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Dynamic Harmonic Balance and its Application to Analysis of Convergence of Second-Order Sliding Mode Control Algorithms

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Abstract—**The harmonic balance (HB) principle is extended to transient processes and named the dynamic HB. Based on the dynamic HB, first the differential equations for the amplitude and frequency of the self-excited oscillations in the Lure system in the vicinity of the periodic solution are derived. It is then applied to analysis of motions in the vicinity of the origin of the complex plane for systems with second-order sliding mode control algorithms, therefore, describing the process of convergence of state variables in the system. An example is provided.**

I. INTRODUCTION

HARMONIC balance principle is a convenient tool for finding parameters of self-excited periodic motions. Due to this convenience, it is widely used in many areas of science and engineering. For a system with one nonlinearity and linear dynamics (Lure system), it can be illustrated by drawing the Nyquist plot of the linear dynamics and the plot of the negative reciprocal of the describing function (DF) [1] of the nonlinearity in the complex plane and finding the point of intersection of the two plots, which would correspond to the self-excited periodic motion in the system. Therefore, the harmonic balance principle treats the system as a loop connection of the linear dynamics and of the nonlinearity. It is also possible to reformulate the harmonic balance, so that the format of the system analyzed is not a loop connection but the denominator of the closed-loop system. This would imply a different interpretation of the harmonic balance, which would allow one to extend the harmonic balance principle to analysis of not only self-excited periodic motions but also other types of oscillatory motions.

One of the types of the systems that exhibit vanishing oscillatory motion is the conventional and second-order sliding mode (SM) control system. There are a number of second-order SM (SOSM) algorithms available now, the most popular of which are "twisting", "super-twisting", "twisting-as-a-filter" [2], [3], "sub-optimal" [4], [5], and a number of other algorithms [6]. The problem of convergence rate is a valid problem in the conventional SM control and "terminal SM" [7], [8] control too. Therefore, some common approach to the problem of the convergence rate assessment, including qualitative (finite-time or asymptotic) and quantitative assessment, is of high importance.

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The frequency-domain approach to assessment of convergence rate would provide a number of advantages over the direct solution/estimates of the system differential equations. The most important one would be the unification of the treatment of all the algorithms based on some frequency-domain characteristics. This in turn may lead to formulation of some criteria that should be satisfied for a SOSM algorithm to provide a finite-time convergence, which can also lead to relatively simple rules that would allow one to develop new SOSM algorithms.

In publication [9], convergence of the second-order system with twisting algorithm was presented. Also, a frequency-domain criterion of finite-time convergence that involves the so-called *phase deficit* was formulated. However, the approach was limited to second-order systems and it was a fundamental limitation, which didn't allow the author to extend it to higher-order systems.

In the present paper, a different frequency-domain approach to analysis of convergence is presented, which is suitable for analysis of high-order systems. Also, the *harmonic balance* principle is extended to the case of transient oscillations. The paper is organized as follows. At first the harmonic balance principle is considered and its different representation is proposed. Then a system comprising a second-order plant and an asymptotic SOSM (relay) controller is analyzed with the use of the approach proposed. Such characteristics as frequency and amplitude of oscillations as functions of time are derived. After that a system comprising the twisting SOSM controller and a second-order plant is analyzed with the use of the proposed approach. Finally, an approach to analysis of the type of convergence based on the frequency-domain characteristics is considered.

II. HARMONIC BALANCE FOR TRANSIENT OSCILLATIONS

Consider the system that includes linear dynamics given by the following equations:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}\end{aligned}, \quad (1)$$

where $\mathbf{x} \in R^n$, $y \in R^1$, $u \in R^1$, $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times 1}$, $\mathbf{C} \in R^{1\times n}$, and a single-valued odd-symmetric nonlinearity $f(y)$:

$$u = -f(y), \quad (2)$$

We shall refer to (1) as to the linear part of the system. One can see that the system (1), (2) is a Lure system. The transfer function of the linear part is $W_l(s) = \mathbf{C}(\mathbf{I}s - \mathbf{A})^{-1}\mathbf{B}$, which can also be presented as a ratio of two polynomials $W_l(s) = P(s)/Q(s)$. We shall assume that the linear part has relative degree higher than *two*, so that the Nyquist plot of system (1) has a point of intersection with the real axis at some finite frequency. Assume also an autonomous mode, so that the input to the nonlinearity is the output of the linear dynamics, and the output of the nonlinearity is the input to the linear dynamics. The conventional HB condition (for periodic motion) is formulated as

