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# On the Stability of an Interconnected System of Euler-Bernoulli Beam and Heat Equation with Boundary Coupling

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***Abstract*— We study the stability of an interconnected system of Euler-Bernoulli beam and heat equation with boundary coupling, where the boundary temperature of the heat equation is fed as the boundary moment of the Euler-Bernoulli beam and, in turn, the boundary angular velocity of the Euler-Bernoulli beam is fed into the boundary heat flux of the heat equation. We show that the spectrum of the closed-loop system consists only of two branches: one along the real axis and the another along two parabolas symmetric to the real axis and open to the imaginary axis. The asymptotic expressions of both eigenvalues and eigenfunctions are obtained. With a careful estimate for the resolvent operator, the completeness of the root subspaces of the system is verified. The Riesz basis property and exponential stability of the system are then proved. Finally we show that the semigroup, generated by the system operator, is of Gevrey class $\delta > 2$.**

## I. Introduction

Engineering applications give rise to fluid-structure interactions, composite laminates in smart materials and structures, structural-acoustic systems, and other interactive physical process, which are modeled by partial differential equation (PDE) cascades or interconnected PDEs. Control design and stability analysis for such systems have become active over the past decades, see [5], [6], [8], [18], [19], [21], [22] and the references therein.

The stability and controllability analysis for a heat-wave system, arising from the fluid-structure interaction, were treated in [21], [22]. Feedback controllers for several classes of coupled PDEs and structural-acoustic models were introduced in [8]. The stability and Riesz basis property of the composite laminates and the sandwich beam with boundary controls were analyzed in [18], [19].

We consider Euler-Bernoulli beam and heat equation (see Figure 1) governed by the equations:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w(0,t) = w(1,t) = 0, & t \geq 0, \\ w_{xx}(1,t) = 0, & t \geq 0, \\ w_{xx}(0,t) = f_1(t), & t \geq 0, \\ y_1(t) = -w_{xt}(0,t), & t \geq 0, \\ w(x,0) = w_0(x), & 0 \leq x \leq 1, \\ w_t(x,0) = w_1(x), & 0 \leq x \leq 1, \end{cases} \tag{1}$$

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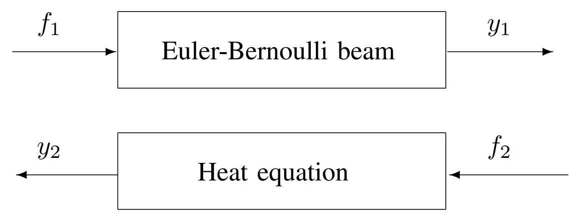


Fig. 1. Euler-Bernoulli beam (1) and heat equation (2)

and

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u(1,t) = 0, & t \geq 0, \\ u_x(0,t) = f_2(t), & t \geq 0, \\ y_2(t) = -u(0,t), & t \geq 0, \\ u(x,0) = u_0(x), & 0 \leq x \leq 1, \end{cases} \tag{2}$$

where the Euler-Bernoulli beam is hinged at the right hand, the right side of the heat equation is kept at zero temperature, $f_1(t)$ and $f_2(t)$ are the boundary controls applied at the left ends of the beam and the heat respectively, $y_1(t)$ and $y_2(t)$ are the observations, and $(w_0(x), w_1(x))$ and $u_0(x)$ are the initial conditions. We denote the two dynamic systems with the mappings

$$\mathbf{E} : f_1 \mapsto y_1$$

and

$$\mathbf{H} : f_2 \mapsto y_2.$$

It is well known that the feedback law

$$f_1(t) = -y_1(t) \tag{3}$$

achieves exponential stability of the Euler-Bernoulli beam system, as well as that the feedback law

$$f_2(t) = -y_2(t) \tag{4}$$

guarantees exponential stability of the heat equation. In this paper we study the case where the two subsystems are interconnected via the feedback laws (see Figure 2)

$$f_1(t) = -y_2(t) \tag{5}$$

and

$$f_2(t) = y_1(t). \tag{6}$$

The interconnection (5), (6) can be interpreted in three ways. The first interpretation of (5), (6) is as

$$f_1(t) = (-\mathbf{H}y_1)(t),$$

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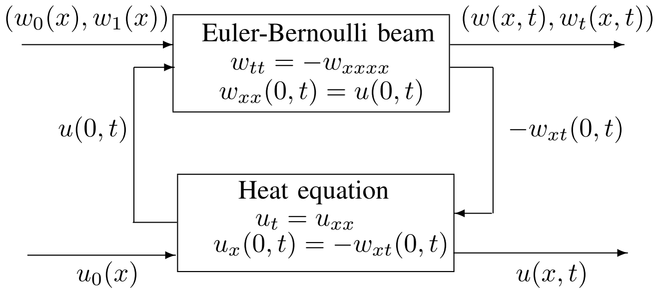


