2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Distance-based Formation Control Using Euclidean Distance Dynamics Matrix: General Cases

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***Abstract*— We propose a formation control law based on inter-agent distances for a general group of single-integrator modeled agents on the plane. By attempting to directly control the Euclidean distance matrix of the group, we derive the proposed control law from the time-derivative of the matrix. Accordingly, if the initial and desired formations of the group are generically rigid, then the desired formation of the group is locally asymptotically stable. The stability analysis, in which Lyapunov direct method is applied to the distance dynamics of the group, is straightforward. Simulation results demonstrates comparable effectiveness of the control law to an existing law.**

I. INTRODUCTION

Distance-based formation control, in which agents are stabilized based on inter-agent distance information without any available common directional sense, have recently attracted a significant amount of interest [1]–[6].

Stability analysis in the realm of distance-based formation control is, due to the absence of an available common directional sense for agents, complicate in general. Such complicacies principally arise from the non-compactness of the equilibrium set and the presence of the undesired equilibrium subset. To counter such complicacies, researchers have utilized several schemes. For instance, Krick *et al.* have applied center manifold theory to demonstrate the local asymptotic stability of the desired formation of a general agent group [1]. In [2], Dörfler and Francis have introduced link dynamics, which is similar to edge dynamics exploited in [4], [5], and proposed a differential geometric approach for stability analysis of the desired formation of agents. Cao *et al.* have applied Lyapunov's direct method to the edge dynamics of three agent groups, proving globally exponential stability of the desired triangular formation of three agents with a directed information graph [4], [5].

In the meanwhile, though most existing distance-based formation control laws have been designed by the gradient of artificial potential functions, Oh and Ahn have attempted to achieve the desired formation of a three-agent group by the direct control of the Euclidean distance dynamics of associated the realization of the group in [6].

In this paper, attempting to extend the results in [6] to a general group of agents, we propose a formation control law for a group of single-integrator modeled agents on the plane, based on the direct control of inter-agent distances. Accordingly, contributions of this paper are as follows. First, we consider the direct control of the Euclidean distance matrix of a general group of agents. Then, the proposed control law is derived from the time-derivative of the matrix, which is a new strategy to design a formation control law. Second, by utilizing distance dynamics, we can overcome complicacies arising from the non-compactness of the equilibrium sets, similar to [2], [4], [5]. Consequently, the stability analysis presented in this paper is relatively straightforward. Finally, we provide a partial result on the equilibrium sets by exploiting the properties of gradient systems; that is, a neighborhood exists such that any point in the desired equilibrium set is isolated from the undesired equilibrium set in the neighborhood.

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The outline of this paper is as follows. The mathematical background, problem formulation, and the control strategy are provided in Section II. In Section III, the proposed control law is derived. Local asymptotic stability of the group desired formation by the control law is analyzed in Section IV, and simulation results are presented in Section V. Conclusion is then discussed in Section VI.

II. PRELIMINARIES

The mathematical background, problem formulation, and the control strategy in this paper are presented in this section. Details on graph rigidity and Euclidean distance matrices are found in [7]–[9] and [10]–[12], respectively.