$$W_l(j\Omega)N(a) = -1, \quad (3)$$

where Ω is the frequency and a is the amplitude of the self-excited periodic motion at the input to the nonlinearity, $N(a)$ is the describing function of the nonlinearity. Find the closed-loop transfer function $W_{cl}(s)$ of system (1), (2) using the replacement of the nonlinearity with the DF $u = -N(a)\cdot y$:

$$W_{cl}(s) = \frac{W_l(s)N(a)}{1 + W_l(s)N(a)} = \frac{P(s)N(a)}{Q(s) + P(s)N(a)} \quad (4)$$

Let us note that (3) is equivalent to

$$R(a, j\Omega) = Q(j\Omega) + P(j\Omega)N(a) = 0, \quad (5)$$

which means that the denominator of the closed-loop transfer function turns into zero when the frequency and the amplitude become equal to the frequency and the amplitude of the periodic motion. Equation (5) is also sometimes used for finding a periodic solution via algebraic methods. However, equation (5) usually is not attributed to the denominator of the closed-loop transfer function but considered a direct result of (3). Assuming that $R(a,s)$ can be represented in the following form $R(a,s) = (s - s_1)(s - s_2)\cdot\ldots\cdot(s - s_n)$, where s_i are roots of the characteristic polynomial, we must conclude that there must be at least one pair of complex conjugate root with zero real parts. It would imply the existence of the conservative component in $W_{cl}(s)$. Indeed, we can consider the existence of non-vanishing oscillations as a result of the existence of the component $(s^2 + \rho^2)$ in the denominator of $W_{cl}(s)$, where ρ is a parameter that depends on the amplitude a. However, one can notice that even if a damped oscillation occurs, so that there exists a pair of complex conjugate roots s_i, s_{i+1} then $(s - s_i)(s - s_{i+1}) = 0$, and the characteristic polynomial becomes zero, with $s = \sigma \pm j\Omega$, where σ is the decay (Note: strictly speaking, we have a decaying oscillation only if $\sigma < 0$; yet we will refer to this variable as to the decay even if $\sigma \geq 0$).

We now show that a linear system response to the harmonic signal with decaying amplitude $u(t) = e^{\sigma t}\sin(\Omega t)$ is also a harmonic signal with the same values of the frequency and amplitude decay. Indeed, this is a result of the property of the Laplace transform that states that $L[e^{-a}f(t)] = F(s+a)$. Therefore, for the system input $u(t)$, the Laplace transform will be $L[u(t)] = \Omega/[(s-\sigma)^2 + \Omega^2]$, which will result in the system output (in the Laplace domain) $Y(s) = \Omega W(s)/[(s-\sigma)^2 + \Omega^2]$, where $W(s)$ is the transfer function. The substitution $s' = s - \sigma$ yields $Y(s') = \Omega W(s' + \sigma)/[(s')^2 + \Omega^2]$, which means that $y'(t) = L^{-1}[Y(s')]$ is a sinusoid of frequency Ω, amplitude $|W(\sigma + j\Omega)|$, and having the phase shift $\arg W(\sigma + j\Omega)$. In turn, the output signal is $y(t) = e^{\sigma t}y'(t)$, i.e. a decaying sinusoid. Therefore, for our analysis of propagation of the decaying sinusoids through linear dynamics we can use the same transfer functions, in which the Laplace variable should be replaced with $(\sigma + j\Omega)$.

The describing function N in the case of a transient oscillation may become a function of not only amplitude but of its derivatives too [10] (we disregard possible dependence of the DF on the frequency).

Considering that conditions (3) and (5) are equivalent, and the equality of the denominator of the closed-loop transfer function to zero (for some s) implies the fulfillment of (3), we can rewrite (3) for the transient oscillation as follows.

$$N(a, \dot{a}, \ldots)W_l(\sigma + j\Omega) = -1, \quad (6)$$

The use of the derivatives of the amplitude as arguments of the DF is inconvenient because it results in the necessity of consideration of additional variables (derivatives of the amplitude) which are not present otherwise. It is more convenient to consider σ and its derivatives than the derivatives of the amplitude. Also, we limit our consideration of the describing function arguments to the first derivative of the amplitude (or equivalently, to σ) only. We show below that for some nonlinearities the DF is a function of the amplitude only – like in the conventional DF analysis. Therefore, we can write the condition of the existence of a transient or steady oscillation as follows:

$$N(a, \sigma)W_l(\sigma + j\Omega) = -1, \quad (7)$$

We shall refer to (7) as to the *dynamic harmonic balance* condition (equation).