Fig. 2. Block diagram for the closed-loop system (7)

namely, as replacing the unit-gain static feedback (3) of the Euler-Bernoulli beam by a dynamic feedback law governed by the heat equation. The second interpretation of (5), (6) is as

$$f_2(t) = (\mathbf{E}(-y_2))\,(t),$$

namely, as replacing the unity-gain static feedback (4) of the heat equation by a dynamic feedback law governed by the Euler-Bernoulli beam. The third interpretation of (5), (6) is simply as a coupled PDE system given in Figure 2.

Under the feedback laws (5), (6), the interconnected system of Euler-Bernoulli beam and heat equation is:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_t(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w(1,t) = w_{xx}(1,t) = 0, & t \geq 0, \\ u(1,t) = 0, & t \geq 0, \\ w(0,t) = 0, & t \geq 0, \\ u(0,t) = w_{xx}(0,t), & t \geq 0, \\ u_x(0,t) = -w_{xt}(0,t), & t \geq 0, \\ w(x,0) = w_0(x), & 0 \leq x \leq 1, \\ w_t(x,0) = w_1(x), & 0 \leq x \leq 1, \\ u(x,0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \tag{7}$$

The energy function for (7) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[w_t^2(x,t) + w_{xx}^2(x,t) + u^2(x,t)\right] dx. \tag{8}$$

Then we have

$$\frac{d}{dt}E(t) = -\int_0^1 u_x^2(x,t)dx \leq 0$$

and $E(t)$ is non-increasing.

We provide a detailed spectral analysis for the system (7). We show that there are two branches of eigenvalues of (7): one is along the real axis, and another is along the two parabolas symmetric to the real axis and open to the imaginary axis. The latter branch of eigenvalues generated by the beam is very similar to the case studied in [4], where the well-posedness and exponential stability of an Euler-Bernoulli beam with non-monotone boundary feedback $w_{xxx}(0,t) = -kw_{xt}(0,t)$, proposed early in [11], were considered for the feedback gain $k > 0$ with $k \neq 1$. Later on, its Gevrey regularity was treated in [1], [14].

In this paper, the asymptotic expressions of the eigenvalues and eigenfunctions, the Riesz basis property and exponential stability of (7) are studied. Moreover, we show that the $C_0$-semigroup, generated by the system operator, is of Gevrey class $\delta > 2$. (Gevrey regularity is described in terms of the bounds on all derivatives of the semigroups. The differentiability of the Gevrey semigroup is slightly weaker than that of an analytic semigroup [1], [14], [17].)

We proceed as follows. In Section 2 we formulate the problem as an evolution equation in Hilbert energy space. The $C_0$-semigroup approach is used to prove the well-posedness of the system. Section 3 is devoted to the spectral analysis and the asymptotic expressions of eigenvalues and eigenfunctions are presented. The Riesz basis property and exponential stability are established in Section 4. Finally, Gevrey regularity of the semigroup is obtained in Section 5.

## II. Well-posedness of the system (7)

We consider the system (7) in the energy space

$$\mathcal{H} = H_L^2(0,1) \times L^2(0,1) \times L^2(0,1)$$

where $H_L^2(0,1) = \{f|\ f \in H^2(0,1), f(0) = f(1) = 0\}$ and the norm in $\mathcal{H}$ is induced by the following inner product

$$\langle X_1, X_2\rangle = \int_0^1 \left[f_1''\overline{f_2''} + g_1\overline{g_2} + h_1\overline{h_2}\right] dx, \tag{9}$$

where $X_i = (f_i, g_i, h_i) \in \mathcal{H}$, $i = 1, 2$. Define the system operator by

$$\begin{cases} \mathcal{A}(f,g,h) = (g, -f^{(4)}, h''), (f,g,h) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \begin{cases} (f,g,h) \in (H^4 \times H_L^2 \times H^2) \cap \mathcal{H} \\ h(1) = f''(1) = 0, \\ g'(0) = -h'(0), f''(0) = h(0) \end{cases} \end{cases} \tag{10}$$

Then (7) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases} \dfrac{dX(t)}{dt} = \mathcal{A}X(t), t > 0, \\ X(0) = X_0. \end{cases} \tag{11}$$

where $X(t) = (w(\cdot,t), w_t(\cdot,t), u(\cdot,t))$ and $X_0 = (w_0, w_1, u_0)$. We have the following result directly.

