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Bounding self-induced oscillations via invariant level sets of piecewise quadratic Lyapunov functions*

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Abstract— **A Lyapunov approach is developed in this paper for estimation of the magnitude of self-induced oscillations for systems with piecewise linear elements. The oscillatory trajectories are bounded by invariant level sets of a piecewise quadratic Lyapunov function. An optimization problem with bilinear-matrix-inequality constraints is formulated to minimize the invariant level set and to obtain tight bound for oscillatory trajectories. Several examples demonstrate the effectiveness of the new method on analysis of self-induced oscillations.**

Keywords: Self-induced oscillation, chaos, piecewise linear systems, invariant set, piecewise quadratic function

I. Introduction

Nonlinear oscillations are ubiquitous in physical systems and have been studied for systems of various types in biology, chemistry, circuits, communications, biophysics, plasma physics, power electronics, etc, (e.g., see [2], [4], [16]). Some oscillations are natural phenomena, some are artificially created, e.g., for transmitting information [8] or mixing substances [20] and some are undesirable and need to be eliminated or suppressed (e.g., [3]). In many cases, the oscillations demonstrate chaotic behaviors.

An important problem in nonlinear oscillations is to estimate the magnitude of the oscillation. In many chaotic systems, the oscillation occurs within a global attractor (e.g., see [5], [12], [13], [23]). If we can determine a bounded set which contains the global attractor, then we are certain that the system has no other equilibrium points, periodic solutions, or chaotic attractors outside this bounded set. Thus we can focus our study inside the bounded set. Estimating the bounds for chaotic oscillations is also useful for chaos control and chaos synchronization.

The concept of invariant set plays an important role in estimating the bounds for periodic or chaotic attractors. The positively invariant set can be effectively derived from the level set of a Lyapunov function. In [21], Lyapunov functions are used to study the bounds for trajectories of the Lorenz equations. Later in [17], [18], [22], quadratic Lyapunov functions are used to construct ellipsoidal invariant sets for estimating the bounds for various types of Lorenz systems and other types of chaotic systems. In [9], [10], piecewise quadratic Lyapunov functions are used to construct invariant sets for bounding oscillating trajectories for systems with one piecewise linear element which is continuous and odd-symmetric.

In this work, we will study more general systems whose piecewise linear element is continuous but may not be odd-symmetric. The reason for studying systems with piecewise linear elements is that, they are numerically tractable and they can be used to approximate most nonlinear systems. Furthermore, many typical nonlinear oscillation patterns can be realized with Chua's circuit family [5], [23], or generalized Chua's circuit [19]. These circuits have three energy storage elements and one piecewise linear resistive element. They can be used to generate limit cycles, double scroll and multiscroll chaotic attractors. More recently in [14], [15], multiscroll chaotic attractors are generated with third order circuits whose nonlinear element contains saturated function series. The nonlinear elements in [14], [15] may not be odd-symmetric.

A natural and effective way to incorporate the piecewise linear property of the nonlinear element is to use piecewise quadratic Lyapunov functions, which were initially developed in [11] for stability analysis of piecewise linear systems. A great advantage of piecewise quadratic Lyapunov functions is that they can be constructed by solving optimization problems constrained by linear matrix inequalities (LMIs) or bilinear matrix inequalities (BMIs).

The piecewise quadratic Lyapunov functions were first used in [9], [10], to estimate the bounds for oscillatory attractors via invariant sets. The Lyapunov approach in [9], [10] is based on the idea of representing a piecewise linear function as the sum of a linear function and a family of saturation functions. This treatment turns a piecewise linear system into a standard saturated system. However, this approach is only applicable to the case where the nonlinearity is odd-symmetric. In this work, we will present a new method that is applicable to general piecewise linear systems. Moreover, less conservative conditions will be derived for the invariance of the level set of the Lyapunov function. When applied to the examples in [9], [10], tighter bound for the magnitude of oscillations will be obtained.