Assume now that the characteristic polynomial of the closed-loop system (with parametric dependence on the amplitude of the oscillations) has a pair of complex conjugate roots with negative real parts. Then a vanishing oscillation of certain frequency and amplitude occurs. The idea of considering equations of vanishing oscillations is similar to the one of the Krylov-Bogoliubov method [11]. However, the latter can only deal with small "deviations" from the harmonic oscillator and is limited to second-order systems. In the present approach, the "equivalent damping" is not limited to small values. Let us consider instantaneous values of the frequency, amplitude and decay and formulate the *dynamic harmonic balance principle* as follows.

At every time, a single-frequency mode transient oscillation can be described as a process of variable (instantaneous) frequency, amplitude and decay, which must satisfy equation (7).

Note: In (7) and the formulation given above, we consider only transient oscillations with zero mean and single-frequency mode when the characteristic polynomial (5) has only one pair of complex conjugate roots.

The overall motion can now be obtained from the dynamics HB as follows:

$$y(t) = a(t)\, e^{\sigma(t)t} \sin \Psi(t)\,, \qquad (8)$$

where $a(t)$, $\sigma(t)$ are obtained from the following differential equation:

$$\dot a(t) = a(t)\sigma(t), \quad a(0) = a_0\,, \qquad (9)$$

and $\Psi(t)$ is the phase computed as follows: $\Psi(t) = \int_0^t \Omega(\tau) d\tau + \phi$, where $\Omega(t)$ is obtained from (7), ϕ is selected to satisfy initial conditions.

III. Analysis of Motions in the Vicinity of a Periodic Solution

Carry out frequency-domain analysis of the transient process of the convergence to the periodic motion in the vicinity of a periodic solution in system (1), (2), using the dynamic harmonic balance condition (7). We can write the conventional harmonic balance condition, which can also be obtained from (7) when $\sigma = 0$, as follows:

$$N(a_0) W_l(j\Omega_0) = -1\,, \qquad (10)$$

where Ω_0 and a_0 are the frequency and the amplitude of the periodic solution. Write the dynamics harmonic balance condition for the increments from the periodic solution:

$$N(a_0 + \Delta a, \sigma) W_l(\sigma + j(\Omega_0 + \Delta\Omega)) = -1\,, \qquad (11)$$

We now take the derivative from both sides of (11) with respect to Δa (or a) in the point $a = a_0$:

$$\left(\left.\frac{\partial N(a,\sigma)}{\partial a}\right|_{a=a_0} + \left.\frac{\partial N(a,\sigma)}{\partial \sigma}\right|_{\sigma=0} \left.\frac{\mathrm{d}\sigma}{\mathrm{d}a}\right|_{a=a_0} \right) W_l(j\Omega_0) + N(a_0) \left.\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right|_{s=j\Omega_0} \left.\frac{\mathrm{d}s}{\mathrm{d}a}\right|_{a=a_0} = 0\,, \qquad (12)$$

At first limit our analysis only to the nonlinearities the describing function of which does not depend on σ (for example, the ideal relay nonlinearity). Later the same analysis can be applied to nonlinearities that depend on σ.

Express the derivative $\left.\frac{\mathrm{d}s}{\mathrm{d}a}\right|_{a=a_0}$ from equation (12):

$$\left.\frac{\mathrm{d}s}{\mathrm{d}a}\right|_{a=a_0} = -\frac{\left.\frac{\mathrm{d}N(a)}{\mathrm{d}a}\right|_{a=a_0} W_l(j\Omega_0)}{N(a_0) \left.\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right|_{s=j\Omega_0}}\,. \qquad (13)$$

Considering that $s = \sigma + j\Omega$, we can rewrite equation (13) as follows:

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}a}\right|_{a=a_0} + j\left.\frac{\mathrm{d}\Omega}{\mathrm{d}a}\right|_{a=a_0} = -\frac{\left.\frac{\mathrm{d}N(a)}{\mathrm{d}a}\right|_{a=a_0} W_l(j\Omega_0)}{N(a_0) \left.\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right|_{s=j\Omega_0}}\,. \qquad (14)$$