*Theorem 1:* Let $\mathcal{A}$ be given by (10). Then $\mathcal{A}^{-1}$ exists and is compact. Hence, $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only. Moreover $\mathcal{A}$ is dissipative in $\mathcal{H}$ and $\mathcal{A}$ generates a $C_0$-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$.

## III. Spectral Analysis

Let us now consider the eigenvalue problem of $\mathcal{A}$. $\mathcal{A}X = \lambda X$, where $X = (f,g,h) \in D(\mathcal{A})$, if and only if $g(x) = \lambda f(x)$, and $f, h$ satisfy the following eigenvalue problem:

$$\begin{cases} f^{(4)}(x) + \lambda^2 f(x) = 0, \\ h''(x) - \lambda h(x) = 0, \\ f(0) = f(1) = f''(1) = h(1) = 0, \\ f''(0) = h(0), \\ \lambda f'(0) = -h'(0). \end{cases} \tag{12}$$

A direct computation yields the following lemma.

*Lemma 1:* Let $\mathcal{A}$ be defined by (10). Then for each $\lambda \in \sigma(\mathcal{A})$, we have $\mathrm{Re}\lambda < 0$.

Due to Lemma 1 and the fact that the eigenvalues are symmetric about the real axis, we consider only those $\lambda$ which are located in the second quadrant of the complex plane:

$$\lambda := i\rho^2,\ \rho \in \mathcal{S} := \left\{\rho \in \mathbb{C} \mid 0 \leq \arg\rho \leq \frac{\pi}{4}\right\}. \tag{13}$$

Note that for any $\rho \in \mathcal{S}$, we have

$$\mathrm{Re}(-\rho) \leq \mathrm{Re}(i\rho) \leq \mathrm{Re}(-i\rho) \leq \mathrm{Re}(\rho), \tag{14}$$

and

$$\begin{cases} \mathrm{Re}(-\rho) = -|\rho|\cos(\arg\rho) \leq -\frac{\sqrt{2}}{2}|\rho| < 0, \\ \mathrm{Re}(i\rho) = -|\rho|\sin(\arg\rho) \leq 0. \end{cases} \tag{15}$$

Moreover, if we denote $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with

$$\begin{cases} \mathcal{S}_1 := \{\rho \in \mathbb{C} \mid \frac{\pi}{8} < \arg\rho \leq \frac{\pi}{4}\}, \\ \mathcal{S}_2 := \{\rho \in \mathbb{C} \mid 0 \leq \arg\rho \leq \frac{\pi}{8}\}, \end{cases} \tag{16}$$

then we have

$$\begin{cases} \mathrm{Re}(i\rho) \leq -|\rho|\sin\left(\frac{1}{8}\pi\right) < 0, & \forall\rho \in \mathcal{S}_1, \\ \mathrm{Re}(-\sqrt{i}\rho) \leq -|\rho|\cos\left(\frac{3}{8}\pi\right) < 0, & \forall\rho \in \mathcal{S}_2. \end{cases} \tag{17}$$

Now substituting $\lambda = i\rho^2$ into (12), we have the eigenvalue system of (7) in $\rho$:

$$\begin{cases} f^{(4)}(x) - \rho^4 f(x) = 0, \\ h''(x) - i\rho^2 h(x) = 0, \\ f(0) = f(1) = f''(1) = h(1) = 0, \\ f''(0) = h(0), \\ i\rho^2 f'(0) = -h'(0). \end{cases} \tag{18}$$