II. System description and piecewise quadratic Lyapunov functions

A. Systems with a piecewise linear element

Most systems with one nonlinear element $\psi(\cdot)$ can be described as:

$$\dot{x} = Ax + B\psi(Kx) \tag{1}$$

where $x \in \mathbb{R}^n$. We consider the case where $\psi(\cdot)$ is a piecewise linear function with $\psi(0) = 0$. Throughout the paper we assume that $\psi(\cdot)$ is a continuous function with $J + N + 1$ partitions: $(-\infty, \underline{a}_N], (\underline{a}_N, \underline{a}_{N-1}], \cdots, (\underline{a}_1, \bar{a}_1)$, $[\bar{a}_1, \bar{a}_2)$, $\cdots, [\bar{a}_{J-1}, \bar{a}_J)$, $[\bar{a}_J, \infty)$, where $0 \in (\underline{a}_1, \bar{a}_1)$, $\underline{a}_j < 0$, $\bar{a}_j > 0$, for all j. The slopes in each interval are $\underline{c}_N, \underline{c}_{N-1}, \cdots, c_0, \bar{c}_1, \cdots, \bar{c}_{J-1}, \bar{c}_J$, respectively.

Given $h > 0$, denote

$$\Omega_h = \{x \in \mathbb{R}^n : Kx \in [-h, h]\}.$$

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Assume that the oscillatory trajectories are inside Ω_h. We may start with a large h and then reduce it for better estimation. For simplicity, we assume that $h > \max\{-\underline{a}_N, \bar{a}_J\}$. Denote $\underline{a}_{N+1} = -h, \bar{a}_{J+1} = h$, and

$$\begin{aligned}
&\Omega_0 = \{x \in \mathbb{R}^n : Kx \in [\underline{a}_1, \bar{a}_1]\}\\
&\Omega_1 = \{x \in \mathbb{R}^n : Kx \in [\bar{a}_1, \bar{a}_2]\}\\
&\vdots\\
&\Omega_J = \{x \in \mathbb{R}^n : Kx \in [\bar{a}_J, \bar{a}_{J+1}]\}\\
&\Omega_{J+1} = \{x \in \mathbb{R}^n : Kx \in [\underline{a}_2, \underline{a}_1]\}\\
&\vdots\\
&\Omega_{J+N} = \{x \in \mathbb{R}^n : Kx \in [\underline{a}_{N+1}, \underline{a}_N]\}
\end{aligned}$$

Let

$$\begin{aligned}
&\bar{f}_1 = \bar{a}_1 c_0, \quad \underline{f}_1 = \underline{a}_1 c_0\\
&\bar{f}_j = \bar{f}_{j-1} + \bar{c}_{j-1}(\bar{a}_j - \bar{a}_{j-1}), \ \ j = 2, \cdots, J\\
&\underline{f}_j = \underline{f}_{j-1} + \underline{c}_{j-1}(\underline{a}_j - \underline{a}_{j-1}), \ \ j = 2, \cdots, N.
\end{aligned}$$

Then

$$\psi(Kx) = \begin{cases} c_0 Kx & x \in \Omega_0\\ \bar{f}_j + \bar{c}_j(Kx - \bar{a}_j) & x \in \Omega_j, \ j = 1, \cdots, J\\ \underline{f}_j + \underline{c}_j(Kx - \underline{a}_j), & x \in \Omega_{J+j}, \ j = 1, \cdots, N \end{cases}$$

And the system (1) can be described as follows

$$\dot{x} = \begin{cases} (A + BKc_0)x, \quad x \in \Omega_0\\ (A+BK\bar{c}_j)x + B(\bar{f}_j - \bar{a}_j\bar{c}_j), \quad x \in \Omega_j,\\ \qquad\qquad j = 1, \cdots, J\\ (A+BK\underline{c}_j)x + B(\underline{f}_j - \underline{a}_j\underline{c}_j), \quad x \in \Omega_{J+j},\\ \qquad\qquad j = 1, \cdots, N \end{cases} \tag{2}$$

B. The piecewise quadratic Lyapunov function

A piecewise quadratic Lyapunov function was introduced in [11] for the stability analysis of piecewise linear systems. Another form of piecewise quadratic Lyapunov function was defined in [6] for systems with saturation/deadzone, where the function was used to investigate global and regional stability, and some other performances such as the reachable set and the nonlinear L_2 gain. It turns out that the two forms of piecewise quadratic function in [11] and [6] are actually equivalent for systems with saturation/deadzone. In [9], [10], the definition in [6] was adopted since the systems have odd-symmetric nonlinear element and thus can be described as systems with saturation.

In this paper, we use the definition in [11] to deal with systems with more general piecewise-linear elements.