Equation (14) is a complex equation. It can be split into two equations for the real and imaginary parts. However, only real parts of (14) give an equation that has a solution. Once it is solved and $a(t)$ is found, $\Omega(t)$ can be found too. Considering that

$$\frac{1}{W(s)} \left.\frac{\mathrm{d}W(s)}{\mathrm{d}s}\right|_{s=j\Omega_0} = \left.\frac{\mathrm{d}\ln W(s)}{\mathrm{d}s}\right|_{s=j\Omega_0} = \left.\frac{\mathrm{d}\arg W(j\omega)}{\mathrm{d}\omega}\right|_{\omega=j\Omega_0} - j\left.\frac{\mathrm{d}\ln|W(j\omega)|}{\mathrm{d}\omega}\right|_{\omega=j\Omega_0},$$

and

$$-\frac{1}{N(a)} \left.\frac{\mathrm{d}N(a)}{\mathrm{d}a}\right|_{a=a_0} = \left.\frac{\mathrm{d}\ln \tilde N(a)}{\mathrm{d}a}\right|_{a=a_0} = \left.\frac{\mathrm{d}\ln|\tilde N(a)|}{\mathrm{d}a}\right|_{a=a_0} + j\left.\frac{\mathrm{d}\arg \tilde N(a)}{\mathrm{d}a}\right|_{a=a_0},$$

where $\tilde N(a) = -N^{-1}(a)$, we can write for the real part of (14):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}a} = \mathrm{Re}\left\{ \frac{\frac{\mathrm{d}\ln|\tilde N(a)|}{\mathrm{d}a} + j\frac{\mathrm{d}\arg \tilde N(a)}{\mathrm{d}a}}{\frac{\mathrm{d}\arg W(j\omega)}{\mathrm{d}\omega} - j\frac{\mathrm{d}\ln|W(j\omega)|}{\mathrm{d}\omega}} \right\},$$

which can be rewritten as follows (we skip for brevity the notation of the point in which the derivative is taken):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}a} = \frac{\frac{\mathrm{d}\ln|\tilde N(a)|}{\mathrm{d}a}\frac{\mathrm{d}\arg W(j\omega)}{\mathrm{d}\omega} - \frac{\mathrm{d}\arg \tilde N(a)}{\mathrm{d}a}\frac{\mathrm{d}\ln|W(j\omega)|}{\mathrm{d}\omega}}{\left(\frac{\mathrm{d}\arg W(j\omega)}{\mathrm{d}\omega}\right)^2 + \left(\frac{\mathrm{d}\ln|W(j\omega)|}{\mathrm{d}\omega}\right)^2} \qquad (15)$$

As a “side” product of our analysis, stability of a periodic solution can be assessed from as follows: $\left.\frac{\mathrm{d}\sigma}{\mathrm{d}a}\right|_{a=a_0} < 0$.

IV. Convergence Rate of SOSM System with Twisting Algorithm

It is worth noting that the dynamic harmonic balance condition (7) is valid not only for a Lure system but for system having a few nonlinearities, such as systems with second-order sliding mode (SOSM) control algorithms. However, the describing function of the whole control algorithm has to be obtained and used in equation (7) –

similar to the analysis of periodic motions [12]. We now analyze convergence of the so-called twisting algorithm [2], [3], which is defined as follows:

$$u = -c_1 \cdot \operatorname{sgn} y - c_2 \cdot \operatorname{sgn} \dot{y}, \qquad (16)$$

where c_1 and c_2 are amplitudes of the two relays in the control law.

Firstly, we formulate the following lemma, which will be instrumental below.

Lemma 1 (given without proof, which can be based upon consideration of time being function of *z*). For the first-order nonlinear differential equation

$$\dot{z} = -g(z), \qquad (17)$$

where $g(z) > 0$ for all $z > 0$, and $g(0) = 0$, and the initial condition $z(0) = z_0 > 0$ the following holds. If there exists function $h(z)$, such that $h(z) \le g(z)$ for all $z \in [0; z_0]$, $h(z) > 0$, and $h(0)=0$, so that a finite-time convergence to zero in the equation

$$\dot{z} = -h(z) \qquad (18)$$

takes place ($z(T_h)=0$, $z(t) \in [0; z_0]$) then the finite-time convergence to zero in the original equation takes place too, with the convergence time $T_g \le T_h$.