Let

$$\begin{cases} f(x) = c_1 e^{\rho x} + c_2 e^{-\rho x} + c_3 e^{i\rho x} + c_4 e^{-i\rho x}, \\ h(x) = d_1 e^{\sqrt{i}\rho x} + d_2 e^{-\sqrt{i}\rho x}, \end{cases} \tag{19}$$

where $c_s, s = 1,2,3,4$ and $d_1, d_2$ are constants. Substituting these into the boundary conditions of (18), we have

$$\begin{cases} c_1 + c_2 + c_3 + c_4 = 0, \\ c_1 e^{\rho} + c_2 e^{-\rho} + c_3 e^{i\rho} + c_4 e^{-i\rho} = 0, \\ c_1\rho^2 e^{\rho} + c_2\rho^2 e^{-\rho} - c_3\rho^2 e^{i\rho} - c_4\rho^2 e^{-i\rho} = 0, \\ d_1 e^{\sqrt{i}\rho} + d_2 e^{-\sqrt{i}\rho} = 0, \\ c_1\rho^2 + c_2\rho^2 - c_3\rho^2 - c_4\rho^2 - d_1 - d_2 = 0, \\ c_1 i\rho^3 - c_2 i\rho^3 - c_3\rho^3 + c_4\rho^3 \\ \qquad + d_1\sqrt{i}\rho - d_2\sqrt{i}\rho = 0. \end{cases} \tag{20}$$

Then (18) has the nontrivial solution if and only if the characteristic determinant $\det\Delta(\rho) = 0$, where $\Delta(\rho) =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ e^{\rho} & e^{-\rho} & e^{i\rho} & e^{-i\rho} & 0 & 0 \\ \rho^2 e^{\rho} & \rho^2 e^{-\rho} & -\rho^2 e^{i\rho} & -\rho^2 e^{-i\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & e^{-\sqrt{i}\rho} \\ \rho^2 & \rho^2 & -\rho^2 & -\rho^2 & -1 & -1 \\ i\rho^3 & -i\rho^3 & -\rho^3 & \rho^3 & \sqrt{i}\rho & -\sqrt{i}\rho \end{bmatrix}. \tag{21}$$

*Lemma 2:* Let $\lambda = i\rho^2$ with $\rho \in \mathcal{S}$ and let $\Delta(\rho)$ be given by (21). Then the following asymptotic expansion holds:

$$\begin{aligned} -2^{-1}\rho^{-5}e^{-\rho}\det\Delta(\rho) &= a_1 e^{i\rho}e^{\sqrt{i}\rho} + a_2 e^{i\rho}e^{-\sqrt{i}\rho} \\ &+ a_3 e^{-i\rho}e^{\sqrt{i}\rho} + a_4 e^{-i\rho}e^{-\sqrt{i}\rho} + \mathcal{O}(e^{-|\rho|}), \end{aligned} \tag{22}$$

where

$$\begin{cases} a_1 = 1 + \sqrt{2} + i(1 + \sqrt{2}), \\ a_2 = \sqrt{2} - 1 + i(\sqrt{2} - 1), \\ a_3 = 1 - \sqrt{2} - i(1 + \sqrt{2}), \\ a_4 = -1 - \sqrt{2} - i(\sqrt{2} - 1). \end{cases} \tag{23}$$

Moreover, when $\rho \in \mathcal{S}_1$ and $\rho \in \mathcal{S}_2$, $\det\Delta(\rho)$ has more accurate asymptotic expansions respectively: for $\rho \in \mathcal{S}_1$,

$$-\frac{1}{2}\rho^{-5}e^{-\rho}e^{i\rho}\det\Delta(\rho) = a_3 e^{\sqrt{i}\rho} + a_4 e^{-\sqrt{i}\rho} + \mathcal{O}(e^{-c_1|\rho|}) \tag{24}$$

and for $\rho \in \mathcal{S}_2$,

$$-\frac{1}{2}\rho^{-5}e^{-\rho}e^{-\sqrt{i}\rho}\det\Delta(\rho) = a_1 e^{i\rho} + a_3 e^{-i\rho} + \mathcal{O}(e^{-c_2|\rho|}) \tag{25}$$

where $c_1$ and $c_2$ are positive constants.

