= \epsilon\chi_k\tau - |b_k|_\theta \\ \le \epsilon\tau\beta(|b_k|_\theta) - |b_k|_\theta &\le (\epsilon\tau L - 1)|b_k|_\theta < 0, \end{aligned}$$

which is a contradiction.

Next, we will show that there exists a finite $\ell_o \ge 0$, such that $\mathrm{S}_{\ell_o,\infty}$ consists of only $(\mathcal{AB})$, i.e.,

$$\mathrm{S}_{\ell_o,\infty} \in \mathcal{AB}\cdots\mathcal{AB}\cdots.$$

If $\mathrm{S}_1 \in \mathcal{A}$, let $\ell_o = 1$; if $\mathrm{S}_1 \in \mathcal{B}$, let $\ell_o = 2$. Then, it suffices to prove that there does not exist two adjacent segments $\mathrm{S}_{k,1}$ such that (i) $\mathrm{S}_{k,1} \in \mathcal{AA}$; or (ii) $\mathrm{S}_{k,1} \in \mathcal{BB}$. In (2.3), since $\chi_k \ge 0$, the direction of $\hat{\theta}$ is determined by $(-1)^k$. As a result, (ii) is obviously true. For (i), $\mathrm{S}_{k,1} \in \mathcal{AA}$ happens only when $\chi_k = 0$ or $\chi_{k+1} = 0$, which implies $\hat{\theta}(t) = \theta$ for all $t \in [k\tau, (k+1)\tau]$ or $t \in [(k+1)\tau, (k+2)\tau]$, i.e., $\mathrm{S}_k \in \mathcal{C}$ or $\mathrm{S}_{k+1} \in \mathcal{C}$.

From Lemma 3.1, we know that, the trajectory $\hat{\theta}(t)$ starting from the ℓ_o-th segment consists of only the units of $\underline{\mathcal{AB}}$ and $\underline{\mathcal{AB}}\overline{\mathcal{AB}}$. Now, denote the value $\hat{\theta}$ at the head of the $r-$th unit as z_r. From the definition of $\underline{\mathcal{AB}}$ and Lemma 3.2, we note that $|z_r|_\theta$ is monotonically decreasing. So, we have

$$\lim_{r\to\infty} |z_r|_\theta = \rho \ge 0.$$

If $\rho = 0$, the proof is complete obviously. Otherwise, we will consider the case of $\rho > 0$. Let $\mathrm{S}_{k,3} \in \underline{\mathcal{AB}}\overline{\mathcal{AB}}$ for some k. Clearly, we have $|\hat{\theta}(k\tau - 2\alpha\tau)|_\theta \ge (1-2\alpha)\rho$ and $\chi_k \ge \beta((1-2\alpha)\rho)$. From (3.7) of Lemma 3.2, we have

$$\begin{aligned} |\mathrm{S}^-_{k,3}|_\theta - |\mathrm{S}^+_{k,3}|_\theta &\ge \epsilon\tau(1-\varepsilon)\chi_k \\ &\ge \epsilon\tau(1-\varepsilon)\beta((1-2\alpha)\rho) > 0. \end{aligned}$$

As a result, there are only finite units of class $\underline{\mathcal{AB}}\overline{\mathcal{AB}}$, otherwise, we have $|z_r|_\theta < 0$ for a sufficiently large r. In the other words, there exists an integer ℓ, such that $\mathrm{S}_{2k,1} \in \underline{\mathcal{AB}}$ for any $k \ge \ell$. The proof is thus complete by using Lemma 3.3. ■

To demonstrate the FBA law's performance and effectiveness, we will study an academic example to examine the influence of the control parameters τ and ϵ on parameter convergence performance. Actually, the FBA law has been tested for more systems including control of low-velocity friction model and synchronization of Hindmarsh-Rose model neurons, etc. In these applications, a variety of nonlinear functions have been examined including hyperbolic, sinusoidal, exponential, and polynomial functions. Due to the space limit, the details of these applications are not given here.

Example 3.1: We consider the following nonlinear system in the form of (2.1):

$$\dot{x} = \begin{bmatrix} -x_1 + 0.2\sin x_2 \\ -2x_2 - x_2^3 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix} [\tanh(\theta r) + u], \qquad r(t) = 0.2\sin(10t)$$

Without the unknown term $\tanh(\theta r)$, the system is globally asymptotically stable. To deal with the term $\tanh(\theta r)$, the controller is designed as follows $u = -\tanh(\hat{\theta} r)$ where $\hat{\theta}$ is updated as (2.3) with $\alpha = 1/5$ and different sets of parameters τ and ϵ. The simulation results are given in Fig. 4 with the true value $\theta = 2$ and the initial states $x(0) = [10, -2]$ and $\hat{\theta}(0) = 0$ or 4. It is illustrated in the top graph that the state $x(t)$ converges to the equilibrium point, and in the middle and bottom graphs that $\hat{\theta}$ converges to θ.

Next, we will discuss how the parameters τ, ϵ, and α affect the closed loop system's performance. We will mainly examine the first two parameters because α is not designed as an effective tuning parameter and it is chosen as $\alpha = 1/5 < 1/4$. On one hand, to guarantee the stability of the closed-loop system, τ should be sufficiently large to make Assumption 1 satisfied and in turn ϵ should be sufficiently small to make (3.12) satisfied. On the other hand, a too large τ or a too small ϵ obviously induces a long settling time for the system. This is a typical tradeoff between steady and transient performances. To achieve the balance to match the

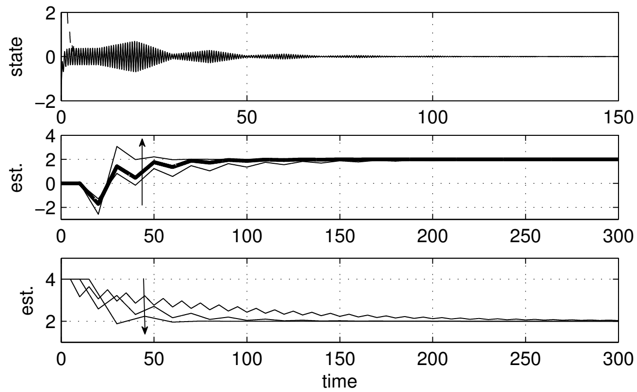

Fig. 4. Profiles of asymptotic stabilization and parameter convergence. Above: profile of states x with $\tau = 10$ and $\epsilon = 0.4$; middle: profile of parameter convergence with $\tau = 10$ for $\epsilon = 0.3, 0.4, 0.6$ (along arrow line); bottom: profile of parameter convergence with $\epsilon = 0.4$ for $\tau = 5, 10, 15$ (along arrow line).

design requirements, these two parameters need a careful tuning. In the middle graph of Fig. 4, for a given $\tau = 10$, a larger ϵ corresponds to a shorter settling time, but may induce some offshoot. For the case with the curve marked in bold, the asymptotic stability of the plant is demonstrated in the top graph. In the bottom graph of Fig. 4, for a given $\epsilon = 0.4$, a larger τ corresponds to a shorter settling time. But for a small τ, there is more space to increase ϵ to shorten the settling time. ∎

IV. Conclusion

Engineers always look after simple but effective controllers. Such a controller is proposed in this paper to deal with the adaptive problem for nonlinearly parameterized systems. The adaptation law has a uniform format with two major tuning parameters and it is effective for a wide range of systems with various nonlinearities. By properly tuning the parameters, the system stability and parameter convergence can be achieved with a satisfactory transient performance. This novel design methodology is expected to be a new attempt to the development of adaptive control for nonlinearly parameterized systems.

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