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Consensus Acceleration of Multi-agent Systems via Model Prediction

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***Abstract*— A fastest consensus problem of topology fixed networks has been formulated as an optimal linear iteration problem and efficiently solved by Xiao and Boyd [1]. Considering a kind of predictive mechanism, we show that the consensus evolution can be further accelerated while physically maintaining the network topology. The underlaying mechanism is that an effective prediction is able to convert the network status along temporal dimension to that in spatial dimension and hence induce a network with a virtually denser topology. With this topology, an even faster consensus is expected to occur. The result is motivated by the predictive mechanism widely existing in biological swarms, flocks, and synchronization networks.**

***Index Terms*— Multi-agent systems, synchronization, consensus, prediction**

I. Introduction

The collective motion of autonomous individuals is currently a subject of intensive research that has potential applications in biology, physics, and engineering. One of the most remarkable characteristics of collective behaviors such as flocks of birds, schools of fish, or swarms of locusts, is the emergence of an ordered collective behaviors of the whole group [2], [3], [4]. One essential problem is called consensus problem [5], [6], [7], where a group of self-propelled agents agree upon certain quantities of interest such as attitude, position, temperature, voltage, etc. The consensus theory of multi-agent systems has emerged as a challenging new area of research in recent years. This is mainly due to its broad applications in sensor network data fusion, load balancing, swarms/flocks, unmanned air vehicles (UAVs), attitude alignment of satellite clusters, congestion control of communication networks, multi-agent system (MAS) control, etc. [8], [9], [10].

Convergence rate or speed is an important performance index in the analysis of consensus problems. Among the early works on consensus problems, Tsitsiklis [11] proposed a decentralized method to eliminate the disagreement within the group and hence derived the conditions for asymptotic agreement of all agents' decisions. In [5], Olfati-Saber and Murray presented a theoretical framework for consensus problems that the second smallest interactive eigenvalue of the interaction graph Laplacian matrix, namely algebraic connectivity [12], quantifies the convergent rate to consensus. To improve the convergence rate towards consensus, they further proposed a method based on the addition of a few long links to a regular lattice, thus transforming it into a small-world network [13], [14]. In [15], consensus results were derived for several moving direction alignment models such as the famous Vicsek model [2]. Specifically, a weak joint-connectivity condition of the agents on some time intervals was proved to be sufficient for moving direction consensus. In [6], [7], the joint-connectivity condition guaranteeing consensus [5] was further relaxed into the existence of a rooted directed spanning tree over time. The most recent research includes that the existence of consensus behavior for a class of MASs was systematically addressed in [16], a finite-time consensus protocol based on Lyapunov method was given in [17] to improve the consensus speed, and a class of constrained consensus and optimization problem was studied in [18].

In summary, the aforementioned works focused on choosing proper interaction graphs possessing sufficiently strong algebraic connectivity to guarantee consensus, but not on finding optimal protocols to *maximize* the convergence rate for a fixed network topology. The convergence rate optimization problem was studied in [1], [19], [20], [21], etc. In particular, it was claimed by Xiao and Boyd [1] that, when the network topology is symmetric, the problem of finding the fastest converging linear iteration can be cast as a semidefinite programming problem, and thus can be efficiently and globally solved. Along this research line, we conjecture that Xiao and Boyd's fastest consensus evolution can be further accelerated while physically maintaining the network topology provided that a predictive mechanism is included. Specifically, an effective prediction is able to convert the network status along temporal dimension to that in spatial dimension, which corresponds to adding more virtual connections into the network and hence substantially intensify its algebraic connectivity. With this "denser links" topology, a faster consensus is expected to occur. The contribution of this observation is two-folds. First, it allows us to reveal the roles of prediction mechanisms during temporal evolution of flocking/swarming behaviors, which universally exist in natural bio-groups. Secondly, from an industrial application point of view, it may be applicable in some relevant prevailing engineering areas such as autonomous robot formations, sensor networks, and UAVs [8], [9], [10]. Each agent typically has limited communication energy or links, and thus a predictive mechanism can used to accelerate the emergence of coordinated motions without increasing the communication capacity.

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As a matter of fact, the prediction mechanism is well accepted in the biology literature that individuals typically possess some predictive intelligence allowing them to predict the future movements of their neighbors using the past observations [22], [23], [24], [25]. All these natural evidences have inspired and motivated engineers to extract the underlying mechanism of predictive intelligence to improve consensus performance in engineering design. For instance, a predictive mechanism was proposed by Ferrari-Trecate *et al.* [26], in which decentralized model predictive control schemes were designed by taking into account the constraints on the agents' input to guarantee consensus under mild assumptions. This paper provides another application instance of the prediction mechanism in the problem of fast consensus.