*Proof:* Due to the space limitation, we omit the details of the proof here. ■

*Theorem 2:* Let $\mathcal{A}$ be defined by (10). The spectrum $\sigma(\mathcal{A})$ has two families:

$$\sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda}_{2n}, n \in \mathbb{N}\}, \tag{26}$$

where $\lambda_{1n}$ and $\lambda_{2n}$ have the following asymptotic expansions:

$$\begin{cases} \lambda_{1n} = -\left[n\pi + \frac{1}{2}\theta_1\right]^2 + \mathcal{O}(e^{-c_1 n}), \\ \lambda_{2n} = \left[n\pi + \frac{1}{2}\theta_2\right]\ln r \\ \qquad + \frac{1}{4}\left[(2n\pi + \theta_2)^2 - (\ln r)^2\right]i + \mathcal{O}(e^{-c_2 n}), \end{cases} \tag{27}$$

and

$$\begin{cases} \theta_1 = \pi - \arctan 2\sqrt{2}, \quad \theta_2 = \arctan\frac{\sqrt{2}}{2}, \\ r = \frac{\sqrt{3}}{1+\sqrt{2}} < 1, \quad \ln r < 0. \end{cases} \tag{28}$$

Therefore,

$$\mathrm{Re}\lambda_{1n}, \mathrm{Re}\lambda_{2n} \to -\infty, \quad \text{as} \quad n \to \infty. \tag{29}$$

*Proof:* Since this is a direct computation, for the space limitation, we omit the details here. ■

We now get the asymptotic behavior of the eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$ respectively.

*Theorem 3:* Let $\mathcal{A}$ be defined by (10), let $\sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda}_{2n}, n \in \mathbb{N}\}$ be the spectrum of $\mathcal{A}$. Then there are two families of approximate normalized eigenfunctions of $\mathcal{A}$:

(i) One family $\{\Phi_{1n} = (f_{1n}, \lambda_{1n} f_{1n}, h_{1n}), n \in \mathbb{N}\}$, where $\Phi_{1n}$ is the eigenfunction of $\mathcal{A}$ with respect to the eigenvalue $\lambda_{1n}$, has the following asymptotic expression:

$$\begin{bmatrix} f''_{1n}(x) \\ \lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{bmatrix} = \begin{bmatrix} -2\sqrt{i}\left[\varphi_{1n1}(x) + \overline{\varphi_{1n1}(x)}\right] \\ 2\sqrt{i}\left[\varphi_{1n1}(x) - \overline{\varphi_{1n1}(x)}\right] \\ a_3\varphi_{1n2}(x) + a_4\overline{\varphi_{1n2}(x)} \end{bmatrix} \tag{30}$$

where $\varphi_{1n1}(x), \varphi_{1n2}(x)$ have the following forms:

$$\begin{cases} \varphi_{1n1}(x) = e^{i\sqrt{i}[n\pi + \frac{1}{2}\theta_1]x} + \mathcal{O}(e^{-c_1 n}), \\ \varphi_{1n2}(x) = e^{i[n\pi + \frac{1}{2}\theta_1]x} + \mathcal{O}(e^{-c_1 n}), \end{cases} \tag{31}$$

and $a_3, a_4, \theta_1$ are constants given by (23) and (28) respectively;

(ii) The another family $\{\Phi_{2n} = (f_{2n}, \lambda_{2n} f_{2n}, h_{2n}), \overline{\Phi}_{2n} = (\overline{f}_{2n}, \overline{\lambda}_{2n}\overline{f}_{2n}, \overline{h}_{2n}), n \in \mathbb{N}\}$, where $\Phi_{2n}$ and $\overline{\Phi}_{2n}$ are the eigenfunctions of $\mathcal{A}$ with respect to the complex conjugate eigenvalue pairs $\lambda_{2n}$ and $\overline{\lambda}_{2n}$ respectively, has the following asymptotic expression:

$$\begin{bmatrix} f''_{2n} \\ \lambda_{2n} f_{2n} \\ h_{2n} \end{bmatrix} = \begin{bmatrix} \varphi_{2n1} - \varphi_{2n2} \\ +[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}]\varphi_{2n3} \\ i\varphi_{2n2} - i\varphi_{2n1} \\ +i[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}]\varphi_{2n3} \\ \left[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}\right]\varphi_{2n4} \end{bmatrix} \tag{32}$$

where $\varphi_{2nj}(x)$, $j = 1, 2, 3, 4$, are given by

$$\begin{cases} \varphi_{2n1}(x) = e^{\frac{1}{2}[\ln r + (2n\pi + \theta_2)i](1-x)} + \mathcal{O}\left(e^{-c_2 n}\right), \\ \varphi_{2n2}(x) = e^{-\frac{1}{2}[\ln r + (2n\pi + \theta_2)i](1-x)} + \mathcal{O}\left(e^{-c_2 n}\right), \\ \varphi_{2n3}(x) = e^{\frac{1}{2}[i \ln r - (2n\pi + \theta_2)]x} + \mathcal{O}\left(e^{-c_2 n}\right), \\ \varphi_{2n4}(x) = e^{\frac{1}{2}\sqrt{i}[i \ln r - (2n\pi + \theta_2)]x} + \mathcal{O}\left(e^{-c_2 n}\right) \end{cases} \tag{33}$$

and $\theta_2, r$ are constants given by (28).