A. Mathematical background

For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ and $\mathcal{E}$ are sets of N vertices and e edges, respectively, a realization of $\mathcal{G}$ in $\mathbb{R}^n$ is a function p that maps the vertices of $\mathcal{G}$ to points in $\mathbb{R}^n$. The realization p is represented by a stacked vector $[p_1^T \dots p_N^T]^T \in \mathbb{R}^{nN}$. The pair $(\mathcal{G}, p)$ is a framework of $\mathcal{G}$ in $\mathbb{R}^n$. Ordering the e edges of $\mathcal{G}$ in some way, an edge function $f_\mathcal{G} : \mathbb{R}^{n\times N} \to \mathbb{R}^e$ of the framework $(\mathcal{G}, p)$ is defined by $f_\mathcal{G}(p_1, \dots, p_N) = [\dots \|p_i - p_j\|^2 \dots]^T$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ for all $(i, j) \in \mathcal{E}$. Then, two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are equivalent if $f_\mathcal{G}(p) = f_\mathcal{G}(q)$; $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are congruent if $\|p(i) - p(j)\| = \|q(i) - q(j)\|$ for all $i, j \in \mathcal{V}$. Two realizations p and q are congruent if $\|p(i) - p(j)\| = \|q(i) - q(j)\|$ for all $i, j \in \mathcal{V}$. The framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^n$ if a neighborhood U of p in $\mathbb{R}^{nN}$ exits such that ${f_\mathcal{K}}^{-1}(f_\mathcal{K}(p)) \cap U = {f_\mathcal{G}}^{-1}(f_\mathcal{G}(p)) \cap U$, where $\mathcal{K}$ is the complete graph with N vertices. The framework $(\mathcal{G}, p)$ is globally rigid in $\mathbb{R}^n$ if ${f_\mathcal{K}}^{-1}(f_\mathcal{K}(p)) = {f_\mathcal{G}}^{-1}(f_\mathcal{G}(p))$, and a framework $(\mathcal{G}, p)$ is generic if $\mathrm{rank}[\nabla f_\mathcal{G}(p)] = \max\{\mathrm{rank}[\nabla f_\mathcal{G}(q)] : q \in \mathbb{R}^{nN}\}$.

Here, the rigidity matrix $R_{\mathcal{G}}(p)$ of $(\mathcal{G}, p)$ is defined by $1/2\nabla f_{\mathcal{G}}(p)$.

A matrix $D = [d_{ij}] \in \mathbb{R}^{N\times N}$ is called a Euclidean distance matrix (EDM) if and only if there exist points $p_1, \ldots, p_N$ in $\mathbb{R}^n$ such that $d_{ij} \triangleq \|p_i - p_j\|^2$. Note that EDMs are symmetric matrices with zero-diagonal and non-negative off-diagonal elements. Triangular inequality holds for elements of EDMs such that $\sqrt{d_{ij}} \leq \sqrt{d_{ik}} + \sqrt{d_{kj}}$ for all distinct i, j and k in $\{1, \ldots, N\}$. The dimension n of the affine span of points $p_1, \ldots, p_N$ that generate an EDM D is the embedding dimension of D, and $p = [p_1^T \ldots p_N^T]^T \in \mathbb{R}^{nN}$ is a realization of D. If $p \in \mathbb{R}^{nN}$ is a realization of D, then any matrix $[(Qp_1 + b)^T \ldots (Qp_N + b)^T]^T \in \mathbb{R}^{nN}$ is also a realization of D, where $Q \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ denote an orthogonal matrix and a vector, respectively.

B. Problem Formulation

For a group of N mobile agents, we model the group by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of agents and connectivity among agents, respectively. We refer to the graph $\mathcal{G}$ as the information graph of the group. We assume that every agent is modeled by a single-integrator,

$$\dot{p}_i = u_i,\ i \in \mathcal{V}, \tag{1}$$

where $p_i \in \mathbb{R}^2$ and $u_i \in \mathbb{R}^2$ denote the position and control inputs, respectively, of agent i on the plane. Each agent i has the following measurements:

$$d_{ji} = \|p_j - p_i\|^2,\ j \in \mathcal{N}_i,\ i \in \mathcal{V}, \tag{2}$$

where $\mathcal{N}_i$ is the set of neighbors of agent i. Note, however, that though we assume that every agent is capable of sensing relative displacements to all neighbors, it is recognized that agents have nonidentical local coordinate systems due to the absence of an available common directional sense for agents.

For a given realization $p^* = [p_i^{*T} \ldots p_N^{*T}]^T \in \mathbb{R}^{2N}$, the overall goal of the group is to form a realization that is congruent to the realization. Here, we refer to the set of all frameworks $(\mathcal{G}, p)$, where p is congruent to p^*, as the desired formation of the group:

$$F^* \equiv \{(\mathcal{G}, p) \mid \|p_i - p_j\| = \|p_i^* - p_j^*\|,\ \forall i, j \in \mathcal{V}\}. \tag{3}$$

Thus the overall group goal is the stabilization of p such that $(\mathcal{G}, p) \in F^*$. Note that the EDM of any realization p of $(\mathcal{G}, p) \in F^*$ is identical because EDMs are invariant under linear motions of its realizations. The subtask of agent i is the stabilization of p_i such that

$$d_{ij} = \|p_j^* - p_i^*\|^2,\ j \in \mathcal{N}_i\ i \in \mathcal{V}. \tag{4}$$

The consistency between the overall group goal and the subtasks is critical in formation control of mobile agents; the achievement of all subtasks has to lead to the success of the overall goal. The rigidity of information graphs is related to such consistency since rigidity guarantees the congruency of equivalent realizations. In other words, if information graphs are locally (globally) rigid, then the desired formation (3) is locally (globally) achievable by attaining all subtasks (4). Thus, we assume that any framework $(\mathcal{G}, p) \in F^*$ is rigid.