The rest of this paper is organized as follows. The main problem is formulated in Section II. The consensus analysis for networks with general multi-integrator dynamics, together with the predictive protocol, is given in Section III. Specifically, the consensus evolution can be significantly accelerated by minimizing the spectral radius through the predictive protocol. Numerical simulation is studied in Section IV to illustrate the effectiveness of the protocol proposed in this paper. Finally, some conclusions are drawn in Section V. In this paper, we let $I_a \in \mathbb{R}^{a\times a}$ (the subscript a may be omitted for notation concise when no confusion is caused) be an identity matrix and $\mathbf{1} \in \mathbb{R}^n$ a vector with all elements one. The symbol $\otimes$ is the Kronecker product operator.

II. Problem Description

The main objective of this paper is to find a predictive algorithm to improve the convergence speed towards consensus. The network to be studied here consists of multiple agents with an interconnected structure represented by a weighted directed graph (called *digraph* for short). In this representation of network, the n nodes of the digraph represent the n agents of the network and a weighted edge a_{ij} indicates the existence of a communication link from agent j to agent i in the network. Specifically, let $\xi_i \in \mathbb{R}$ be the state of the $i-$th agent, which is governed by an s-order dynamics as follows

$$\xi_i^{(s)} = -\sum_{j=1}^{n} a_{ij} p_s(\xi_i - \xi_j), \quad i = 1, \cdots, n, \quad s \geq 1 \qquad (1)$$

where $p_s(\xi) := \gamma_o \xi + \gamma_1 \dot{\xi} + \cdots + \gamma_{s-1}\xi^{(s-1)}$, $\gamma_i \in \mathbb{R}$. Without loss of generality, we let $\gamma_o = 1$ by assuming γ_o is absorbed by a_{ij}. For the network (1), the matrix $A \in \mathbb{R}^{n\times n}$ with a_{ij} being its (i,j)-entry is called the associated weighted *adjacency matrix*. This matrix has the following properties: (i) its entries are nonnegative, i.e., $a_{ij} \geq 0$, where $a_{ij} = 0$ means no communication link exists from j to i; (ii) there is no self-cycle, i.e., $a_{ii} = 0, \forall i = 1, \ldots, n$. With a state vector $\xi := [\xi_1, \cdots, \xi_n]^{\mathsf{T}}$, the system (1) can be put in a compact form of

$$\xi^{(s)} = -Lp_s(\xi), \; L = \{l_{ij}\} \in \mathbb{R}^{n\times n},$$
$$l_{ii} = \sum_{l=1}^{n} a_{il}, \quad l_{ij} = -a_{ij}, \forall i \neq j \qquad (2)$$

where L is called the *graph Laplacian matrix*. Obviously, L has an eigenvector $\mathbf{1}$ corresponding to an eigenvalue 0, i.e., $L\mathbf{1} = 0$.

In this paper, we will consider the system (2) in the discrete-time domain with a sampling period ϵ. Using the following Euler's approximation $\dot{\xi}(t) = [\xi(t+\epsilon) - \xi(t)]/\epsilon$, $\cdots$, $\xi^{(s)}(t) = [\xi^{(s-1)}(t+\epsilon) - \xi^{(s-1)}(t)]/\epsilon$, and denoting $x(k) := \xi(k\epsilon)$, we have

$$x(k+s) = f(z(k)), \quad z(0) = z_o$$
$$z(k) := [x^{\mathsf{T}}(k), x^{\mathsf{T}}(k+1), \cdots, x^{\mathsf{T}}(k+s-1)]^{\mathsf{T}} \in \mathbb{R}^{ns}, \qquad (3)$$

where

$$f(z(k)) := -\Phi_s^{\mathsf{T}} z(k) - L\sum_{i=0}^{s-1} \epsilon^{(s-i)}\gamma_i \Phi_i^{\mathsf{T}} z(k),$$
$$\Phi_i = \phi_i \otimes I, \; i = 1, \cdots, s. \qquad (4)$$

The vector $\phi_i \in \mathbb{R}^s$, $i = 1, \cdots, s$, is a constant vector representing the coefficients[1] of a polynomial of $(a-1)^i$, i.e.,

$$(a-1)^i \equiv \begin{bmatrix} 1 & a & \cdots & a^{s-1} \end{bmatrix} \phi_i, \; i = 0, \cdots, s-1,$$
$$(a-1)^s \equiv \begin{bmatrix} 1 & a & \cdots & a^{s-1} \end{bmatrix} \phi_s + a^s.$$

For a causal system (3), at the current instant $(k+s-1)\epsilon$, the next state $x(k+s)$ is determined by the past/current state $z(k)$. In this paper, we will investigate an extra control algorithm to improve the network performance in the following setting:

$$x(k+s) = f(z(k)) + v(k+s-1) \qquad (5)$$

where $v(k+s-1)$ is the additional input at the instant $(k+s-1)\epsilon$.