Based on the special partition of the state-space by $J + N$ parallel hyperplanes, $Kx = \bar{a}_j, j = 1, \cdots, J$, $Kx = \underline{a}_j, j = 1, \cdots, N$, we choose

$$F_0 = \begin{bmatrix} I_n \\ 0_{(J+N)\times n} \end{bmatrix}$$

$$F_1 = \begin{bmatrix} I_n & 0\\ K & -\bar{a}_1\\ 0_{(J+N-1)\times n} & 0 \end{bmatrix}, \cdots, F_N = \begin{bmatrix} I_n & 0\\ K & -\bar{a}_1\\ \vdots & \vdots\\ K & -\bar{a}_J\\ 0_{N\times n} & 0 \end{bmatrix}$$

$$F_{J+1} = \begin{bmatrix} I_n & 0\\ 0_{J\times n} & 0\\ K & -\underline{a}_1\\ 0_{(N-1)\times n} & 0 \end{bmatrix}, \cdots, F_{J+N} = \begin{bmatrix} I_n & 0\\ 0_{J\times n} & 0\\ K & -\underline{a}_1\\ \vdots & \vdots\\ K & -\underline{a}_N \end{bmatrix}$$

where the 0's are zero blocks with compatible dimensions, as in the sequel. Let $P \in \mathbb{R}^{(J+N+n)\times(J+N+n)}$ be a symmetric matrix. Define

$$V(x) = \begin{cases} x^{\mathrm{T}} F_0^{\mathrm{T}} P F_0 x, & x \in \Omega_0\\ \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} F_j^{\mathrm{T}} P F_j \begin{bmatrix} x\\ 1 \end{bmatrix}, & x \in \Omega_j, j > 0 \end{cases} \tag{3}$$

Then $V(x)$ is a continuous piecewise quadratic function. We will be interested in V satisfying $V(x) > 0$ for $x \neq 0$.

We will use an invariant level set of V to bound the oscillatory trajectories. The following issues need to be addressed:

- What is the condition for a level set to be invariant?
- What is the condition for the level set to be within Ω_h?
- How to measure the magnitude of a certain output variable inside the level set?

After these issues have been addressed, we will form a BMI optimization problem to estimate the magnitude of oscillation.

III. Estimating magnitude of oscillations via invariant set

Consider the system (1) again and the equivalent description (2). Assume that $\psi(\cdot)$ is continuous and the system is not stable at the origin, i.e., $A + BKc_0$ is not Hurwitz, so that self-induced oscillation is possible. A trajectory may diverge to the infinity, or stay within a bounded set. In the later case, it may converge to a single non-zero equilibrium point, or an oscillatory attractor (e.g., limit cycle, chaos). To estimate the magnitude of oscillation, we need to find a set that bounds the oscillatory trajectories as tightly as possible.

An effective way to bound a trajectory that does not diverge to infinity is to use invariant set. A set is called invariant if every trajectory starting from it stays inside.

A. Level set of V and matrix conditions for invariance

Without loss of generality, we consider the 1-level set of V, defined as

$$L_V := \{x \in \mathbb{R}^n : V(x) \leq 1\}.$$

The boundary of L_V is denoted as ∂L_V. Other level set where $V(x)$ is below another number can be converted into a 1-level set by scaling the matrix P (since $V(x)$ depends on P via a linear relationship, see (3)). For now we assume that $L_V \subset \Omega_h$.

Denote the one-sided directional derivative of V at x along z as

$$\dot{V}(x; z) := \lim_{\Delta t \to 0, \Delta t > 0} \frac{V(x + z\Delta t) - V(x)}{\Delta t}.$$

The directional derivative of V at x along $\dot{x}$, which is $\dot{V}(x; \dot{x})$, will be simply denoted as $\dot{V}(x)$. For system (1), if $\dot{V}(x) < 0$ for all $x \in \partial L_V$, then whenever a trajectory reaches ∂L_V, $V(x)$ is strictly decreasing, implying that the trajectory is entering L_V. Because of the same reason, every trajectory starting from within L_V has to stay inside. In short, the invariance of L_V is ensured by $\dot{V}(x) < 0$ for all $x \in \partial L_V$.

Denote $V_0(x) = x^{\mathrm{T}} F_0 P F_0 x$ and $V_j(x) = \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} F_j^{\mathrm{T}} P F_j \begin{bmatrix} x \\ 1 \end{bmatrix}$. Then $V(x) = V_j(x)$ for $x \in \Omega_j$. For $x \in \mathrm{int}(\Omega_j)$ (i.e., $Kx \neq \underline{a}_i$ or $Kx \neq \bar{a}_i$ for any i), the partial derivative $\partial V/\partial x$ exists and $\partial V/\partial x = \partial V_j/\partial x$. Thus,

$$\dot{V}(x) = (\partial V_j/\partial x)^{\mathrm{T}} \dot{x}, \quad \forall x \in \mathrm{int}(\Omega_j).$$

Lemma 1: Assume that $V(x) > 0$ for all $x \in \Omega_h \setminus \{0\}$ and $L_V \subset \Omega_h$. If

$$\dot{V}(x) = (\partial V_j/\partial x)^{\mathrm{T}} \dot{x} < 0, \quad \forall x \in \mathrm{int}(\Omega_j) \cap \partial L_V, \ \forall j, \tag{4}$$

then L_V is an invariant set.