The formation control problem to be addressed in this paper then can be formulated as follows:

Problem 2.1: For a group of N ($N \geq 3$) agents with an information graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, assume that every agent, modeled by a single-integrator (1), has the measurements (2). Then, assuming that subtasks (4) are assigned to all agents for the desired formation (3), which is a set of rigid frameworks, design a control law of every agent to achieve the assigned subtasks.

In general, formation control consists of dividing the overall goal into subtasks, assigning the subtasks to agents, and the stabilization of agents achieving the subtasks. In this paper, we focus on control law design under the assumption that subtasks are assigned.

C. Control Strategy

For brevity, we refer to the EDM associated with the realization of a group as the EDM of the group. In addition, the EDM of the initial or desired formation denotes the EDM associated with the realization of the initial or desired formation, respectively. The EDM of the group denotes the EDM associated with the realization of the current formation.

For the EDM D of a group of agents under the assumptions of *Problem 2.1*, the time-derivative of D is obtainable element-wisely if the agent positions are differentiable by time:

$$\begin{aligned}\dot{d}_{ij} =& \frac{\partial d_{ij}}{\partial t}\frac{\partial p_i}{\partial t} + \frac{\partial d_{ij}}{\partial t}\frac{\partial p_j}{\partial t} \\ =& 2(p_i - p_j)^T(u_i - u_j),\ \forall (i,j) \in \mathcal{E}.\end{aligned} \tag{5}$$

We refer to (5) as the overall distance dynamics of the group.

Since $\dot{d}_{ij}$ is a function of the position and control inputs of agents i and j, we attempt to achieve the desired formation via direct EDM control. However, it is generally not possible to independently control every element of the EDM while maintaining the embeddability of the EDM in $\mathbb{R}^2$ [12] because arbitrary adjustment of elements causes the EDM to be unrealizable in $\mathbb{R}^2$. Thus, we define Euclidean distance dynamics matrices as *Definition 2.1* [6].

Definition 2.1 (Euclidean Distance Dynamics Matrix [6]): For an $N \times N$ EDM D^0 embeddable in n-dimensional Euclidean space, a function $S : [0, \infty) \to \mathbb{R}^{N\times N}$ is a Euclidean distance dynamics matrix (EDDM) of D^0 if and only if $D : [0, \infty) \to \mathbb{R}^{N\times N}$ that is defined as

$$D(t) \triangleq D^0 + \int_0^t S(\tau)d\tau,$$

where integration of S is defined as element-wise integration, is an EDM embeddable in n-dimensional Euclidean space for all $t \geq 0$.

Hence, the control law design consists of the design of an EDDM and the derivation of a control law from the EDDM.

III. Control Law Design via EDDM

According to [6], for a group of three agents under the assumptions of *Problem 2.1*, an EDDM of the group can be designed as

$$S(t) = -k_s(D(t) - D^*),\ k_s > 0, \tag{6}$$

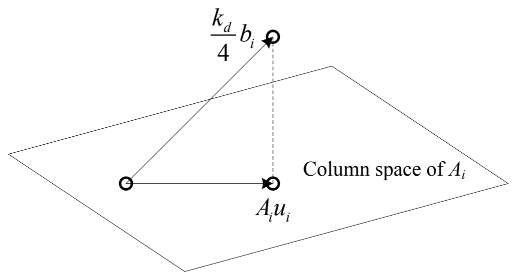


Fig. 1: Projection of $\frac{k_d}{4}b_i$ onto the column space of A_i.