The design of $v(k+s-1)$ naturally depends on the past/current state $z(k)$. However, it is conjectured that the network performance can be further improved if a certain predictive information of the future evolution can be utilized. To this end, we predict the future states of the network which evolutes from the past/current state $z(k)$ through the nominal dynamics (3), i.e.,

$$\tilde{x}(k+s-1+i) = f([x(k-1+i), \cdots, x(k+s-1),$$
$$\tilde{x}(k+s), \cdots, \tilde{x}(k+s-2+i)), \; i = 1, \cdots, h$$

where h is the prediction horizon. It is noted that the states $\tilde{x}(k+s), \cdots, \tilde{x}(k+s+h-1)$ calculated above are not necessarily the real network evolution trajectory, nevertheless, they supply valuable information for the design of $v(k+s-1)$. Now, it is ready to give the structure of $v(k+s-1)$ as follows:

$$v(k+s-1)$$
$$= L\left[\sum_{i=1}^{s} \alpha_i x(k+i-1) + \sum_{i=s+1}^{s+h} \alpha_i \tilde{x}(k+i-1)\right] \qquad (6)$$

where $\alpha = [\alpha_1, \cdots, \alpha_{s+h}]^{\mathsf{T}} \in \mathbb{R}^{s+h}$ represents the gains to be determined.

[1] See Pascal's triangle for such coefficients.

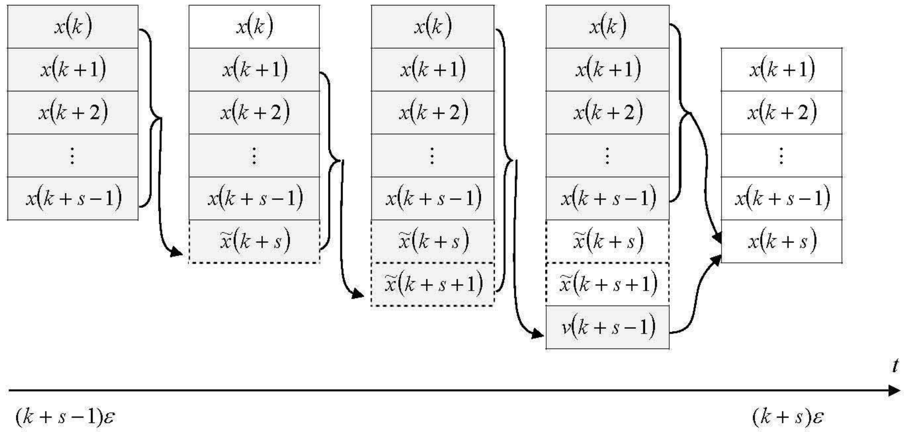


Fig. 1. Schematic diagram of the network (5) with the input (6) for $h = 2$. At the instant $t = (k + s - 1)\epsilon$, the network state is $z(k)$. The prediction $\tilde{x}(k + s)$ is computed based on $[x^{\mathsf{T}}(k), x^{\mathsf{T}}(k + 1), \cdots, x^{\mathsf{T}}(k + s - 1)]^{\mathsf{T}}$, and $\tilde{x}(k + s + 1)$ based on $[x^{\mathsf{T}}(k + 1), \cdots, x^{\mathsf{T}}(k + s - 1), \tilde{x}^{\mathsf{T}}(k + s)]^{\mathsf{T}}$. After $h(= 2)$ predictive steps, $v(k + s - 1)$ can be computed. The computation is conducted between $t = (k + s - 1)\epsilon$ and $t = (k + s)\epsilon$. As a result, the network state is updated to $z(k + 1)$ at the instant $t = (k + s)\epsilon$.

Remark 2.1: An important feature of the controller (6) is that the digraph of the network is maintained because of the matrix L multiplying from left. To implement the controller (6), each agent predicts the state $\tilde{x}(k + s)$ at the instant $(k+s-1)\epsilon$ and propagates it through the digraph, and then predicts and propagates the next state $\tilde{x}(k+s+1)$, until $\tilde{x}(k+s+h-1)$. After the h steps of predictions are complete, the network has the sufficient information to implement the controller (6). The schematic diagram of the network (5) with the input (6) is given in Fig. 1 for $h = 2$. ∎

The network performance to be investigated in this paper is the speed of the network's consensus, i.e., how fast all agents achieve a state agreement. The precise definition is given below.

Definition 2.1: The multi-agent network (5) is said to reach consensus if

$$\lim_{k\to\infty} x(k) - \wp(k, z_o)\mathbf{1} = 0$$

for a trajectory $\wp(k, z_o)$ depending on the initial state z_o, which is called the group decision trajectory. ∎

To close this section, we list our main objective as follows.

Consensus Acceleration Problem: To find an effective controller $v(k + s - 1)$ in the form of (6) such that, if the nominal network (5) with $v(k+s-1) = 0$ reaches consensus, so does the network (5) with the controller $v(k + s - 1)$. Moreover, the group decision trajectories $\wp(k, z_o)$ for both networks are identical and the consensus is accelerated by the controller $v(k + s - 1)$.

III. Main Result

To analyze the consensus performance of the network (5), we can rewrite it in the form of

$$z(k + 1) = P_\epsilon z(k) + Bv(k + s - 1) \tag{7}$$

where

$$P_\epsilon = \begin{bmatrix} [\,0 \quad I\,] \\ -\Phi_s^{\mathsf{T}} - L\sum_{i=0}^{s-1} \epsilon^{(s-i)}\gamma_i\Phi_i^{\mathsf{T}} \end{bmatrix}, \; B = \begin{bmatrix} 0_{(s-1)n\times n} \\ I \end{bmatrix}.$$

The following lemma will be used.