Unlike stability analysis for which it is required that $\dot{V}(x) < 0$ for almost all $x \in L_V \setminus \{0\}$, the condition for the invariance of L_V is more relaxed.

We will use a different approach than that in [11] to derive BMI conditions for $V(x) > 0$ and $\dot{V}(x) < 0$ in each $\Omega_j, j = 0, \cdots, J+N$. In fact, the special structure of the parallel partitions allows each Ω_j to be exactly described with one quadratic inequality. This will be used to derive a less conservative condition for $V(x) > 0$ and $\dot{V}(x) < 0$ within Ω_h for the piecewise linear system.

First we see that Ω_0 can also be written as

$$\Omega_0 = \left\{ x \in \mathbb{R}^n : \left| Kx - \frac{\bar{a}_1 + \underline{a}_1}{2} \right|^2 \leq \left(\frac{\bar{a}_1 - \underline{a}_1}{2} \right)^2 \right\}$$

This can be equivalently described as

$$\Omega_0 = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} M_0 \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \right\}, \tag{5}$$

where

$$M_0 = \begin{bmatrix} 2K^{\mathrm{T}}K & -(\bar{a}_1 + \underline{a}_1)K^{\mathrm{T}} \\ -(\bar{a}_1 + \underline{a}_1)K & 2\bar{a}_1 \underline{a}_1 \end{bmatrix} \tag{6}$$

Similarly, for $j = 1, \cdots, J+N$,

$$\Omega_j = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} M_j \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0 \right\}, \tag{7}$$

where for $j = 1, \cdots, J$,

$$M_j = \begin{bmatrix} 2K^{\mathrm{T}}K & -(\bar{a}_j + \bar{a}_{j+1})K^{\mathrm{T}} \\ -(\bar{a}_j + \bar{a}_{j+1})K & 2\bar{a}_j \bar{a}_{j+1} \end{bmatrix}$$

and for $j = 1, \cdots, N$,

$$M_{J+j} = \begin{bmatrix} 2K^{\mathrm{T}}K & -(\underline{a}_j + \underline{a}_{j+1})K^{\mathrm{T}} \\ -(\underline{a}_j + \underline{a}_{j+1})K & 2\underline{a}_j \underline{a}_{j+1} \end{bmatrix}$$

We first consider the set Ω_0 which contains 0 in its interior. To obtain simple matrix condition, we denote

$$A_0 = A + BKc_0.$$

Then by (2),

$$\dot{x} = A_0 x, \quad \text{for } x \in \Omega_0. \tag{8}$$

Recall that $V(x) = x^{\mathrm{T}} F_0^{\mathrm{T}} P F_0 x$ for $x \in \Omega_0$. Thus $V(x) > 0$ for $x \in \Omega_0 \setminus \{0\}$ can be equivalently stated as

$$F_0^{\mathrm{T}} P F_0 > 0, \tag{9}$$

since Ω_0 contains the origin in its interior.

To examine $\dot{V}(x)$ for $x \in \partial L_V \cap \mathrm{int}(\Omega_0)$, we note that

$$\dot{V}(x) = \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} A_0^{\mathrm{T}} F_0^{\mathrm{T}} P F_0 + F_0^{\mathrm{T}} P F_0 A_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and

$$\partial L_V \cap \mathrm{int}(\Omega_0) = \left\{ x : \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} F_0^{\mathrm{T}} P F_0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \right.$$
$$\left. \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} M_0 \begin{bmatrix} x \\ 1 \end{bmatrix} < 0 \right\} \tag{10}$$

By S-procedure ([1], page 23), $\dot{V}(x) < 0$ for all $x \in \partial L_V \cap \mathrm{int}(\Omega_0)$, if there exist $\beta_0 \geq 0, \zeta_0 \in \mathbb{R}$ such that

$$\begin{bmatrix} A_0^{\mathrm{T}} F_0^{\mathrm{T}} P F_0 + F_0^{\mathrm{T}} P F_0 A_0 & 0 \\ 0 & 0 \end{bmatrix} < \beta_0 M_0 + \zeta_0 \begin{bmatrix} F_0^{\mathrm{T}} P F_0 & 0 \\ 0 & -1 \end{bmatrix} \tag{11}$$

Note that ζ_0 can be either positive or negative since the first constraint for describing $\partial L_V \cap \mathrm{int}(\Omega_0)$ in (10) is an equality.