where D^0 and D^* are the EDMs of the initial and desired formations of the group, respectively. Moreover, the control law of agent i can be derived from the constraints

$$(p_j - p_i)^T u_i = \frac{k_s}{4}\tilde{d}_{ij}, \tag{7}$$

$$(p_i - p_j)^T u_j = \frac{k_s}{4}\tilde{d}_{ij}, \tag{8}$$

where $\tilde{d}_{ij} = d_{ij} - d^*_{ij}$, for all $(i,j) \in \mathcal{E}$. Note that (7) and (8) are derived from (5) and (6). Consequently, since the constraints for the control input of agent i is a system of two linear equations, the control input u_i can be designed as

$$u_i = \frac{k_s}{4}A_i^{-1}b_i,\ \forall i \in \mathcal{V}, \tag{9}$$

where

$$A_i = \begin{bmatrix} (p_j - p_i)^T \\ (p_k - p_i)^T \end{bmatrix}, \tag{10}$$

$b_i = [\tilde{d}_{ji}\,\tilde{d}_{ki}]^T$, $j = i + 1(\text{mod } 3)$ and $k = i + 2(\text{mod } 3)$, under the condition that A_i is nonsingular.

Although S in (6) is an EDDM of a three-agent group, it cannot be considered an EDDM of N ($N > 3$) agent group in general because the convex combination of two EDMs embeddable in n-dimensional space is embeddable in at most $2n$-dimensional space by the following *Theorem 3.1*.

Theorem 3.1: [11] The set of all $N \times N$ EDMs is convex in $N \times N$ zero-diagonal matrix space. Furthermore, the convex combination of two $N \times N$ EDMs embeddable in k- and l-dimensional Euclidean spaces, respectively, is embeddable in at most $(k+l)$-dimensional Euclidean space.

According to *Theorem 3.1*, if D^1 and D^2 are $N \times N$ EDMs embeddable in $\mathbb{R}^2$, then the convex combination $D = \alpha D^1 + (1-\alpha)D^2 \in \mathbb{R}^{N\times N}$ where $0 \le \alpha \le 1$ is also an EDM embeddable in at most $\mathbb{R}^4$. As a special case of the theorem, if $N = 3$, then the convex combination $D = \alpha D^1 + (1-\alpha)D^2 \in \mathbb{R}^{3\times 3}$ is an EDM embeddable in $\mathbb{R}^2$ because every realization of three points is definitely embeddable in $\mathbb{R}^2$. However, if $N > 3$, then such an argument cannot be applicable. For example, Fig. 1 shows that agent i with three neighbors has relative-distance constraints that are incompatible in $\mathbb{R}^2$. Such situations inevitably occur if an agent has more than three neighbors. To resolve such situations and design an EDDM for a general group of agents, we find a solution u_i such that $A_i u_i$ becomes the projection of $(k_d/4)b_i$ onto the column space of A_i as depicted in the figure.

We now design a control law for a group of N-agents under the assumptions of *Problem 2.1*. From (7) and (8), the constraints for the control input of agent i lead to a possibly over-determined system of linear equations,

$$A_i u_i = \frac{k_s}{4}b_i,\ i \in \mathcal{V}, \tag{11}$$

where

$$A_i = \begin{bmatrix} (p_{i_1} - p_i)^T \\ \cdots \\ (p_{i_{|\mathcal{N}_i|}} - p_i)^T \end{bmatrix}, \tag{12}$$

$$b_i = \begin{bmatrix} \tilde{d}_{i_1 i} \\ \cdots \\ \tilde{d}_{i_{|\mathcal{N}_i|} i} \end{bmatrix},$$

and $i_1 \cdots i_{|\mathcal{N}_i|} \in \mathcal{N}_i$.

Since the projection of $\frac{k_d}{4}b_i$ onto the column space of A_i is equivalent to solving $\text{argmin}_{u_i}\|A_i u_i - (k_s/4)b_i\|$, the control input of agent i can be derived from (11) as

$$u_i = \frac{k_s}{4}(A_i^T A_i)^{-1}A_i^T b_i,\ \forall i \in \mathcal{V}, \tag{13}$$

where $(A_i^T A_i)^{-1}$ is the pseudo-inverse of A_i in (11), if $A_i^T A_i$ is nonsingular. Notice that a $[\dot{d}_{ij}] \in \mathbb{R}^{n\times n}$ is an EDDM of the group, if u_i and u_j are designed as (13).