Lemma 3.1: *Consider the system $z(k + 1) = Pz(k)$, $z(0) = z_o$, $z \in \mathbb{R}^{ns}$ where the matrix P satisfies the following condition.*

C1: *There exist a nonsingular matrix $T \in \mathbb{R}^{s\times s}$ and a block diagonal matrix $J \in \mathbb{R}^{s\times s}$ with each block being Jordan block corresponding to the eigenvalue 1, such that*

$$PR = RJ, \quad R := T \otimes \mathbf{1} \in \mathbb{R}^{ns\times s}. \tag{8}$$

If all the remaining $s(n - 1)$ eigenvalues of P are inside the unit circle, then,

$$\lim_{k\to\infty} z(k) - \wp_a(z_o, k) \otimes \mathbf{1} = 0 \tag{9}$$

for a family of trajectories $\wp_a(z_o, k) \in \mathbb{R}^s$ depending on the initial state z_o. ∎

Proof: Consider the Jordan normal form of the matrix P and there exists a matrix $N \in \mathbb{R}^{ns\times(ns-s)}$ such that $PN = N\bar{J}$ where $\bar{J} \in \mathbb{R}^{(ns-s)\times(ns-s)}$ is a block diagonal matrix with each block being Jordan block corresponding to the $s(n-1)$ eigenvalues inside the unit circle. Since the columns of R are linearly independent, we can find a nonsingular matrix $M = [R,\ N]$. As a result,

$$P = M \begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix} M^{-1}$$

and hence

$$0 = \lim_{k\to\infty} P^k - M \begin{bmatrix} J^k & 0 \\ 0 & \bar{J}^k \end{bmatrix} M^{-1}$$

$$= \lim_{k\to\infty} P^k - M \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} M^{-1} = \lim_{k\to\infty} P^k - RJ^kR^{\dagger}$$

where $R^\dagger$ is the upper s rows of M^{-1}, which obviously satisfies

$$R^\dagger R = I_s,\ R^\dagger P = JR^\dagger.$$

From the following calculation,

$$0 = \lim_{k\to\infty} z(k) - P^k z_o = \lim_{k\to\infty} z(k) - (T\otimes \mathbf{1})J^k R^\dagger z_o$$
$$= \lim_{k\to\infty} z(k) - TJ^k R^\dagger z_o \otimes \mathbf{1},$$

we have (9) with $\wp_a(z_o,k) = TJ^k R^\dagger z_o$. ■

Remark 3.1: We define $\rho_s(P)$ as the $(s+1)$-th largest norm of the ns eigenvalues of P, i.e., $\rho_s(P) = \|\lambda_{s+1}\|$ where $[\lambda_1,\cdots,\lambda_{ns}]^\intercal$ is the spectrum of P with $\|\lambda_{i+1}\| \le \|\lambda_i\|, \forall i = 1,\cdots,ns-1$. Obviously, in the above lemma, P has s eigenvalues of 1 and $s(n-1)$ eigenvalues inside the unit circle, then $\rho_s(P)$ is the largest norm of the $s(n-1)$ eigenvalues. In other words, $\rho_s(P)$ is the spectral radius of $\bar{J}$, which characterizes the convergence speed of $\bar{J}^k \to 0$ as $k\to\infty$ and hence that of $z(k) \to \wp_a(z_o,k)$. ▮

The above lemma implies a general consensus result as follows.

Theorem 3.1: **[Consensus]** *Consider the system $z(k+1) = P_\epsilon z(k)$ as described in (7) with $v(k+s-1) = 0$. If the matrix P_ϵ has $s(n-1)$ eigenvalues inside the unit circle, then the system achieves consensus, i.e.,*

$$\lim_{k\to\infty} x(k) - \wp(z_o,k)\otimes \mathbf{1} = 0 \tag{10}$$

for a trajectory $\wp(z_o,k)\in\mathbb{R}$ depending on the initial state z_o. ▮

Proof: We define a constant matrix $Q\in\mathbb{R}^{s\times s}$:

$$Q = \begin{bmatrix} \begin{bmatrix} 0 & I_{s-1} \end{bmatrix} \\ -\phi_s^\intercal \end{bmatrix}.$$

The following calculation

$$|\lambda I - Q| = \lambda^s + [1\ \lambda\ \cdots \lambda^{s-1}]\phi_s = (\lambda-1)^s$$

shows that Q has an eigenvalue $\lambda = 1$ with algebraic multiplicity s. Then, there exists a nonsingular matrix $T\in\mathbb{R}^{s\times s}$ and a block diagonal matrix $J\in\mathbb{R}^{s\times s}$ with each block being Jordan block corresponding to the eigenvalue 1, such that $QT = TJ$.