Next we consider $\Omega_j, j = 1, \cdots, J+N$. For $j = 1, \cdots, J$, denote

$$A_j = \begin{bmatrix} A + BK\bar{c}_j & B(\bar{f}_j - \bar{a}_j \bar{c}_j) \\ 0_{1\times n} & 0 \end{bmatrix},$$

and for $j = 1, \cdots, N$, denote

$$A_{J+j} = \begin{bmatrix} A + BK\underline{c}_j & B(\underline{f}_j - \underline{a}_j \underline{c}_j) \\ 0_{1\times n} & 0 \end{bmatrix}.$$

Then by (2), we have

$$\frac{d}{dt} \begin{bmatrix} x \\ 1 \end{bmatrix} = A_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \text{for } x \in \Omega_j, j = 1, \cdots, J+N. \tag{12}$$

Recall that $V(x) = \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} F_j^{\mathrm{T}} P F_j \begin{bmatrix} x \\ 1 \end{bmatrix}$ for $x \in \Omega_j$. Thus for all $x \in \mathrm{int}(\Omega_j)$,

$$\dot{V}(x) = \begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} (A_j^{\mathrm{T}} F_j^{\mathrm{T}} P F_j + F_j^{\mathrm{T}} P F_j A_j) \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

By S procedure, $V(x) > 0$ for $x \in \Omega_j$ and $\dot{V}(x) < 0$ for all $x \in \partial L_V \cap \mathrm{int}(\Omega_j)$, if there exist $\alpha_j, \beta_j \geq 0, \zeta_j \in \mathbb{R}$ such that

$$\begin{aligned} & F_j^{\mathrm{T}} P F_j + \alpha_j M_j > 0, \\ & A_j^{\mathrm{T}} F_j^{\mathrm{T}} P F_j + F_j^{\mathrm{T}} P F_j A_j \\ & \qquad < \beta_j M_j + \zeta_j \left(F_j^{\mathrm{T}} P F_j - \begin{bmatrix} 0_n & 0 \\ 0 & 1 \end{bmatrix} \right) \end{aligned} \tag{13}$$

In summary, we have the following result.

Proposition 1: Given a symmetric matrix $P \in \mathbb{R}^{(J+N+n)\times(J+N+n)}$. Let L_V be the 1-level set of the piecewise quadratic Lyapunov function as defined in (3). Suppose that $L_V \subset \Omega_h$. If there exist scalars $\alpha_j, \beta_j \geq 0$, $\zeta_j \in \mathbb{R}$, $j = 0, 1, 2, \cdots, J+N$, so that the matrix inequalities (9), (11) and (13) are satisfied, then L_V is an invariant set.

B. LMI conditions for set inclusion

In Proposition 1, we assumed that $L_V \subset \Omega_h$. In what follows, we give matrix conditions for $L_V \subset \Omega_h$.

It is easy to see that L_V is strictly inside Ω_h if and only if $V(x) > 1$ for all x in the hyperplanes $Kx = \pm h$, equivalently, if and only if $V(x) - Kx/h > 0$ for $Kx - h = 0$ and $V(x) + Kx/h > 0$ for $Kx + h = 0$. By S procedure, this is satisfied if there exist $\eta_J, \eta_{J+N} \in \mathbb{R}$ such that

$$F_J^{\mathrm{T}} P F_J - \frac{1}{2h} \begin{bmatrix} 0 & K^{\mathrm{T}} \\ K & 0 \end{bmatrix} + \eta_J \begin{bmatrix} 0 & K^{\mathrm{T}} \\ K & -2h \end{bmatrix} > 0 \tag{14}$$

$$F_{J+N}^{\mathrm{T}} P F_{J+N} + \frac{1}{2h} \begin{bmatrix} 0 & K^{\mathrm{T}} \\ K & 0 \end{bmatrix} + \eta_{J+N} \begin{bmatrix} 0 & K^{\mathrm{T}} \\ K & 2h \end{bmatrix} > 0 \tag{15}$$

Recall that the planes $Kx = \pm h$ are in Ω_J and Ω_{J+N} respectively.