IV. Stability Analysis

First, let us consider the properties of $A_i^T A_i$, which are arranged as,

$$\begin{aligned} A_i^T A_i &= \sum_{j\in\mathcal{N}_i}(p_j - p_i)(p_j - p_i)^T \\ &= \sum_{j\in\mathcal{N}_i}\begin{bmatrix} (x_j - x_i)^2 & (x_j - x_i)(y_j - y_i) \\ (x_j - x_i)(y_j - y_i) & (y_j - y_i)^2 \end{bmatrix}, \end{aligned} \tag{14}$$

where $p_i = (x_i, y_i)^T \in \mathbb{R}^2$ and $p_j = (x_j, y_j)^T \in \mathbb{R}^2$. The matrix A_i then has the following properties.

Lemma 4.1: For a group of N agents under the assumptions of *Problem 2.1*, if the formation of the group is generic, then $A_i^T A_i$ in (14) and its inverse matrix are positive-definite for all $i \in \mathcal{V}$.

Proof: The first leading principal minor of $A_i^T A_i$ is nonnegative:

$$\sum_{k\in\mathcal{N}_i}(x_k - x_i)^2 \ge 0,\ \forall i \in \mathcal{V}. \tag{15}$$

The equality in (15) holds if and only if agent i and all its neighbors are collocated. Moreover, the second leading principal minor of $A_i^T A_i$ is nonnegative by the Cauchy-Schwarz inequality:

$$\sum_{k\in\mathcal{N}_i}(x_k - x_i)^2 \sum_{k\in\mathcal{N}_i}(y_k - y_i)^2 - \left(\sum_{k\in\mathcal{N}_i}(x_k - x_i)(y_k - y_i)\right)^2 \ge 0,\ \forall i \in \mathcal{V}. \tag{16}$$

The equality in (16) holds if and only if $[x_{i_1} - x_i \dots x_{i_{|\mathcal{N}_i|}} - x_i]^T$ and $[y_{i_1} - y_i \dots y_{i_{|\mathcal{N}_i|}} - y_i]^T$, where $i_1, \dots, i_{|\mathcal{N}_i|} \in \mathcal{N}_i$, are linearly dependent. Thus, all leading principal minors of $A_i^T A_i$ are positive if the formation of the group is generic. It follows from Sylvester's criterion [13] that $A_i^T A_i$ is positive definite. Furthermore, the matrix $(A_i^T A_i)^{-1}$ is also positive definite by the positive definiteness of $A_i^T A_i$. ■

While the desired equilibrium set of the group are given by

$$E_d = \{p \in \mathbb{R}^{2N} \mid \|p_i - p_j\| = \|p_i^* - p_j^*\|, \forall (i,j) \in \mathcal{E}\}, \tag{17}$$

the equilibrium set E of the group with the control law (13) is given by

$$E = \{p \in \mathbb{R}^{2N} \mid (A_i^T A_i)^{-1} A_i^T b_i = 0, \forall i \in \mathcal{V}\}. \tag{18}$$

Clearly, any point $p \in E_d$ is also an element of E since if $\|p_i - p_j\| = \|p_i^* - p_j^*\|$ for all $(i,j) \in \mathcal{E}$, then $b_i = 0$ for all $i \in \mathcal{V}$, which means $E_d \subset E$. In order to employ Lyapunov's direct method for the stability analysis of the desired formation of the group, we need to investigate if the desired equilibrium set is isolated from undesired equilibrium points. To analyze the equilibrium set, let us first consider the following Lojasiewicz's inequality [14].

Theorem 4.1 (Lojasiewicz's Inequality [14]): Let f be a real analytic function on a neighborhood of z in $\mathbb{R}^n$. Then, there exist constants $c > 0$ and $\rho \in [0, 1)$ such that

$$\|\nabla f(x)\| \geq c\|f(x) - f(z)\|^\rho \tag{19}$$

in some neighborhood of z.

Then, we have the following *Lemma 4.2*.