Next, we have

$$P_\epsilon R = \begin{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \\ -\Phi_s^\intercal - L\sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\Phi_i^\intercal \end{bmatrix}(T\otimes\mathbf{1})$$
$$= QT\otimes\mathbf{1},$$
$$RJ = T\otimes \mathbf{1}J = TJ\otimes\mathbf{1},$$

where the first equation comes from

$$\begin{bmatrix} 0 & I \end{bmatrix}(T\otimes\mathbf{1}) = \begin{bmatrix} 0 & I_{s-1} \end{bmatrix} T\otimes\mathbf{1}$$

$$(-\Phi_s^\intercal - L\sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\Phi_i^\intercal)(T\otimes\mathbf{1})$$
$$= -\Phi_s^\intercal(T\otimes\mathbf{1}) - L\sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\Phi_i^\intercal(T\otimes\mathbf{1})$$
$$= -\phi_s^\intercal T\otimes\mathbf{1} - L\sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\phi_i^\intercal T\otimes\mathbf{1}$$
$$= -\phi_s^\intercal T\otimes\mathbf{1} - \sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\phi_i^\intercal T\otimes(L\mathbf{1}) = -\phi_s^\intercal T\otimes\mathbf{1}$$

and the second equation is direct. As a result, $P_\epsilon R = RJ$.

Let $\ell\in\mathbb{R}^n$ be the left eigenvector of L corresponding to the eigenvalue 0 satisfying $\ell^\intercal\mathbf{1} = 1$, and $R^\dagger = T^{-1}\otimes\ell^\intercal$. On one hand, we have $R^\dagger R = (T^{-1}\otimes\ell^\intercal)(T\otimes\mathbf{1}) = I_s$. On the other hand, the following calculation

$$\begin{aligned} R^\dagger P_\epsilon &= T^{-1}\otimes\ell^\intercal \begin{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \\ -\Phi_s^\intercal - L\sum_{i=0}^{s-1}\epsilon^{(s-i)}\gamma_i\Phi_i^\intercal \end{bmatrix} \\ &= (T^{-1}Q)\otimes\ell^\intercal \\ JR^\dagger &= J(T^{-1}\otimes\ell^\intercal) = JT^{-1}\otimes\ell^\intercal \end{aligned}$$

verifies $R^\dagger P_\epsilon = JR^\dagger$.

Let $\wp(z_o,k) = [1\ \ 0\ \ \cdots\ \ 0]TJ^kR^\dagger z_o = [1\ 0\ \cdots\ 0]TJ^kT^{-1}\otimes\ell^\intercal z_o = [1\ 0\ \cdots\ 0]Q^k\otimes\ell^\intercal z_o$. From Lemma 3.1, we have $\lim_{k\to\infty} x(k) - \wp(z_o,k)\otimes\mathbf{1} = 0$. The proof is thus complete. ■

Remark 3.2: The condition that the matrix P_ϵ has $s(n-1)$ eigenvalues inside the unit circle has been examined for some special cases. For example, when $s=1$ or $s=2$, we have

$$P_\epsilon = I - \epsilon L \ \text{ or }\ P_\epsilon = \begin{bmatrix} 0 & I \\ -I-\epsilon^2 L + \epsilon\gamma_1 L & 2I - \epsilon\gamma_1 L \end{bmatrix},$$

respectively. These single- and double-integrator networks have been examined in [5], [7]. ▮

In what follows, we will discuss the influence caused by the controller (6). To this end, we will rewrite (6) in a more compact form. We note that

$$\begin{aligned} \tilde{z}(k+1) &= P_\epsilon z(k) \\ \tilde{z}(k+i) &= P_\epsilon\tilde{z}(k+i-1),\ i = 2,\cdots,h \end{aligned}$$

for $\tilde{z}(k+i) := [x^\intercal(k+i),\cdots,x^\intercal(k+s-1),\tilde{x}^\intercal(k+s),\cdots,\tilde{x}^\intercal(k+s+i-1)]^\intercal$, $i=1,\cdots,h$, and hence

$$\tilde{x}(k+s+i-1) = B^\intercal P_\epsilon^i z(k),\ i = 1,\cdots,h,$$

where B is given after (7). As a result, the system (7) becomes

$$z(k+1) = (P_\epsilon + \bar{P}(\alpha))z(k) \tag{11}$$

where

$$\bar{P}(\alpha) = BL\left[\bar{\alpha}^\intercal\otimes I + B^\intercal[P_\epsilon,\cdots,P_\epsilon^h](\tilde{\alpha}\otimes I)\right],$$
$$\alpha := [\bar{\alpha}^\intercal,\tilde{\alpha}^\intercal]^\intercal,\ \bar{\alpha} := [\alpha_1,\cdots,\alpha_s]^\intercal,\ \tilde{\alpha} := [\alpha_{s+1},\cdots,\alpha_{s+h}]^\intercal.$$

From the definition of $\bar{P}$, it is clear that $\bar{P}(0) = 0$. Then, we aim to find a parameter vector α^* as follows:

$$\alpha^* = \min_{\alpha \in \mathbb{R}^{s+h}} \rho_s(P_\epsilon + \bar{P}(\alpha)) \tag{12}$$

with which the main result on consensus acceleration is given below.

