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Optimal train control: analysis of a new local optimization principle

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***Abstract*— It is known that the optimal driving strategy for a train takes the form of a *power-speedhold-coast-brake* strategy unless the track contains steep grades. In such cases the predominant *speedhold* mode must be interrupted by phases of *power* on steep uphill sections and phases of *coast* on steep downhill sections. The Freightmiser device is used by Pacific National to provide on-board advice to train drivers about energy-efficient driving strategies. Freightmiser uses a fast and efficient numerical algorithm to solve a key local energy minimization problem and hence find the optimal switching points. Although the numerical algorithm converges to a feasible solution there is no direct proof that the solution is unique. We explain the basic ideas behind the local energy minimization principle and use an extended perturbation analysis to derive various equivalent forms of the necessary conditions.**

I. Background

A. Review of optimal train control

In a recent paper Howlett *et al* [13] summarize the development of the modern theory of optimal train control [1], [4], [5], [6], [7], [8], [10], [15], [17]. The problem is to minimise the energy required to drive a train from one station to the next within a given time interval. The optimal strategy is essentially a *power–speedhold–coast–brake* strategy except that the *speedhold* mode must be interrupted by a *power* phase for each steep uphill section and by a *coast* phase for each steep downhill section. Thus the optimal strategy is a switching strategy. Howlett *et al* [13] have recently shown that the precise switching points in a globally optimal strategy can be determined for each steep section by solving a special local optimization problem. Specialized control algorithms developed by the Scheduling and Control Group at the University of South Australia for TTG Transportation Technology have been used to develop the Freightmiser system which provides on-board advice to freight train drivers about energy-efficient driving strategies. Freightmiser is currently used by Pacific National, Australia's largest interstate rail freight operator, on all mainline freight services. General methods of computational control could also be used to compute optimal driving strategies but our specialized methods are more accurate and more efficient.

B. Computational methods for optimal control

If the Hamiltonian is linear in the control variables and the control variables have simple bounds then the optimal control is a combination of *bang–bang* control and *singular* arcs. In such cases the computational problem is reduced to one of finding optimal *switching points* efficiently and accurately. More generally one may approximate the optimal control by a switching sequence using only a limited number of well-defined control regimes.

For instance Lee *et al* [16] use the Control Parametrization Enhancing Technique to show that certain optimal discrete-valued control problems are equivalent to optimal control problems involving a new control function with pre-fixed switching points. The transformed problems are essentially combinatorial optimization problems involving optimal parameter selection. Bengea and DeCarlo [2] find optimal controls by a dense embedding into a larger family of systems where *bang–bang*-type solutions of the larger problem are also solutions to the original problem. Xu and Antsaklis [19] formulate control problems for both continuous-time and discrete-time as a two stage optimization. Kaya and Noakes [14] use time-optimal switching to find a feasible control by shooting from an initial point to a target point with a given number of switches. An optimal control is found by minimizing the sum of arc times. Maurer *et al* [18] consider second-order sufficient conditions which are particularly suited for numerical verification. See [13] for an expanded summary. In train control problems Howlett and Leizarowitz [12] show that optimal continuous controls can be realized by chattering on $\{0, 1\}$ or approximated by a finite sequence of alternating $\{0, 1\}$ controls. In problems involving solar-powered racing cars a similar theory of optimal control has been developed [3], [9].

In principle the globally optimal strategy [10], [15], [17] is a *speedhold* strategy that must be interrupted by switching to maximum *power* to traverse steep uphill sections and by switching to *coast* to traverse steep downhill sections. We propose an efficient computational algorithm to determine optimal switching points. In this paper we show how to calculate the optimal control for an isolated steep gradient, where the hold speed cannot be maintained using either maximum power or coasting. Similar calculations can be used for arbitrary sequences of steep gradients. In practice we use a shooting method to find the optimal location at which to leave a holding phase prior to any sequence of steep track sections [11]. The sequence of controls and switching locations are determined by evolution of an adjoint variable, and depend on the sequence of steep gradients. For any candidate initial switching location, the adjoint profile will prescribe a control sequence that causes the speed to drop to zero before the end of the journey or a control sequence that maintains a non-zero speed beyond the end of the journey or a control that allows a switch to the next *speedhold* phase.