Since $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$, has the following form:

$$\begin{cases} \mathcal{A}^*(f, g, h) = (-g, f^{(4)}, h''), \forall (f, g, h) \in D(\mathcal{A}) \\ D(\mathcal{A}^*) = \begin{cases} (f, g, h) \in (H^4 \times H_L^2 \times H^2) \cap \mathcal{H} \\ h(1) = f''(1) = 0, \\ g'(0) = h'(0), f''(0) = h(0), \end{cases} \end{cases} \tag{34}$$

$\mathcal{A}^*$ is a discrete operator ([2], p.2354), and $\mathcal{A}^*$ has the same eigenvalues as $\mathcal{A}$ ([10], p.26) with the same algebraic multiplicity for the conjugate eigenvalues ([2], p.2354 or [3], p.10). Moreover the eigenfunctions of $\mathcal{A}^*$ can be deducted as the following result..

*Theorem 4:* Let $\mathcal{A}^*$ be defined by (34), let $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}, \overline{\lambda}_{2n}, n \in \mathbb{N}\}$. Then there are two families of approximate normalized eigenfunctions of $\mathcal{A}^*$:

(i) One family $\{\Psi_{1n} = (f_{1n}, -\lambda_{1n} f_{1n}, h_{1n}), n \in \mathbb{N}\}$, where $\Psi_{1n}$ is the eigenfunction of $\mathcal{A}$ with respect to the eigenvalue $\lambda_{1n}$, has the following asymptotic expression:

$$\begin{bmatrix} f''_{1n}(x) \\ -\lambda_{1n} f_{1n}(x) \\ h_{1n}(x) \end{bmatrix} = \begin{bmatrix} -2\sqrt{i}\left[\varphi_{1n1}(x) + \overline{\varphi_{1n1}(x)}\right] \\ -2\sqrt{i}\left[\varphi_{1n1}(x) - \overline{\varphi_{1n1}(x)}\right] \\ a_3\varphi_{1n2}(x) + a_4\overline{\varphi_{1n2}(x)} \end{bmatrix} \tag{35}$$

where $\varphi_{1n1}(x), \varphi_{1n2}(x)$ are given by (31), and $a_3, a_4$ are constants given by (23);

(ii) The another family $\{\Psi_{2n} = (f_{2n}, -\lambda_{2n} f_{2n}, h_{2n}), \overline{\Psi}_{2n} = (\overline{f}_{2n}, -\overline{\lambda}_{2n}\overline{f}_{2n}, \overline{h}_{2n}), n \in \mathbb{N}\}$, where $\Psi_{2n}$ and $\overline{\Psi}_{2n}$ are the eigenfunctions of $\mathcal{A}$ with respect to the complex conjugate eigenvalue pairs $\lambda_{2n}$ and $\overline{\lambda}_{2n}$ respectively, has the following asymptotic expression:

$$\begin{bmatrix} f''_{2n} \\ -\lambda_{2n} f_{2n} \\ h_{2n} \end{bmatrix} = \begin{bmatrix} \varphi_{2n1} - \varphi_{2n2} \\ +[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}]\varphi_{2n3} \\ i\varphi_{2n1} - i\varphi_{2n2} \\ -i[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}]\varphi_{2n3} \\ \left[r^{\frac{1}{2}} e^{\frac{1}{2}i\theta_2} - r^{-\frac{1}{2}} e^{-\frac{1}{2}i\theta_2}\right]\varphi_{2n4} \end{bmatrix} \tag{36}$$

where $\varphi_{2nj}(x)$, $j = 1, 2, 3, 4$, are given by (33), and $\theta_2, r$ are constants given by (28).

## IV. Riesz basis property and exponential stability

In this section, we show the Riesz basis generation and exponential stability of the system (11). Before going to show the Riesz basis property, we first list the following two results and due to the space limitation, we omit the proofs here.

*Proposition 1:* Let $\mathcal{A}$ be defined by (10). Then all $\lambda \in \sigma(\mathcal{A})$ with sufficiently large moduli are algebraically simple.