Due to the condition $F_j^{\mathrm{T}} P F_j + \alpha_j M_j > 0$ in (13) and the structure of M_j, it can be shown that V is a convex function when restricted to a plane $Kx = r$ for any constant r. Combining this with the condition that L_V is strictly inside Ω_h, it can be further shown that L_V is simply connected.

C. Maximal output magnitude in the level set

Let the output variable of interest be $y = Cx$. To estimate the maximal output y along an oscillatory trajectory inside L_V, we may compute the minimal $\gamma > 0$ such that $Cx \leq \gamma$ for all $x \in L_V$, which is satisfied if

$$V(x) - 1 \geq 0 \quad \text{for all } x \text{ such that } Cx/\gamma \geq 1. \tag{16}$$

Using S procedure on each set $\Omega_j \cap \{x : Cx/\gamma_j \geq 1\}$, $\gamma_j \leq \gamma$, $j = 0, 1, \cdots, J+N$, the above condition is equivalent to the existence of $\xi_j, \delta_j, \gamma_j \geq 0$, such that

$$\gamma_j \leq \gamma, j = 0, 1, \cdots, J+N \tag{17}$$

$$\begin{bmatrix} F_0^{\mathrm{T}} P F_0 & 0 \\ 0 & -1 \end{bmatrix} + \delta_0 \begin{bmatrix} 0 & -C^{\mathrm{T}}/2\gamma_0 \\ -C/2\gamma_0 & 1 \end{bmatrix} + \xi_0 M_0 \geq 0 \tag{18}$$

$$F_j^{\mathrm{T}} P F_j + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \delta_j \begin{bmatrix} 0 & -C^{\mathrm{T}}/2\gamma_j \\ -C/2\gamma_j & 1 \end{bmatrix} + \xi_j M_0 \geq 0, \quad j = 1, \cdots, J+N \tag{19}$$

D. Estimating the magnitude of oscillation via BMI optimization

In summary, to estimate the maximal output $y = Cx$ along oscillatory trajectories, we need to find an invariant set L_V that bounds the oscillatory trajectories as tightly as possible. To reduce the conservatism of estimation, we perform the analysis in the set $\Omega_h = \{x : |Kx| \leq h\}$, where h is a scalar to be adjusted, so that the condition for set invariance is least conservative. In the previous sections, we obtained

1. The condition for the level set L_V to be invariant by (9), (11) and (13).
2. The condition for L_V to be inside Ω_h by (14), (15).
3. The condition for $Cx \leq \gamma$ for all $x \in L_V$ by (17), (18), (19).

An optimization problem can be formed by minimizing γ so that all the conditions are satisfied, where the optimizing variables include the matrix P defining the Lyapunov function, and the scalars $\alpha_j, \beta_j, \xi_j, \delta_j, \gamma_j \geq 0$, $\zeta_j \in \mathbb{R}$, $j = 0, 1, \cdots, J+N$ and $\eta_J, \eta_{J+N} \in \mathbb{R}$.

The bilinear terms in the matrix inequalities include $\zeta_j P$ in (11) and (13), and $\delta_j C/2\gamma_j$ in (18) and (19). All the other terms are either linear or constant matrices.

The nonlinear terms $\delta_j C/2\gamma_j$ in (18) and (19) can be turned into linear terms by a change of variables. Define new variables $s_j = \delta_j/2\gamma_j$. Then (17), (18), (19) can be replaced with

$$\delta_j \leq 2\gamma s_j, \quad j = 0, 1, \cdots, J+N \tag{20}$$

$$\begin{bmatrix} F_0^{\mathrm{T}} P F_0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -s_0 C^{\mathrm{T}} \\ -s_0 C & \delta_0 \end{bmatrix} + \xi_0 M_0 \geq 0, \tag{21}$$

$$F_j^{\mathrm{T}} P F_j + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -s_j C^{\mathrm{T}} \\ -s_j C & \delta_j \end{bmatrix} + \xi_j M_0 \geq 0, \quad j = 1, \cdots, J+N \tag{22}$$

And the optimization problem can be descibed as

$$\begin{aligned} & \inf \gamma, \\ & s.t. \;\; \delta_j \leq 2\gamma s_j, \quad j = 0, 1, \cdots, J+N \\ & (9), (11), (13), (14), (15), (21), (22) \\ & P = P^{\mathrm{T}}, \alpha_j, \beta_j, \xi_j, \delta_j, s_j \geq 0, \; j = 0, \cdots, J+N. \end{aligned} \tag{23}$$

For fixed ζ_j's, the above is a generalized eigenvalue problem. To simplify computation, we may at first assume that $\zeta_j = \zeta$ for all j and use a one dimensional sweep to find ζ which minimizes γ. Then use this ζ as the initial value for every ζ_j and apply a standard nonlinear optimization, such as "fminsearch" to optimize ζ_j. To use "fminsearch", a function $J(\zeta_0, \zeta_1, \cdots, \zeta_{J+N})$ is defined as the minimal γ for the problem (23) with these given ζ_j's. This approach is effective on the examples in the following section. One may also try more general algorithms for solving BMI problems, e.g., see [7].