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The novelty of our method is that we use a local optimization for each steep section to find the precise switching points for the globally optimal strategy.

C. Mathematical basis

Howlett and Pudney [6] have shown that the motion of a train with distributed mass can be reduced to the motion of a point-mass train. Let $x \in [0, X]$ denote the position of the train on the track where $x = 0$ is the initial position and $x = X$ is the final position. The equation of motion for a point-mass train can be written as

$$vv' \;=\; \frac{p}{v} - q - r(v) + g \tag{1}$$

where $v = v(x)$ is the speed of the train and $v' = v'(x)$ denotes the derivative of v with respect to x, $p = p(x) \in [0, P]$ is the controlled power per unit mass where P is the maximum power, $q = q(x) \in [0, Q]$ is the controlled braking force per unit mass where Q is the maximum braking force, $r(v)$ is the resistance force per unit mass and $g = g(x)$ is the component of gravitational acceleration due to the track gradient. The equations are naturally formulated with x as the independent variable because the track gradient depends on position. The elapsed time $t = t(x) \in [0, T]$, where T is the total time taken for the journey, satisfies the differential equation

$$t' \;=\; \frac{1}{v} \tag{2}$$

where $v = v(x)$ is the solution to the equation of motion and where $t' = t'(x)$ denotes the derivative with respect to x. The cost of the strategy is

$$J = \int_0^X \frac{p}{v}\, dx \tag{3}$$

which is the energy required to drive the train.

The optimal control problem is formulated as follows.

Problem: Let $T \geq T_{\min}$ where $T_{\min}$ is the minimum possible journey time. Find controls $p : [0, X] \to [0, P]$ and $q : [0, X] \to [0, Q]$ and a corresponding speed profile $v : [0, X] \to [0, \infty)$ and time function $t : [0, X] \to [0, T]$ satisfying the differential constraints (1) and (2) and the additional speed constraints $v(0) = v(X) = 0$ and time constraints $t(0) = 0$ and $t(X) = T$ so that the cost (3) is minimized.

The function $r : [0, \infty) \to [0, \infty)$ is the frictional resistance per unit mass. Define related functions $\varphi(v) = v \cdot r(v)$ and $\psi(v) = v^2 \cdot r'(v)$. The value $\varphi(v)$ is the power per unit mass required to travel at constant speed v on a level track. The following property is important.

Property 1: The function φ is non-negative and strictly convex with $\varphi(0) = 0$ and $\varphi(v)/v \to \infty$ as $v \to \infty$.

Howlett and Cheng [8] show that if Property 1 holds then the function ψ is strictly increasing. In this case there are only four possible optimal driving modes *power, speedhold, coast, brake* and only one holding speed $v = V$ for each optimal strategy. In an optimal strategy the *speedhold* phase must be interrupted by segments of *power* on steep uphill segments and by phases of *coast* on steep downhill segments. A section of track is said to be *steep uphill* at speed V and position x if

$$\frac{P}{v} - r(v) + g(x) < 0$$

for all $v \geq V$. Hence the train is unable to maintain speed $v \geq V$ under maximum power with $p = P$ and $q = 0$. A section of track is said to be *steep downhill* at speed V and position x if

$$-r(v) + g(x) > 0$$

for all $v \leq V$. Therefore the train is unable to maintain speed $v \leq V$ when coasting with $p = 0$ and $q = 0$. For a steep uphill section $[b, c]$ it is known [10] that a necessary condition for optimal switching can be written as

$$\int_a^d \frac{\psi(v) - \psi(V)}{v^3}\; I \; dx \;=\; 0 \tag{4}$$

where v is the speed profile for a *power* phase on an interval $(a, d) \supset [b, c]$ with $v(a) = v(d) = V$ and where