We shall prove the following statement.

Theorem 1. In the system (1) controlled by the twisting controller (16), asymptotic convergence takes only if $c_2 = 0$, and finite-time convergence takes place only if $c_2 > 0$.

Proof. Prove the theorem via assuming that the conventional harmonic balance condition holds in the origin and showing that this is a valid assumption for $c_2 = 0$, which leads to the conclusion about the asymptotic convergence, and invalid assumption for $c_2 > 0$ necessitating finite-time convergence (proof by contradiction). Write the conventional harmonic balance condition of the following form for the origin:

$$\left[N_1(a_{01}) + j\Omega_0 \cdot N_2(a_{02})\right] W_l(j\Omega_0) = -1, \qquad (19)$$

where $a_{01} \to 0$, $a_{02} \to 0$, $\Omega_0 \to \infty$, subscript "0" denotes the variable in the origin. We investigate convergence of the transient process in the vicinity of the origin by giving the amplitude a small increment and analyzing the type of convergence from this disturbed initial point. We shall write the *dynamic harmonic balance* equation for an incremented from the origin point:

$$\left[N_1(a_{01} + \Delta a_1) + (\Delta\sigma + j(\Omega_0 + \Delta\Omega)) \cdot N_2(a_{02} + \Delta a_2)\right] \cdot W_l(\Delta\sigma + j(\Omega_0 + \Delta\Omega)) = -1 , \qquad (20)$$

Taking the derivative with respect to a_1 from both sides of (20) yields:

$$\left.\frac{\partial N_1}{\partial a_1}\right|_{a_1=0} \cdot W_l(j\Omega_0) + j\Omega_0 \left.\frac{\partial N_2}{\partial a_1}\right|_{a_1=0} \cdot W_l(j\Omega_0) + \left.\frac{\partial W_l}{\partial s}\right|_{s=j\Omega_0} \frac{ds}{da_1} N_1(a_{01}) + N_2(a_{02}) \frac{ds}{da_1} W_l(j\Omega_0) + j\Omega_0 N_2(a_{02}) \left.\frac{\partial W_l}{\partial s}\right|_{s=j\Omega_0} \frac{ds}{da_1} = 0 \qquad (21)$$

where $s = \Delta\sigma + j(\Omega_0 + \Delta\Omega)$ is the Laplace variable; the derivatives $\frac{\partial N_1}{\partial a_1}$ and $\frac{\partial N_2}{\partial a_1}$ can be obtained by differentiating the describing functions of the two relay nonlinearities [12], respectively, as follows:

$$\frac{\partial N_1}{\partial a_1} = -\frac{4c_1}{\pi a_1^2}, \quad \frac{\partial N_2}{\partial a_1} = -\frac{4c_2}{\pi\Omega a_1^2}.$$

Express the quantity $\frac{ds}{da_1}$ from equation (21).

$$\frac{ds}{da_1} = \frac{1}{a_1\left\{\left.\frac{\partial \ln W_l}{\partial s}\right|_{s=j\Omega_0} + \frac{c_2}{\Omega_0}\frac{c_1 - jc_2}{c_1^2 + c_2^2}\right\}}.$$

Considering that $s = \Delta\sigma + j(\Omega_0 + \Delta\Omega)$ and, therefore, $\frac{ds}{da_1} = \frac{d(\Delta\sigma)}{da_1} + j\frac{d(\Delta\Omega)}{da_1} = \frac{d\sigma}{da_1} + j\frac{d\Omega}{da_1}$, we can write the following expression for the derivative of the decay:

$$\frac{d\sigma}{da_1} = \mathrm{Re}\frac{1}{a_1\left\{\left.\frac{\partial \ln W_l}{\partial s}\right|_{s=j\Omega_0} + \frac{c_2}{\Omega_0}\frac{c_1 - jc_2}{c_1^2 + c_2^2}\right\}} = \frac{1}{a_1}\frac{\frac{\partial \arg W_l}{\partial \ln\Omega_0} + \frac{c_1 c_2}{c_1^2 + c_2^2}}{\left\{\frac{\partial \arg W_l}{\partial \ln\Omega_0} + \frac{c_1 c_2}{c_1^2 + c_2^2}\right\}^2 + \left\{\frac{\partial \ln|W_l|}{\ln\Omega_0} + \frac{c_2^2}{c_1^2 + c_2^2}\right\}^2}, \qquad (22)$$