Theorem 3.2: **[Consensus Acceleration]** *Consider the system (11) where the matrix P_ϵ has $s(n-1)$ eigenvalues inside the unit circle. Then, both the nominal system with $\alpha = 0$ and the controlled system with $\alpha = \alpha^*$ as in (12) achieve consensus, i.e.,*

$$\lim_{k\to\infty} x(k) - \wp(z_o, k) \otimes \mathbf{1} = 0 \tag{13}$$

for the same group decision trajectory $\wp(z_o, k) \in \mathbb{R}$. Moreover, the consensus is accelerated in the sense of

$$\rho_s(P_\epsilon + \bar{P}(\alpha^*)) \le \rho_s(P_\epsilon)$$

where the equality holds only when $\alpha^* = 0$. ∎

Proof: Because α^* is given according to (12), it is obvious that $\rho_s(P_\epsilon + \bar{P}(\alpha^*)) \le \rho_s(P_\epsilon) < 1$. Let R and $R^\dagger$ be those defined in the proof of Theorem 3.1. By Lemma 3.1, it suffices to show that $PR = RJ$ and $R^\dagger P = JR^\dagger$ to prove the achievement of consensus to an identical group decision trajectory, for $P = P_\epsilon + \bar{P}(\alpha^*)$. We have shown that $P_\epsilon R = RJ$ and $R^\dagger P_\epsilon = JR^\dagger$, so what left is to show $\bar{P}(\alpha)R = 0$ and $R^\dagger \bar{P}(\alpha) = 0$. In fact, they are true from the following calculation:

$$\begin{aligned}
&\bar{P}(\alpha)R \\
=\;& BL\left[\bar{\alpha}^\intercal \otimes IR + B^\intercal[P_\epsilon, \cdots, P_\epsilon^h](\tilde{\alpha} \otimes I)R\right] \\
=\;& BL\left[(\bar{\alpha}^\intercal \otimes I)(T \otimes \mathbf{1}) + B^\intercal[P_\epsilon, \cdots, P_\epsilon^h](\tilde{\alpha} \otimes I)R\right] \\
=\;& BL\left[(\bar{\alpha}^\intercal T) \otimes \mathbf{1} + B^\intercal[P_\epsilon R, \cdots, P_\epsilon^h R](\tilde{\alpha} \otimes I)\right] \\
=\;& BL\left[(\bar{\alpha}^\intercal T) \otimes \mathbf{1} + B^\intercal(T \otimes \mathbf{1})[J, \cdots, J^h](\tilde{\alpha} \otimes I)\right] \\
=\;& BL[(\bar{\alpha}^\intercal T) \otimes \mathbf{1} + ([0, \cdots, 0, 1]T) \\
&\otimes \mathbf{1}[J, \cdots, J^h](\tilde{\alpha} \otimes I)] = 0
\end{aligned}$$

$$\begin{aligned}
R^\dagger \bar{P}(\alpha) &= (T^{-1} \otimes \ell^\intercal) \\
BL\left[\bar{\alpha}^\intercal \otimes I + B^\intercal[P_\epsilon, \cdots, P_\epsilon^h](\tilde{\alpha} \otimes I)\right] &= 0
\end{aligned}$$

where we note that $L\mathbf{1} = 0$ and $\ell^\intercal L = 0$. ∎

Generally, it is difficult to find an analytical solution to the optimization problem (12). Because an optimal α^* is found off-line and the dimension of α, i.e., $s + h$ is usually a small number, we can pick a set $\mathbb{A}$ of a reasonable size containing the origin, and exhaustively search for the optimal $\alpha^* \in \mathbb{A} \subset \mathbb{R}^{s+h}$. Nevertheless, for some special cases, there exists a more effective search algorithm. For instance, we consider the case of $s = 1$ as studied in [1]. Let $\lambda_i \ne 0$ be the eigenvalue of L and ℓ_i the corresponding left eigenvector, i.e., $\ell_i L = \lambda_i \ell_i$ for $i = 1, \cdots, n-1$. With $s = 1$, we have $\bar{P}(\alpha) = L[P_\epsilon^0, P_\epsilon^1, \cdots, P_\epsilon^h](\alpha \otimes I)$ and $P_\epsilon = I - \epsilon L$. As a result,

$$\begin{aligned}
&\ell_i[P_\epsilon + \bar{P}(\alpha)] \\
&= \ell_i[(1-\epsilon\lambda_i) + \lambda_i[1, (1-\epsilon\lambda_i), \cdots, ((1-\epsilon\lambda_i))^h]\alpha].
\end{aligned}$$