$$I(x) = \int_a^x \frac{P + \psi(v(\xi))}{v(\xi)^3}\; d\xi. \tag{5}$$

In a recent paper Howlett *et al* [13] have shown that the condition (4) can be rewritten in the form

$$\int_a^d E'(v)\delta v \; dx \;=\; 0 \tag{6}$$

where $E : (0, \infty) \to (0, \infty)$ is defined by the formula

$$E(v) = \frac{\psi(V)}{v} + r(v) \tag{7}$$

and where $\delta v : [0, X] \to [0, \infty)$ is an infinitesimal perturbation of the speed profile. Hence they deduced that the necessary conditions for optimal switching were also necessary conditions for a minimum of the locally defined functional

$$J_\ell(v) = \int_a^d [E(v) - E(V)]\, dx \tag{8}$$

where, once again, v is a solution to the equation of motion in *power* mode and where $a < b < c < d$ and a and d are chosen so that $v(a) = v(d) = V$. Howlett *et al* then show that the cost function in the local minimization can be rewritten in the form

$$J_\ell(v) = \psi(V)\left\{\Delta t - \frac{\Delta x}{V}\right\} + \int_a^d [r(v) - r(V)]\; dx \tag{9}$$

where $\Delta x = d - a$ and $\Delta t = t(d) - t(a)$. Hence $J_\ell(v)$ is the difference between the work done by the proposed strategy and a hypothetical (infeasible) *speedhold* strategy at speed V subject to a weighted penalty term given by the difference between the time taken for the proposed strategy and the time taken for the *speedhold* strategy. This is intuitively reasonable since the overall objective is to find a strategy that minimizes energy consumption subject to completion of the journey within a specified time. Similar arguments and similar formulae define necessary conditions for a *coast* phase on a steep downhill section of an optimal journey.

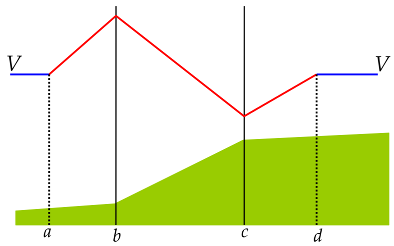

Fig. 1. Optimal speed profile on a steep uphill section.

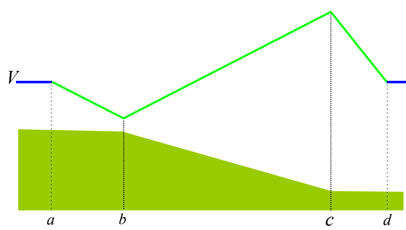

Fig. 2. Optimal speed profile on a steep downhill section.

Figures 1 and 2 show stylized speed profiles for an optimal *power* phase on a steep uphill section and an optimal *coast* phase on a steep downhill section.

II. AN EXTENDED PERTURBATION ANALYSIS

A. Perturbation of the speed profile

Although Freightmiser uses a fast and efficient numerical algorithm to solve the local energy minimization problem, there is no direct proof that the numerical algorithm converges to uniquely determined optimal switching points. We propose an extended perturbation analysis. Let v denote a strategy that satisfies the necessary conditions (6) for optimal switching. For some given $\delta > 0$ let $w : [0, X] \times (-\delta, \delta) \to [0, \infty)$ denote the unique perturbed speed profile satisfying the equation

$$w \frac{\partial w}{\partial x} = \frac{P}{w} - r(w) + g \tag{10}$$

with $w(a, h) = V + h$ (see Fig. 3).