where

$$\frac{\partial \ln|W_l|}{\partial \ln\Omega_0} = \left.\frac{\partial \ln|W_l|}{\partial \ln\omega}\right|_{\omega=\Omega_0} = \lim_{\omega\to\infty}\frac{\partial \ln|W_l|}{\partial \ln\omega} = -r, \qquad (23)$$

r is the relative degree of the plant transfer function, which reflects the fact of the existence of high-frequency asymptotes of the Bode magnitude plot,

$$\frac{\partial \arg W_l}{\partial \ln \Omega_0} = \frac{\partial \arg W_l}{\partial \ln \omega}\bigg|_{\omega=\Omega_0} = \lim_{\omega\to\infty} \frac{\partial \arg W_l}{\partial \ln \omega} = 0- \qquad (24)$$

With account of (23) and (24), formula (22) can be rewritten as follows:

$$\frac{d\sigma}{d\ln a_1} = \frac{\frac{c_1c_2}{c_1^2+c_2^2}+0-}{\left\{\frac{c_1c_2}{c_1^2+c_2^2}\right\}^2 + \left\{-r+\frac{c_2^2}{c_1^2+c_2^2}\right\}^2}. \qquad (25)$$

It follows from formula (25) that if $c_2 = 0$ then $\frac{d\sigma}{d\ln a_1} = 0$ and if $c_2 > 0$ then $\frac{d\sigma}{d\ln a_1} > 0$. Before interpreting these conclusions, we analyze the derivative $\frac{d\sigma}{d\ln a_1}$. By definition $\sigma = \dot{a}_1 / a_1$ and, therefore,

$$\frac{d\sigma}{d\ln a_1} = a_1\frac{d\sigma}{da_1} = a_1\frac{d(\dot{a}_1/a_1)}{da_1} = \frac{d\dot{a}_1}{da_1} - \frac{\dot{a}_1}{a_1} \qquad (26)$$

We can interpret (26) as the second-order differential equation with state variables a_1 and $\dot{a}_1$ without explicit time and draw the phase portrait of this second-order system in the phase plane using the isoclines technique (Fig. 1).

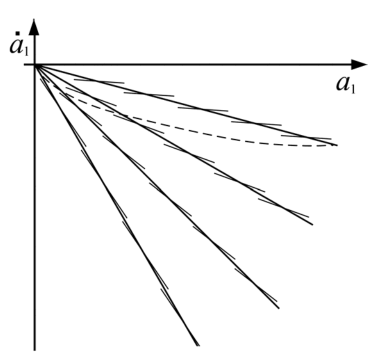

Fig. 1. Phase trajectory of system (25) with isoclines shown

Denote two angles (their tangents) as $\tan\varphi = \frac{d\dot{a}_1}{da_1}$ and $\tan\psi = \frac{\dot{a}_1}{a_1}$; φ gives the slope of the tangent line in a point of the phase portrait and ψ gives the slope of the vector from the origin to a point of the phase portrait. The condition $\frac{d\sigma}{d\ln a_1} = 0$ means that the angles φ and ψ must be equal, which means in turn that the phase portrait is a strait line and $\dot{a}_1 / a_1 = const$. The last condition constitutes the asymptotic convergence of a_1 to 0.

In the case when $c_2 > 0$, the angle ψ (absolute value) is always greater than the angle φ (absolute value). In fact, the difference between the tangents of these angles is equal to $q = \frac{\frac{c_1c_2}{c_1^2+c_2^2}}{\left\{\frac{c_1c_2}{c_1^2+c_2^2}\right\}^2 + \left\{-r+\frac{c_2^2}{c_1^2+c_2^2}\right\}^2}$ in any point of the phase portrait. Therefore, the phase trajectories of the system

$$\frac{d\dot{a}_1}{da_1} - \frac{\dot{a}_1}{a_1} = q, \quad q \ge 0 \qquad (27)$$

look like in Fig. 1 (the example of $c_1 = 10$, $c_2 = 10$, r=2, and q=0.2), with the trajectory schematically shown by the dash line. The fact that the difference between the tangents of the angles φ and ψ must be constant and equal to q in all points (including the origin) results in the infinite slope of the phase trajectories in the origin (the difference of two infinite values still gives q, which would be impossible with finite slopes). Therefore, in the vicinity of the origin, the differential equation for the amplitude is $\dot{a}_1 = h(a_1)$, with function $h(a_1) \le 0$ having infinite slope at $a_1 \to 0$.