Lemma 4.2: For a group of N agents under the assumptions of *Problem 2.1*, if the desired formation of the group is generically rigid, then there exists a neighborhood U of any point in E_d such that $[b_1^T A_1 \dots b_N^T A_N]^T \neq \mathbf{0}$ for all $[\dots \tilde{d}_{ij} \dots]^T \neq \mathbf{0}$ in U.

Proof: Suppose that a formation control law for the group is given by

$$\dot{p} = [b_1^T A_1 \cdots b_N^T A_N]^T. \tag{20}$$

Define a potential function ϕ as

$$\phi(p) \triangleq \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\|^2 - \|p_i^* - p_j^*\|^2)^2. \tag{21}$$

The dynamics in (20) then can be expressed as

$$\dot{p} = -\nabla\phi(p), \tag{22}$$

and the desired equilibrium set in (17) and the equilibrium set in (18) can be expressed as

$$E_d = \{p \in \mathbb{R}^{2N} \mid \phi(p) = 0\}, \tag{23}$$
$$E = \{p \in \mathbb{R}^{2N} \mid \nabla\phi(p) = 0\}. \tag{24}$$

Since ϕ in (21) is a real analytic function on a neighborhood of any point $\bar{p}$ in E_d, there exist constants $c > 0$ and $\rho \in [0, 1)$ such that

$$\begin{aligned} \|\nabla\phi(p)\| \geq& c\|\phi(p) - \phi(\bar{p})\|^\rho \\ =& c\|\phi(p)\|^\rho \end{aligned} \tag{25}$$

in a neighborhood U of $\bar{p}$ by Lojasiewicz's inequality, based on (23). Then, for any point p in U, if $p \notin E_d$, $\|\nabla\phi(p)\| > 0$. That is, $p \notin E$ for all $p \in U$, based on (24). It follows from (20) and (22) that $[b_1^T A_1 \cdots b_N^T A_N]^T \neq \mathbf{0}$ for all $[\cdots \tilde{d}_{ij} \cdots]^T \neq \mathbf{0}$ in U. ■

Consequently, the following *Theorem 4.2* presents the main result for a group of N agents.

Theorem 4.2 (Main Result): For a group of N agents under the assumptions of *Problem 2.1*, if the desired formation of the group is generically rigid, then the desired formation of the group is locally asymptotically stable by the control law (13).

Proof: First, the overall distance error dynamics of the group can be arranged as

$$\begin{aligned} \dot{\tilde{d}}_{ij} =& -\frac{k_s}{4}(p_j - p_i)^T (A_i^T A_i)^{-1} A_i^T b_i \\ & -\frac{k_s}{4}(p_i - p_j)^T (A_j^T A_j)^{-1} A_j^T b_j, \ (i,j) \in \mathcal{E}. \end{aligned}$$

We take a Lyapunov function candidate for the error dynamics as

$$V = \sum_{(i,j)\in\mathcal{E}} \frac{1}{4}\frac{k_s}{4}\tilde{d}_{ij}^2. \tag{26}$$

The derivative of V along the trajectories of the error dynamics then can be arranged as

$$\begin{aligned} \dot{V} =& \sum_{(i,j)\in\mathcal{E}} 2\tilde{d}_{ij}\dot{\tilde{d}}_{ij} \\ =& -\sum_{(i,j)\in\mathcal{E}} k_s\tilde{d}_{ij} \left[(p_j - p_i)^T (A_i^T A_i)^{-1} A_i^T b_i\right] \\ & -\sum_{(i,j)\in\mathcal{E}} k_s\tilde{d}_{ij} \left[(p_i - p_j)^T (A_j^T A_j)^{-1} A_j^T b_j\right] \\ =& -k_s \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} \tilde{d}_{ij}(p_j - p_i)^T (A_i^T A_i)^{-1} A_i^T b_i \\ =& -k_s \sum_{i\in\mathcal{V}} b_i^T A_i (A_i^T A_i)^{-1} A_i^T b_i \\ \leq& 0. \end{aligned}$$

Since the desired formation is generically rigid, $(A_i^T A_i)^{-1}$ is positive definite by *Lemma 4.1*. Moreover, there exists a neighborhood U of any point $\bar{p} \in E_d$ such that $[b_1^T A_1 \ \cdots \ b_N^T A_N]^T \neq \mathbf{0}$ for all $[\cdots \tilde{d}_{ij} \cdots]^T \neq \mathbf{0}$ in U by *Lemma 4.2*. Hence, $\dot{V}$ is negative definite if the desired formation is generically rigid and the initial formation is in U. Note that the control law (13) is well defined in U by *Lemma 4.1*. Thus, it follows from the rigidity of the information graph that the desired formation of the group is locally asymptotically stable. ■

Furthermore, the argument in the proof of *Theorem 4.2* leads to the following *Corollary 4.1*.