By differentiating with respect to h and rearranging the

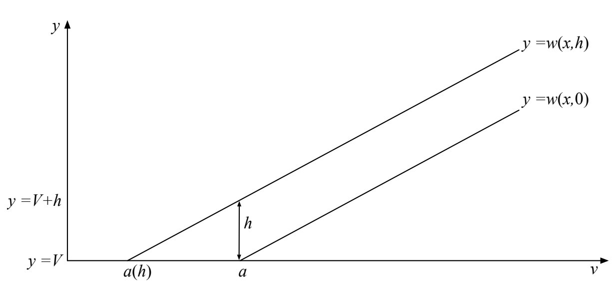

Fig. 3. The perturbed speed profile near a for $h > 0$.

order of differentiation on the left-hand side we have

$$\frac{\partial}{\partial x}\left[w \frac{\partial w}{\partial h}\right] = (-1)\frac{P + \psi(w)}{w^3}\left[w \frac{\partial w}{\partial h}\right] \tag{11}$$

and since $\frac{\partial w}{\partial h}(a, h) = 1$ it follows by separation of variables and direct integration that

$$\frac{\partial w}{\partial h} = \frac{V + h}{w} \exp\left[(-1)\int_a^x \frac{P + \psi(w(\xi))}{w(\xi)^3}\, d\xi\right] \tag{12}$$

and hence, by direct differentiation with respect to h, that

$$\frac{\partial^2 w}{\partial h^2} = \left\{\frac{1}{V + h} - \frac{1}{w}\frac{\partial w}{\partial h} - \int_a^x \frac{d}{dv}\left[\frac{P + \psi(v)}{v^3}\right]\bigg|_{v = w(\xi)} \frac{\partial w}{\partial h}(\xi)\, d\xi\right\}\frac{\partial w}{\partial h} \tag{13}$$

for all $x \in [0, X]$. There exists a point $a(h)$ near $a(0) = a$ and a point $d(h)$ near $d(0) = d$ where $w(a(h), h) = w(d(h), h) = V$. If $v_0(x) = w(x, 0)$ is optimal and we write $\delta^n v_0(x) = \frac{\partial^n w}{\partial h^n}(x, 0)$ then

$$\delta v_0 = \frac{V}{v_0} \exp\left[(-1)\int_a^x \frac{P + \psi(v_0(\xi))}{v_0(\xi)^3}\, d\xi\right] \tag{14}$$

and

$$\delta^2 v_0 = \left\{\frac{1}{V} - \frac{\delta v_0}{v_0} - \int_a^x \frac{d}{dv}\left[\frac{P + \psi(v)}{v^3}\right]_{v = v_0(\xi)} \delta v_0(\xi)\, d\xi\right\}\delta v_0 \tag{15}$$

and we can write

$$w = v_0 + \delta v_0\, h + \delta^2 v_0 \frac{h^2}{2!} + O(h^3). \tag{16}$$

In this notation $\delta^n v_0$ is the n^{th} order infinitesimal perturbation.

B. Perturbation of the local energy functional

We can regard $J_\ell(w)$ as a function of h and write

$$J_\ell(h) = \int_{a(h)}^{d(h)} [E(w) - E(V)]\, dx \tag{17}$$

where $a(h) < b < c < d(h)$ are chosen so that $w(a(h), h) = w(d(h), h) = V$. Note that $a(0) = a$ and $d(0) = d$. If we differentiate with respect to h and then set $h = 0$ we obtain

$$J_\ell'(0) = \int_a^d E'(v_0)\delta v_0\, dx = 0 \tag{18}$$

and

$$J_\ell''(0) = \int_a^d \left[E''(v_0){\delta v_0}^2 + E'(v_0)\delta^2 v_0\right] dx. \tag{19}$$

Since

$$J_\ell(h) = J_\ell(0) + J_\ell'(0)h + \frac{J_\ell''(0)}{2!}h^2 + O(h^3) \tag{20}$$

it follows that the condition $J_\ell''(0) > 0$ would be sufficient to ensure that $J_\ell(h)$ reaches a minimum at $h = 0$. If all turning points for $J_\ell(v)$ are minimum turning points then there can only be one turning point on each steep section. Thus the optimal switching points would be uniquely defined. If this were true on every steep section then the overall optimal strategy would be unique. Although this condition appears to be true in practice where a quadratic resistance $r(v) = r_0 + r_1 v + r_2 v^2$ is used, a direct proof that $J_\ell''(0) > 0$ is not yet available.