Now prove that the equation $\dot{a}_1 = h(a_1)$, with function $h(a_1) \le 0$ having infinite slope at $a_1 \to 0$, features finite-time convergence. Define a majoring nonlinearity as $h_2(a_1) = -\beta a_1^\alpha$ for the function $h(a_1)$ through the selection of values α and β in such a way that in the initial point the following two equalities hold: $h_2(a_1) = h(a_1)$ and $\frac{\mathrm{d}\,h_2(a_1)}{\mathrm{d}\,a_1} = \frac{\mathrm{d}\,h(a_1)}{\mathrm{d}\,a_1}$. This results in the following two equations: $\frac{\mathrm{d}\,\dot{a}_1}{\mathrm{d}\,a_1} = q + \frac{\dot{a}_1}{a_1} = -\alpha\beta a_1^{\alpha-1}$ and $\tan\psi = \frac{\dot{a}_1}{a_1} = -\beta a_1^{\alpha-1}$. Solution of these equations results in the following expressions: $\alpha = 1 - \frac{q}{\tan\psi}$, $\beta = a_{10}^{\alpha-1}\tan\psi$, where a_{10} is the value of a_1 in the initial point. It follows from the last formulas if α and β are selected to ensure the same initial point and the same initial slope for the original and the majoring nonlinearities then in all other points corresponding to any selected ψ the slope (absolute value) of the original nonlinearity is steeper than the slope of the majoring nonlinearity for all $a_1 \in (0; a_{10})$. Therefore, $h(a_1) = h_2(a_1) < 0$ for all $a_1 \in (0; a_{10})$. Since differential equation $\dot{a}_1 = -\beta a_1^\alpha$ has finite-time convergence [7], [8], according to Lemma 1, the original equation $\dot{a}_1 = h(a_1)$ (or $\frac{\dot{a}_1}{a_1} = \sigma \to -\infty$) has finite-time convergence of a_1 to 0.

Because we assumed the existence of a periodic solution in the origin, which requires the fulfillments of the condition $\sigma = 0$ at $a_1 \to 0$, and we also showed that it is the case only if $c_2 = 0$, and therefore, our assumption was not valid for $c_2 > 0$, we can now conclude that asymptotic convergence of the transient oscillation amplitude takes place only if $c_2 = 0$, and finite-time convergence takes place if $c_2 > 0$. ■

A simple illustration of the majoring nonlinearity (corresponding to certain trajectory, which has to be found) is the square root function. If, for example, $h_2(a_1) = -\sqrt{a_1}$, then $\frac{d h_2}{d a_1} - \frac{h_2}{a_1} = \frac{1}{2} a_1^{-1/2}$. Consider only one trajectory that begins in the point where $0.5 a_{10}^{-1/2} = q$ and, therefore, the slopes of $h(a_1)$ and $h_2(a_1)$ are equal. This point corresponds to $a_{10} = 0.25 q^{-2}$. For all $a_1 \in (0; a_{10})$, the slope of $h_2(a_1)$ is steeper, and, therefore $h(a_1)$ is located below $h_2(a_1) = -\sqrt{a_1}$. The nonlinearity $h_2(a_1) = -\sqrt{a_1}$ is, therefore, a majoring nonlinearity for $h(a_1)$ (corresponding to the trajectory that begins in the point $a_{10} = 0.25 q^{-2}$, $\dot{a}_{10} = -0.5 q^{-1}$). The latter analysis is valid only for the considered trajectory. However, the use of function $h_2(a_1) = -\beta a_1^{\alpha}$, as shown above, allows one to design a majoring nonlinearity corresponding to any trajectory.