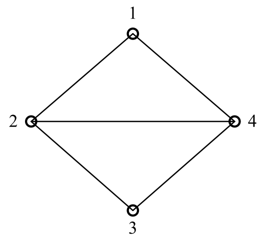


Fig. 2: The information graph of four-agent groups.

Corollary 4.1: For a group of N agents under the assumptions of *Problem 2.1*, if the desired formation of the group is generically rigid, then the desired formation of the group is locally asymptotically stable by the control law of the form

$$u_i = \frac{k_s}{4} P_i A_i^T b_i, \; \forall i \in \mathcal{V}, \tag{27}$$

where $P_i \in \mathbb{R}^{2\times 2}$ is positive definite.

Proof: For the overall distance error dynamics

$$\begin{aligned}\dot{\tilde{d}}_{ij} = &-\frac{k_s}{4}(p_j - p_i)^T P_i A_i^T b_i \\ &-\frac{k_s}{4}(p_i - p_j)^T P_i A_j^T b_j, \; (i,j) \in \mathcal{E}\end{aligned}$$

of the group, we take a Lyapunov function candidate as V in (26). Then, the derivative of V along the trajectories of the error dynamics can be arranged as $\dot{V} = k_s \sum_{i\in\mathcal{V}} b_i^T A_i P_i A_i^T b_i \leq 0$. By using the argument in the proof of *Theorem 4.2*, $\dot{V}$ is negative definite in a neighborhood U of any $\bar{p} \in E_d$. Therefore, the desired formation of the group is locally asymptotically stable. ■

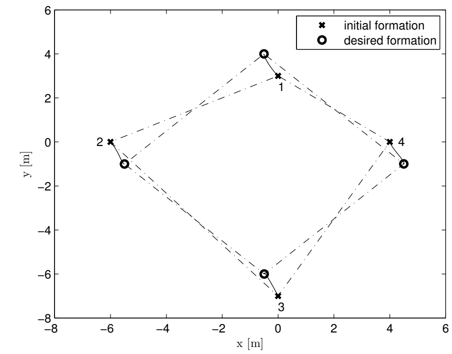
Although *Theorem 4.2* and *Corollary 27* confirm the local asymptotic stability of the desired formation, they does not imply the convergence of $p(t)$ to a fixed point. The proof of such a convergence property can be found in [18].

V. Simulation results

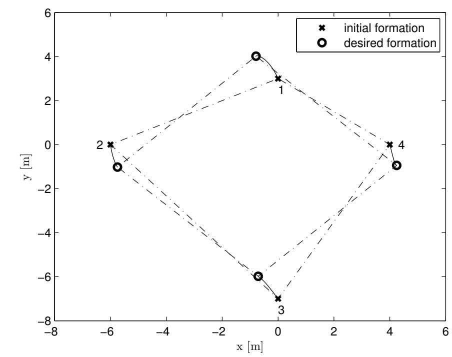
We present the simulation results of formation control for four and ten agents, comparing the proposed control law with the existing one proposed in [1]–[3] for agent groups having undirected information graphs. Since we address only undirected information graph cases, we do not compare with the control law in [4], [5].

For a group of four agents under the assumptions of *Problem 2.1*, the information graph of the group was given as Fig. 2. The initial and desired realization of the group were assumed to be $[(0,3)^T \; (-6,0)^T \; (0,-7)^T \; (4,0)^T]^T$ and $[(0,5)^T \; (-5,0)^T \; (0,-5)^T \; (5,0)^T]$, respectively. Fig. 3 and 4 depict trajectories and squared inter-agent distance errors by the existing and the proposed control laws, respectively. As opposed to three-agent case [6], the proposed control law did not demonstrate a coordination property. For instance, the distance between agents 2 and 4, though initially adjusted to the desired value, varied even by the proposed control law as depicted in Fig. 4(b).