C. The adjusted adjoint equation

The necessary conditions for an optimal strategy have been found by applying the Pontryagin principle. The analysis is outlined in Howlett *et al* [13] where the adjusted adjoint variable $\eta : [0, X] \to (-\infty, \infty)$ for a segment of maximum power is defined as the solution to the differential equation

$$\eta' - \frac{P + \psi(v)}{v^3}\,\eta = \frac{\psi(v) - \psi(V)}{v^3} \tag{21}$$

with $\eta(a) = 0$. In this equation v is the corresponding speed profile. In the case of an optimal profile Howlett *et al* [13] note that $\eta(a) = \eta(d) = 0$. Of course $v(a) = v(d) = V$ and so it is also clear that $\frac{d\eta}{dx}(a) = \frac{d\eta}{dx}(d) = 0$. In fact, for an optimal strategy, Howlett [10] has shown that $x = a$ and $x = d$ are minimum turning points for the solution η.

D. A formula for the perturbed adjusted adjoint variable

The perturbed adjoint variable $\zeta : [0, X] \times (-\delta, \delta) \to (-\infty, \infty)$ is defined as the solution to the differential equation

$$\frac{\partial \zeta}{\partial x} - \frac{P + \psi(w)}{w^3}\,\zeta = \frac{\psi(w) - \psi(V)}{w^3} \tag{22}$$

where w is the perturbed speed and $\zeta(a(h), h) = 0$. It follows from (14) that $w\frac{\partial w}{\partial h}$ is an integrating factor for (22) and hence that

$$\frac{d}{dx}\left[w\frac{\partial w}{\partial h}\zeta\right] = E'(w)\frac{\partial w}{\partial h}. \tag{23}$$

Therefore

$$\begin{aligned}\left[w\frac{\partial w}{\partial h}\,\zeta\right]\bigg|_{x=d(h)} &= \int_{a(h)}^{d(h)} E'(w)\frac{\partial w}{\partial h}\,d\xi \\ &= J_\ell'(h).\end{aligned} \tag{24}$$

If we could show that $\zeta(d(h), h) < 0$ when $h < 0$ and that $\zeta(d(h), h) > 0$ when $h > 0$, then the sign pattern for $J_\ell'(h)$ would show that $J_\ell(h)$ reaches a minimum at $h = 0$.

Figures 4 and 5 show corresponding perturbed profiles for speed w and adjoint ζ on a steep uphill section with a typical freight train. The calculations were made using Freightmiser code. On the lower perturbed speed profile where the *power* phase starts too late, the control switches from *power* to *coast* when the adjoint variable changes from positive to negative. On the upper perturbed speed profile where the *power* starts too early the control remains set to *power* because the adjoint variable remains positive. For the optimal speed profile the adjoint variable returns to zero at precisely the same point as the speed returns to the desired holding speed.

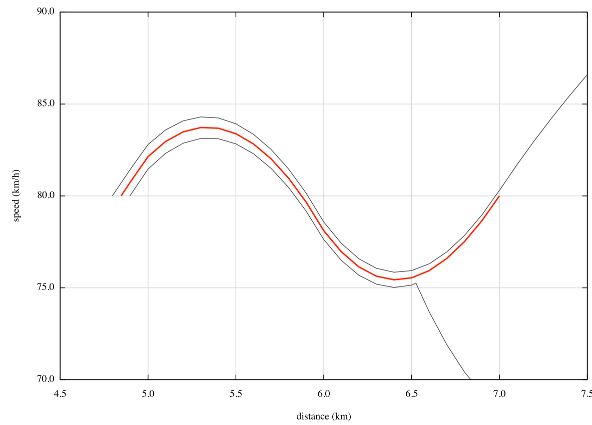

Fig. 4. Perturbed speed profiles on a steep uphill section.