IV. Examples

Example 1: Consider the system for generating a van del pol oscillator,

$$\begin{aligned} \dot{x} &= Ax + B\psi(Kx) \\ &= \begin{bmatrix} 0 & 1 \\ -4000 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -200 \end{bmatrix} \psi([0 \;\; 1]x). \end{aligned}$$

where $\psi(v) = -v + v^3/3$. We approximate the function $\psi(v) = -v + v^3/3$ with a piecewise linear function with 8 breakpoints at $\pm 0.5, \pm 1, \pm 1.5, \pm 2$. This system was used in [9]. A plot of the function can be found in [9].

We choose $y = v = [0 \ \ 1]x$ as the output. By simulating the piecewise linear system, the magnitude of the output along the limit cycle is 2.0033. By simulating the original nonlinear system, the magnitude is 2.0247. By using the optimization algorithm in [9], the output bound on $y = v$ is obtained as 2.2131V.

By solving the optimization problem (23), we obtained a smaller bound for y, 2.0750, which is much closer to the actual magnitude 2.0247 than the estimate in [9]. The resulting invariant set is plotted in Fig. 1 (outer closed curve) along with the limit cycle (the inner dotted closed curve).

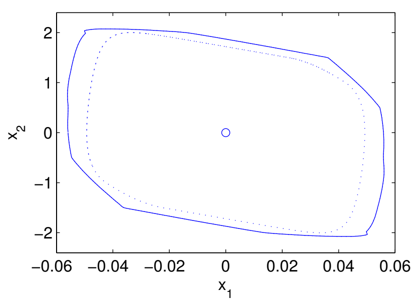

Fig. 1. Limit cycle and an invariant level set.

Example 2: The system in this example is modified from a system in [14] (for generating Fig. 5 in [14]), which has a 3-scroll chaotic attractor, as plotted in Fig. 2. Since

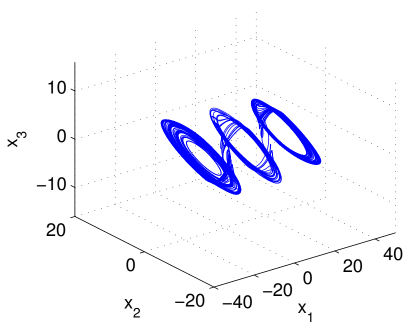

Fig. 2. The 3-scroll chaotic attractor

the chaotic attractor in [14] is not a global attractor, we added two breakpoints for the piecewise linear function and slightly changed the other parameters. In terms of this paper's notation, the system is described by (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.72 & -0.72 & -0.73 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $K = [1 \quad 0 \quad 0]$. The nonlinear function ψ is not symmetric. Its breakpoints are $\underline{a}_2 = -20, \underline{a}_1 = -1, \bar{a}_1 = 1, \bar{a}_2 = 19, \bar{a}_3 = 21, \bar{a}_4 = 34$. The slopes are $\underline{c}_2 = 0.6; \underline{c}_1 = 0; c_0 = 7, \bar{c}_1 = 0.1, \bar{c}_2 = 7, \bar{c}_3 = 0, \bar{c}_4 = 0.4$.

Let us first estimate the magnitude of $y_1 = x_1 = C_1 x$, where $C_1 = [1 \quad 0 \quad 0]$. If we use a quadratic Lyapunov function, the minimal upper bound for y_1 by the algorithm is 60.5. The resulting invariant ellipsoid is plotted in Fig. 3.

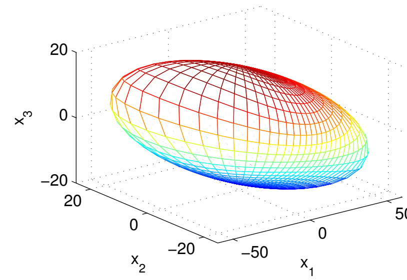

Fig. 3. Invariant ellipsoid by using quadratic function for bound of x_1.