In other words,

$$\begin{aligned}
\zeta_i(\alpha) := (1-\epsilon\lambda_i) + \lambda_i[1, (1-\epsilon\lambda_i), \cdots, (1-\epsilon\lambda_i)^h]\alpha \\
= [1\ \ j](g_i\alpha + \xi_i)
\end{aligned}$$

is the eigenvalue of $P_\epsilon + \bar{P}(\alpha)$ corresponding to the left eigenvector ℓ_i, where

$$\begin{aligned}
\bar{g}_i &= \Re\left\{\lambda_i[1, (1-\epsilon\lambda_i), \cdots, (1-\epsilon\lambda_i)^h]\right\}, \\
\tilde{g}_i &= \Im\left\{\lambda_i[1, (1-\epsilon\lambda_i), \cdots, (1-\epsilon\lambda_i)^h]\right\} \\
\bar{\xi}_i &= 1 - \epsilon\Re\{\lambda_i\},\ \tilde{\xi}_i = -\epsilon\Im\{\lambda_i\}, \\
g_i &= [\bar{g}_i^\intercal, \tilde{g}_i^\intercal]^\intercal,\ \ \xi_i = [\bar{\xi}_i, \tilde{\xi}_i]^\intercal.
\end{aligned}$$

Instead of searching for an α^* to minimize the second largest norm of the eigenvalues as in (12), we aim to find an α^* to minimize the sum of the norms of all eigenvalues, i.e.,

$$\alpha^* = \min_{\alpha \in \mathbb{R}^{s+h}} \sum_{i=1}^{n-1} \|\zeta_i(\alpha)\|^2. \tag{14}$$

We expect the solution to (14) is an effective replacement for that to (12). In (14), a globally optimal α^* can be analytically found as follows. We note (14) can be rewritten as

$$\alpha^* = \min_{\alpha \in \mathbb{R}^{s+h}} \sum_{i=1}^{n-1} \|g_i\alpha + \xi_i\|^2 = \min_{\alpha \in \mathbb{R}^{s+h}} \|G\alpha + \xi\|^2 \tag{15}$$

where

$$G = [g_1^\intercal, \cdots, g_{n-1}^\intercal]^\intercal,\ \ \xi = [\xi_1^\intercal, \cdots, \xi_{n-1}^\intercal]^\intercal.$$

Now, it is ready to give the analytical solution $\alpha^* = -(G^\intercal G)^{-1}G^\intercal \xi$.

IV. A Numerical Example

In this section, we will consider a network with a fixed topology and compare the consensus performance of the network with and without the additional predictive controller (6). To describe the consensus degree in quantity, we define a consensus index as follows

$$\begin{aligned}
i_c(k) = \|[x_1(k) - x_2(k), \cdots, x_{n-1}(k) - x_n(k), \\
x_n(k) - x_1(k)]\|.
\end{aligned}$$

Obviously, $\lim_{t\to\infty} i_c(k) = 0$ as a network reaches consensus.

In particular, we consider the network with 8 agents and 17 edges which is depicted in Fig. 1 of [1]. With the topology and optimal weights given in [1], the second largest norm of the eigenvalues is $\rho_1(P_\epsilon) = 0.6$. The eigenvalues distribution of P_ϵ is given in Fig. 2 (marked as $+$). Consider the controller (6) with different predictive horizons, the optimal α^* to (14) can be analytically calculated and the corresponding $\rho_1(P_\epsilon + \bar{P}(\alpha^*))$ is listed in Table I. For example, when $h = 2$, we have an optimal $\alpha^* = [-0.0033\ \ -0.4142\ \ -0.3875]^\intercal$ and $\rho_1(P_\epsilon + \bar{P}(\alpha^*)) = 0.0783$. The eigenvalues distribution of $P_\epsilon + \bar{P}(\alpha^*)$ in also given in Fig. 2 (marked as $\circ$). With

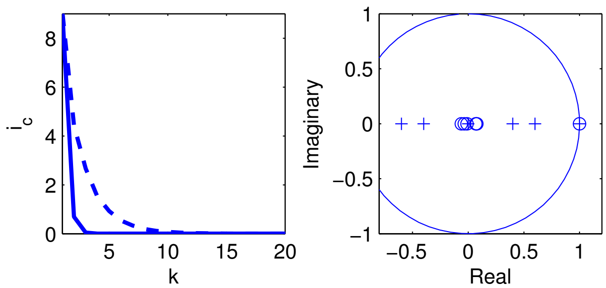
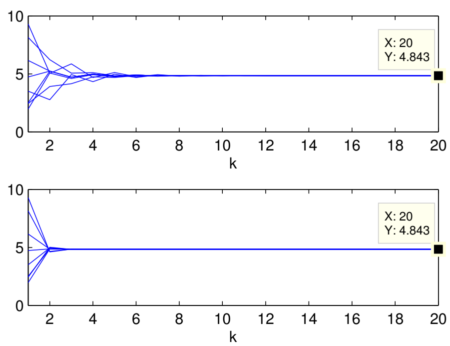


Fig. 2. Left: Consensus index (dashed line: the nominal system with $v = 0$; solid line: the controlled system with v designed in (6)). Right: Eigenvalue distribution (+: P_ϵ; $\circ$: $P_\epsilon + \bar{P}(\alpha^*)$).



(A)

Fig. 3. Network trajectories achieving consensus. Top: the nominal system with $v = 0$; Bottom: the controlled system with v designed in (6).