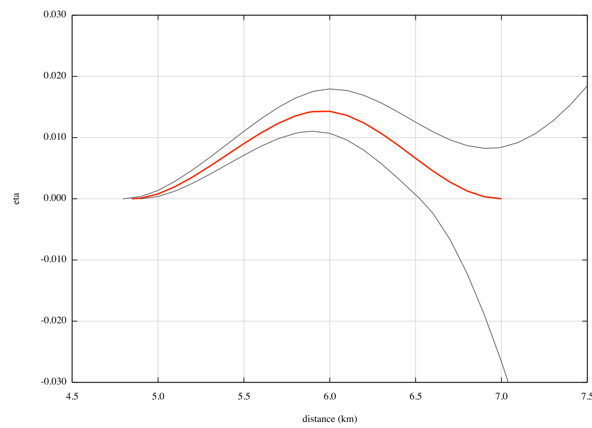

Fig. 5. Perturbed adjoint profiles on a steep uphill section.

E. A formula for the perturbed adjusted adjoint variable when the gradient is continuously differentiable

If g is continuously differentiable then

$$\begin{aligned}\frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial x}\zeta\right] &= \frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial x}\right]\zeta + \left[w\frac{\partial w}{\partial x}\right]\frac{\partial \zeta}{\partial x} \\ &= \zeta\, g' + E'(w)\frac{\partial w}{\partial x}\end{aligned}$$

and hence if we integrate from $x = a(h)$ to $x = d(h)$ then we obtain

$$\left[w\frac{\partial w}{\partial x}\zeta\right]\bigg|_{x=d(h)} = \int_{x=a(h)}^{d(h)} \zeta\, dg. \tag{25}$$

It follows that the integral on the right-hand side of (25) is positive if $\zeta(d(h), h)$ is positive, and is negative if $\zeta(d(h), h)$ is negative. An earlier equation (24) showed that $J_\ell(h)$ has a minimum turning point at $h = 0$ if this condition is true.

F. A formula for the perturbed adjusted adjoint variable when the gradient is piecewise constant

Consider a journey from $x = 0$ to $x = X$ on a track with piecewise constant gradient. Let $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = X$ and suppose the track has gradient γ_j on the interval (x_j, x_{j+1}) for each $j = 0, 1, \ldots, n$. Suppose there is a steep uphill section of track (x_r, x_s) and that the optimal strategy is to start power at some point $a_0 \in (x_{r-1}, x_r)$ and to finish at some point $d_0 \in (x_s, x_{s+1})$. Consider the perturbed speed profile w for sufficiently small values of the perturbation parameter h so that $a(h) \in (x_{r-1}, x_r)$ and $d(h) \in (x_s, x_{s+1})$. The equation of motion takes the form

$$w\frac{\partial w}{\partial x} = \frac{P}{w} - r(w) + \gamma_j$$

on the interval (x_j, x_{j+1}) from which it follows by differentiating both sides that

$$\frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial x}\right] = (-1)\frac{P + \psi(w)}{w^3}\left[w\frac{\partial w}{\partial x}\right]$$

and, if we divide through by $w\frac{\partial w}{\partial x}$, that

$$\frac{\partial}{\partial x}\left[\log_e\left(w\frac{\partial w}{\partial x}\right)\right] = (-1)\frac{P + \psi(w)}{w^3}.$$

Therefore, on the interval (x_j, x_{j+1}), the function $w\frac{\partial w}{\partial x}$ is an integrating factor for the perturbed adjoint equation. Hence, if $[p, q] \subset (x_j, x_{j+1})$, then

$$\begin{aligned} w\frac{\partial w}{\partial x}\zeta\Big|_p^q &= \int_p^q E'(w)\frac{\partial w}{\partial x}\, dx \\ &= E(w(q)) - E(w(p)). \end{aligned} \qquad (26)$$