Fig. 5 presents the information graph of a ten-agent group under the assumptions of *Problem 2.1*. To simulate the group,



(a) By an existing control law proposed in [1]–[3].



(b) By the proposed control law.

Fig. 3: The formation trajectories of four-agent groups.

a realization $10 \times [(\cos(2\pi/10), \sin(2\pi/10))^T \ldots (\cos(10 \times 2\pi/10), \sin(10 \times 2\pi/10))^T]^T$ was given as the desired realization, and the initial realization was assumed to be perturbed from the desired realization by a random variable that was uniformly distributed on $[-2.5, 2.5]$. As depicted in Fig. 6(a) and (b), the proposed control law demonstrated comparable performance to the existing control law.

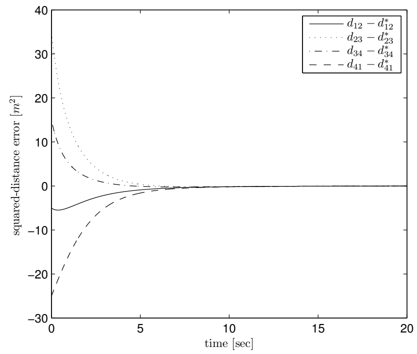
VI. Conclusion

In this paper, as an extension of the results presented in [6], we proposed a formation control law for a general group of agents based on the inter-agent distances. Introducing distance dynamics, we then presented the local asymptotic stability. The proposed control law showed comparable performances to an existing control law in [1].

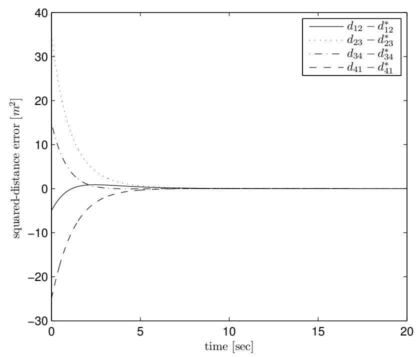
Though several effective displacement-based control laws have been previously proposed, e.g. in [15]–[17], there have yet to be satisfactory results on global stability for a general agent group in distance-based setup. Thus a critical future work is the global stability analysis of a general group.

VII. Acknowledgement

This research was supported by the National Research Foundation of Korea (NRF) (No. 2010−0015778).



(a) By an existing control law proposed in [1]–[3].



(b) By the proposed control law.

Fig. 4: The squared inter-distance errors of four-agent groups.

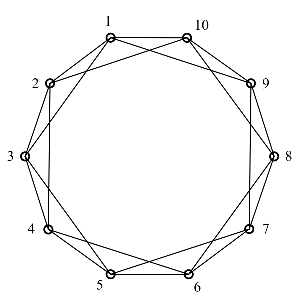
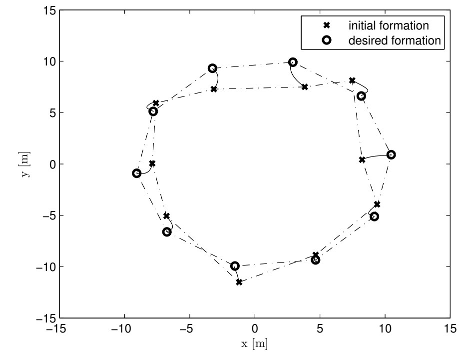
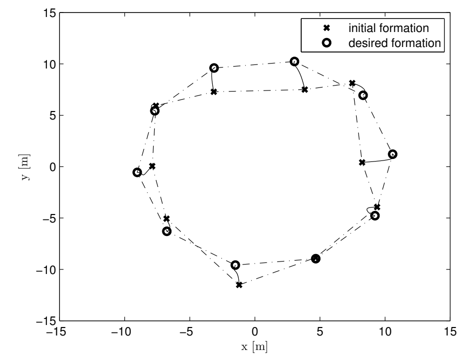


Fig. 5: The information graph of ten-agent groups.



(a) By an existing control law proposed in [1]–[3].



(b) By the proposed control law.

Fig. 6: The formation trajectories of ten-agent groups.

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