$s = 1$, we note $\wp(z_o, k) = \ell^{\mathsf{T}} x(0)$ in Theorem 3.1. With the parameters in the simulation, we have

$$\lim_{k\to\infty} x(k) = \ell^{\mathsf{T}} x(0)\mathbf{1} = (4.84)\mathbf{1}.$$

Both systems with and without (6) can achieve consensus in the sense as shown in Fig. 3 where the system with (6) achieves consensus more quickly. The consensus acceleration is demonstrated through the consensus index in Fig. 2.

TABLE I

THE SECOND LARGEST NORM OF THE EIGENVALUES WITH DIFFERENT h

h	0	1	2	3
$\rho_1(P_\epsilon + \bar{P}(\alpha^*))$	0.6000	0.3257	0.0783	0.0396

V. CONCLUSION

In this paper, we have developed a class of predictive controllers for consensus networks to significantly increase their convergence speed. The controller does not physically change the network topology or request more communication channels. But its effectiveness has been demonstrated through both theoretical analysis and numerical simulation.

REFERENCES

[1] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

[2] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.

[3] G. Grégoire and H. Chaté. Onset of collective and cohesive motion. *Physical Review Letters*, 92(2):025702, 2004.

[4] M. Aldana, V. Dossetti, C. Huepe, V. M. Kenkre, and H. Larralde. Phase transitions in systems of self-propelled agents and related network models. *Physical Review Letters*, 98:095702, 2007.

[5] R. Olfati-Saber and R. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Auto. Contr.*, 49(9):1520–1533, 2004.

[6] W. Ren and R. W. Beard. Consensus seeking in multiagents systems under dynamically changing interaction topologies. *IEEE Trans. Auto. Contr.*, 50(5):655–661, 2005.

[7] W. Ren, R. W. Beard, and E. M. Atkins. Information consensus in multivehicle cooperative control. *IEEE Control System Magazine*, 71(2):71–82, 2007.

[8] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci. Wireless sensor networks: a survey. *Computers Network*, 38(4):393–422, 2002.

[9] P. Ogren, E. Fiorelli, and N. E. Leonard. Cooperative control of mobile sensor networks:adaptive gradient climbing in a distributed environment. *IEEE Trans. Auto. Contr.*, 49(8):1292–1302, 2004.

[10] T. Arai, E. Pagello, and L. E. Parker. Guest editorial advances in multirobot systems. *IEEE Trans. Auto. Contr.*, 18(5):655–661, 2002.

[11] J. Tsitsiklis and M. Athans. Convergence and asymptotic agreement in distributed decision problems. *IEEE Trans. Auto. Contr.*, 29(1):42–50, 1984.

[12] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.

[13] R. Olfati-Saber. Ultrafast consensus in small-world networks. In *Proceedings of American Control Conference*, pages 2371–2378, June 2005.

[14] S. H. Strogatz. Exploring complex networks. *Nature*, 410:268–276, 2001.

[15] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile agents using nearest neighbor rules. *IEEE Trans. Auto. Contr.*, 48(6):988–1001, 2003.

[16] C. Q. Ma and J. F. Zhang. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Trans. Auto. Contr.*, 55(5):1263–1268, 2010.

[17] L. Wang and X. Feng. Finite-time consensus problems for networks of dynamic agents. *IEEE Trans. Auto. Contr.*, 55(4):950–955, 2010.

[18] A. Nedić, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Trans. Auto. Contr.*, 55(4):922–938, 2010.

[19] G. Notarstefano and F. Bullo. Distributed consensus on enclosing shapes and minimum time rendezvous. *In Proc. IEEE Conf. Decision and Control*, pages 4295 – 4300, Dec. 2006.

[20] Y. Kim and M. Mesbahi. On maximizing the second smallest eigenvalue of a state-dependent graph laplacian. *IEEE Trans. Auto. Contr.*, 51(1):116–120, 2006.

[21] S. Patterson, B. Bamieh, and A. E. Abbadi. Convergence rates of distributed average consensus with stochastic link failures. *IEEE Trans. Auto. Contr.*, 55(4):880–892, 2009.

[22] E. F. Woods. Electronic prediction of swarming in bees. *Nature*, 184(4690):842–844, 1959.

[23] J. A. Gottfried, J. O. Doherty, and R. J. Dolan. Encoding predictive reward value in human amygdala and orbitofrontal cortex. *Science*, 301(5636):1104–1107, 2003.

[24] C. Summerfield, T. Egner, M. Greene, E. Koechlin, J. Mangles, and J. Hirsch. Predictive codes for forthcoming perception in the frontal cortex. *Science*, 314(5803):1311–1314, 2006.

[25] D. Melcher. Predictive codes for forthcoming perception in the frontal cortex. *Nature Neuroscience*, 10(7):903–907, 2007.

[26] G. Farrai-Trecate, L. Galbusera, M. P. E. Marcinadi, and R. Scattolini. Model predictive control schemes for consensus in multi-agent systems with single- and double-integrator dynamics. *IEEE Trans. Auto. Contr.*, 54(11):2560–2572, 2009.