The difficulty with an integration over the entire interval $(a(h), d(h))$ is that $w\frac{\partial w}{\partial x}$ is not continuous across gradient changes. Thus the integration must be done separately over each interval of constant gradient to give

$$\left(\frac{P}{w_r} - r(w_r) + \gamma_{r-1}\right)\zeta_r = E(w_r) - E(V),$$

for the first interval $(a(h), x_r)$,

$$\left(\frac{P}{w_{j+1}} - r(w_{j+1}) + \gamma_j\right)\zeta_{j+1} - \left(\frac{P}{w_j} - r(w_j) + \gamma_j\right)\zeta_j = E(w_{j+1}) - E(w_j)$$

for each interval (x_j, x_{j+1}) from $j = r$ to $j = s - 1$, and finally

$$\left(\frac{P}{V} - r(V) + \gamma_s\right)\zeta(d(h), h) - \left(\frac{P}{w_s} - r(w_s) + \gamma_s\right)\zeta_s = E(V) - E(w_s)$$

for the final interval $(x_s, d(h))$ where we have written $w_j = w(x_j, h)$ and $\zeta_j = \zeta(x_j, h)$ for convenience. By adding these terms and simplifying we obtain

$$\left(\frac{P}{V} - r(V) + \gamma_s\right)\zeta(d(h), h) - \sum_{j=r}^{s}\zeta_j(\gamma_j - \gamma_{j-1}) = 0$$

which we can rewrite as

$$\left[w\frac{\partial w}{\partial x}\zeta\right]\Big|_{x=d(h)} = \sum_{j=r}^{s}\zeta_j(\gamma_j - \gamma_{j-1}). \qquad (27)$$

Note the similarity to the earlier formula (25) when the gradient is continuously differentiable. If we set $h = 0$ then $\zeta(x, 0) = \eta(x)$ and we write $\eta_j = \eta(x_j)$. Since $\eta(d) = 0$ we obtain

$$\sum_{j=r}^{s}\eta_j(\gamma_j - \gamma_{j-1}) = 0$$

which is a known necessary condition for optimality ([13]). Note that (26) can be rewritten in the form

$$\left(\frac{P}{w} - r(w) + \gamma_j\right)\zeta(q, h) = E(w) - E(V) - \mu_j$$

where we write $\zeta = \zeta(q, h)$ and $w = w(q, h)$ and regard $q \in (x_j, x_{j+1})$ as a variable and $p \in (x_j, x_{j+1})$ as a fixed point and where μ_j is defined by

$$E(V) + \mu_j(h) = \left(\frac{P}{w(p, h)} - r(w(p, h)) + \gamma_j\right)\zeta(p, h).$$

It follows that ζ is given by the formula

$$\zeta = \frac{E(w) - E(V) - \mu_j}{P/w - r(w) + \gamma_j} \qquad (28)$$

on the interval (x_j, x_{j+1}). The initial condition $\zeta(a(h), h) = 0$ means that $\mu_{r-1} = 0$ and the continuity of ζ at $x = x_j$ means that $\mu_j = \mu_{j-1} + (\gamma_{j-1} - \gamma_j)\zeta_j$ for all $j = r, \ldots, s-1$. When $h = 0$ we write $\eta(x) = \zeta(x, 0)$, $v(x) = w(x, 0)$ and $\lambda_j = \mu_j(0)$ and the formula becomes

$$\eta = \frac{E(v) - E(V) - \lambda_j}{P/v - r(v) + \gamma_j}$$

for $x \in (x_j, x_{j+1})$. Once again the initial condition $\eta(a) = 0$ means that $\lambda_{r-1} = 0$ and the continuity of η at $x = x_j$ means that $\lambda_j = \lambda_{j-1} + (\gamma_{j-1} - \gamma_j)\eta_j$ for all $j = r, \ldots, s-1$. Since $\eta(d) = 0$ for an optimal strategy it follows that $\lambda_s = 0$.