If we use the piecewise quadratic (PWQ) Lyapunov function by solving the optimization problem (23), a smaller upper bound for $y_1 = x_1$ is obtained as 47.7831. The invariant set is plotted in Fig. 4.

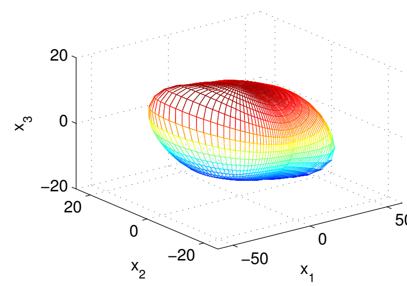

Fig. 4. Invariant set by using PWQ function for bound of x_1

Fig. 5 plots the chaotic trajectory, the boundaries of the two resulting invariant sets, projected to the (x_1, x_2) plane, where the outer ellipsoidal boundary corresponds to the invariant set in Fig. 3 and the smaller asymmetric boundary is projected from the invariant set in Fig. 4.

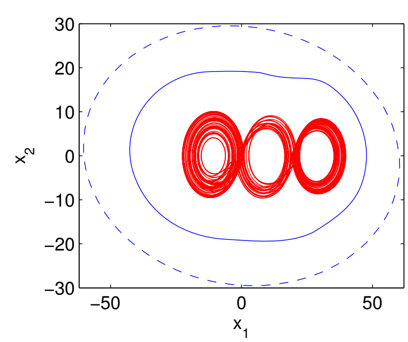

Fig. 5. Projections of a trajectory and two invariant sets

Since the nonlinear element is not symmetric, the 3-scroll chaotic attractor is not symmetric. Thus the maximal x_1 and the maximal $-x_1$ are different. Therefore we chose $y_2 = -x_1 = [-1 \ \ 0 \ \ 0]x$ and obtained an upper bound for

y_2 as 31.8853, which is indeed smaller than the upper bound on x_1. To form a tighter bound for the chaotic attractor, we obtained 2 more invariant sets, by minimizing the upper bounds on x_2 and x_3, which are obtained as 16.2172 and 13.0725.

Fig. 6 plots the boundary of the intersection of the 4 invariant level sets, which is also an invariant set.

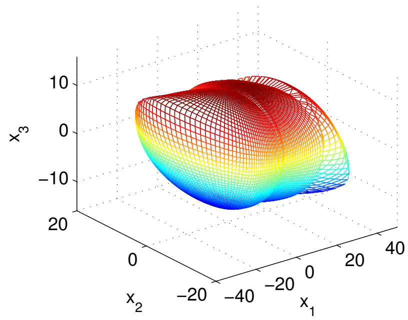

Fig. 6. Intersection of four invariant sets

The projections of the intersection of the four invariant sets are plotted in Fig. 7.

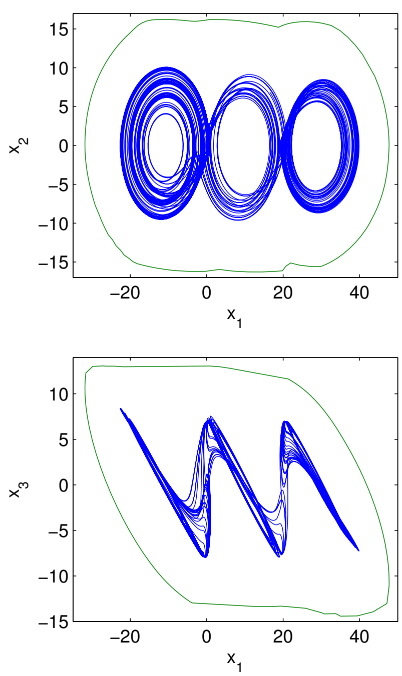

Fig. 7. Projections of the invariant set

The above examples demonstrate that the invariant level sets of piecewise quadratic functions can provide tight bounds for the oscillatory trajectories and yield good estimates for the magnitude of the oscillations. From each example, we see some gap between the attractors and the boundary of the invariant set. This gap is caused by the difference between the trajectory of the system and the shape of the level set. A possible approach to obtain tighter bound is to add extra partitions between $\bar{a}_j$ and $\bar{a}_{j+1}$ (with the same slope $\bar{c}_j$ for ψ). This will increase the dimension of P and the number of parameters to be optimized.

V. Conclusion

We derived a new BMI-based method to estimate the magnitude of self-induced oscillations for systems with one piecewise linear element. Invariant level sets of piecewise quadratic functions are used to bound the oscillatory orbits.

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