V. Example

An example of analysis of the system with the linear plant $W_l(s) = 1/(s^2 + s + 1)$ and the twisting controller with $c_1 = 50$, $c_2 = 5$ is given in Fig. 2. One can see from Fig. 2 that there is a good match between the results based on the presented theory, and the simulations. The instantaneous frequency and amplitude of the "theoretical" plot are close to the values obtained via simulations. However, due to the effect of the accumulation of phase (via integration of the instantaneous frequency) caused by the errors of frequency estimation, the instantaneous error of the system output may not necessarily be monotone decreasing function of time. In overall, the proposed approach provides a good estimate of the SOSM transient dynamics.

VI. Conclusion

The dynamic harmonic balance condition is formulated, and a frequency-domain approach to analysis of transient oscillatory processes is developed. The proposed method is applied to analysis of convergence of SOSM controlled systems. The previously developed approach applicable only to second-order systems is now extended, through the use of the dynamic harmonic balance condition, to high-order systems. The proposed approach may find many applications in various areas of engineering, in solving the problems that involve estimation of the dynamics of establishing of oscillations.

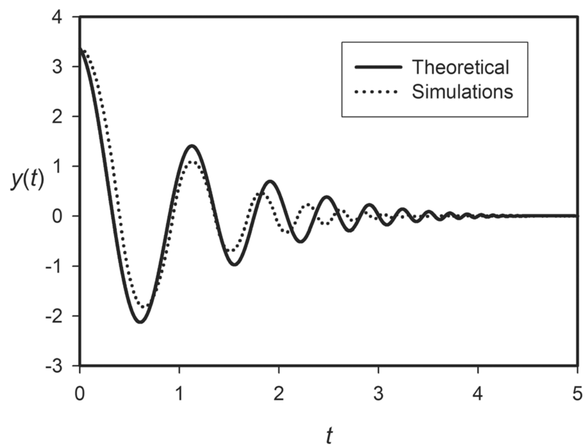

Fig. 2. Example of analysis of twisting SOSM controlled system

References

[1] D.P. Atherton, *Nonlinear Control Engineering - Describing Function Analysis and Design*. Van Nostrand Company Limited: Workingham, Berks, UK, 1975.

[2] A. Levant (Levantovsky, L.V.). "Sliding order and sliding accuracy in sliding mode control", *International Journal of Control*, **58**, No. 6, pp.1247-1263, 1993.

[3] A. Levant, "Universal SISO sliding-mode controllers with finite-time convergence", *IEEE transactions on Automatic Control*, **46**, No. 9, pp. 1447-1451, 2001.

[4] G. Bartolini, A. Ferrara, E. Usai, "Chattering Avoidance by Second-Order Sliding Mode Control", *IEEE Transactions on Automatic Control,* Vol.43, No.2, pp. 241-246, 1998.

[5] G. Bartolini, A. Ferrara, A. Levant, E. Usai, "On Second Order Sliding Mode Controllers", In *Variable Structure Systems, Sliding Mode and Nonlinear* (*Lecture Notes in Control and Information Science, 247*) (K.D. Young and U. Ozguner (eds.)), pp. 329-350. Springer-Verlag, London, 1999.

[6] Y.B. Shtessel, I.A. Shkolnikov, M.J. Brown, "An asymptotic second-order smooth sliding mode control", *Asian Journal of Control*, **5**, No. 4, pp. 498-504, 2003.

[7] Z. Man, A.P. Poplinsky, H.R. Wu, "A robust MIMO terminal sliding mode control for rigid robotic manipulators", *IEEE Trans. Automat. Contr*., **39**, No. 12, pp. 2464-2468, 1994.

[8] X. Yu, Y. Wu, Z. Man, "On global stabilization of nonlinear dynamical systems", In: *Variable Structure Systems, Sliding Mode and Nonlinear Control* (K.D. Toung, U. Ozguner (Eds.)), p. 109-122. Springer, London, 1999.

[9] I. Boiko, "Extension of harmonic balance principle and its application to analysis of convergence rate of second-order sliding mode control algorithms," *Proc. 2008 American Contr. Conf.*, Seattle, 2008, pp. 4691-4696.

[10] A. Gelb, W. E. Vander Velde. *Multiple-Input Describing Functions and Nonlinear System Design*. McGraw Hill, 1968

[11] N.M. Krylov, N.N. Bogoliubov, *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, 1947.

[12] I. Boiko, M., L. Fridman, M.I. Castellanos, "Analysis of second-order sliding mode algorithms in the frequency domain", *IEEE Trans. Automat. Contr*., **49**, No. 6, pp. 946-950, 2004.