G. An alternative integration of the perturbed adjusted adjoint equation for piecewise constant gradient

Consider a track with piecewise constant gradient as described in the previous subsection. For $x \in (x_j, x_{j+1})$ we have

$$w\frac{\partial w}{\partial x} = \frac{P}{w} - r(w) + \gamma_j.$$

If we differentiate with respect to h and reverse the order of differentiation we obtain

$$\frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial h}\right] = (-1)\frac{P + \psi(w)}{w^3}\left[w\frac{\partial w}{\partial h}\right]$$

for $x \in (x_j, x_{j+1})$ whereas if we differentiate with respect to x we deduce that

$$\frac{\partial}{\partial x}\left[w\frac{\partial w}{\partial x}\right] = (-1)\frac{P + \psi(w)}{w^3}\left[w\frac{\partial w}{\partial x}\right].$$

Solution of these two equations shows us that

$$w\frac{\partial w}{\partial h} = C_j w\frac{\partial w}{\partial x}$$

for some constant C_j and $x \in (x_j, x_{j+1})$. We suppose the magnitude of the perturbation $\delta w = \frac{\partial w}{\partial h}$ on the interval $(a(h), x_r)$ is defined by $C_{r-1} = h$. In general we must choose C_j so that $w\delta w$ is continuous which means we require

$$C_{j-1}\left(P/w_j - r(w_j) + \gamma_{j-1}\right) = C_j\left(P/w_j - r(w_j) + \gamma_j\right)$$

and so

$$C_j = h\prod_{i=r}^{j}\frac{P/w_j - r(w_j) + \gamma_{j-1}}{P/w_j - \varphi(w_j) + \gamma_j}$$

for $x \in (x_j, x_{j+1})$ and each $j = r, r+1, \ldots, s$. It follows from (23) that

$$\left[w\delta w\zeta\right]\Big|_{x=d(h)} = \int_{a(h)}^{d(h)} E'(w)\frac{\partial w}{\partial h}dx$$

and hence that

$$\begin{aligned}\left[w\delta w\zeta\right]\Big|_{x=d(h)} &= C_{r-1}\int_{a(h)}^{x_r} E'(w)\frac{\partial w}{\partial x}dx \\ &+\sum_{j=r}^{s-1} C_j\int_{x_j}^{x_{j+1}} E'(w)\frac{\partial w}{\partial x}dx + C_s\int_{x_s}^{d(h)} E'(w)\frac{\partial w}{\partial x}dx\end{aligned}$$

which in turn we can now write as

$$\begin{aligned}\left[w\delta w\zeta\right]\Big|_{x=d(h)} &= C_{r-1}\left\{E(w_r) - E(V)\right\} \\ &+\sum_{j=r}^{s-1} C_j\left\{E(w_{j+1}) - E(w_j)\right\} + C_s\left\{E(V) - E(w_s)\right\}\end{aligned}$$

or equivalently as

$$\begin{aligned}&\left[w\delta w\zeta\right]\Big|_{x=d(h)} \\ &= \sum_{j=r}^{s}\left[C_{j-1} - C_j\right]E(w_j) + \left[C_{r-1} - C_s\right]E(V).\end{aligned}$$

When we set $h = 0$ this gives an alternative form for the optimality conditions.

III. Conclusions and future work

There are two important issues arising from our discussion of the new local energy minimization principle. In the first place, on-board calculations that continually update the recommended speed profile require very fast and very accurate algorithms. The performance of general adaptive Runge-Kutta numerical solution schemes for the relevant differential equations can be enhanced by using various analytic expressions throughout the calculation. For instance, the formula (28) allows us to evaluate the adjoint variable directly from knowledge of the speed. Without this formula it would be necessary to use a numerical solution scheme to solve the adjoint differential equation separately backwards with respect to position.

In the second place, the various formulae can be used to give insight into different ways in which the uniqueness of the solution could be proved. For example we showed that the sign of the perturbed adjusted adjoint variable at $x = d(h)$ is the same as the sign of the first derivative $J_\ell'(h)$ of the local energy functional. A sufficient condition for a minimum of the local energy functional would therefore be that $\zeta(d(h), h) < 0$ when $h < 0$ and $\zeta(d(h), h) > 0$ when $h > 0$. Our future work will look at finding suitable sufficient conditions to enable a direct proof that the algorithms converge to a unique solution.

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