*Theorem 5:* Let $\mathcal{A}$ be defined by (10). Then both the root subspaces of $\mathcal{A}$ and $\mathcal{A}^*$ are complete in $\mathcal{H}$, that is, $\mathrm{Sp}(\mathcal{A}^*) = \mathrm{Sp}(\mathcal{A}) = \mathcal{H}$.

To establish the Riesz basis property of the system (11), we recall the following two lemmas:

*Lemma 3:* An approximately normalized sequence $\{e_i\}_{i=1}^{\infty}$ and its approximately normalized biorthogonal sequence $\{e_i^*\}_{i=1}^{\infty}$ are Riesz bases for a Hilbert space $H$ if and only if ([20], pp.27)

a) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are complete in $H$; and

b) both $\{e_i\}_{i=1}^{\infty}$ and $\{e_i^*\}_{i=1}^{\infty}$ are Bessel sequences in $H$, that is, for any $x \in H$, two sequences $\{\langle x, e_i\rangle\}_{i=1}^{\infty}, \{\langle x, e_i^*\rangle\}_{i=1}^{\infty}$ belong to $\ell^2$.

*Lemma 4:* ([16, Lemma 3.2]) Suppose that a sequence $\{\mu_n\}$ satisfies $\sup_{n\geq 1} Re\mu_n < \infty$ and has asymptotics

$$\mu_n = \alpha(n + i\beta \ln n) + \mathcal{O}(1), \quad \alpha \neq 0, \ n = 1, 2, 3, \cdots \tag{37}$$

where $\beta$ is a real number. Then the sequence $\{e^{\mu_n x}\}_{n=1}^{\infty}$ is a Bessel sequence in $L^2(0, 1)$.

*Lemma 5:* Let $\varphi_{1ns}(x)$ and $\varphi_{2nj}(x)$ $s = 1,2$, $j = 1,2,3,4$ be given by (31) and (33) respectively. Then all $\{\varphi_{1ns}(x)\}_{n=1}^{\infty}$ and $\{\varphi_{2nj}(x)\}_{n=1}^{\infty}$, $s = 1,2$, $j = 1,2,3,4$ are Bessel sequences in $L^2(0,1)$.

*Proof:* By (31), if we take $\alpha = i\sqrt{i}\pi, \beta = 0$ and $\alpha = i\pi, \beta = 0$ in $\varphi_{1n1}(x)$ and $\varphi_{1n2}(x)$ respectively, then it follows from Lemma 4 directly that both $\{\varphi_{1n1}(x)\}_{n=1}^{\infty}$ and $\{\varphi_{1n2}(x)\}_{n=1}^{\infty}$ are Bessel sequences in $L^2(0,1)$.

Similarly, by (33), if we take $\alpha = i\pi, \beta = 0$ in $\varphi_{2n1}(x)$, $\alpha = -i\pi, \beta = 0$ in $\varphi_{2n2}(x)$, $\alpha = -\pi, \beta = 0$ in $\varphi_{2n3}(x)$, and $\alpha = -\sqrt{i}\pi, \beta = 0$ in $\varphi_{2n4}(x)$ respectively, then it follows from Lemma 4 directly that $\varphi_{2nj}(x)$, $j = 1,2,3,4$ are Bessel sequences in $L^2(0,1)$. The proof is complete. ∎

Now we can establish the Riesz basis property of the system (11).

*Theorem 6:* Let $\mathcal{A}$ be defined by (10). Then the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis for $\mathcal{H}$.

*Proof:* Let $\sigma(\mathcal{A}) = \{\lambda_{1n}, \lambda_{2n}, \overline{\lambda}_{2n}\}_{n=1}^{\infty}$ be the eigenvalues of $\mathcal{A}$. By Theorem 2 and Proposition 1, we have that each eigenvalue of $\mathcal{A}$ with sufficient large modulus is simple, and hence there exists an integer $N > 0$ such that all $\lambda_{1n}, \lambda_{2n}, \overline{\lambda}_{2n}$ with $n \geq N$, are algebraically simple. For $n \leq N$, if the algebraic multiplicity of each $\lambda_{sn}$ is $m_{sn}$, $s = 1,2$, we can find the highest order generalized eigenfunction $\Phi_{s,n,1}$ from

$$(\mathcal{A} - \lambda_{sn})^{m_{sn}}\Phi_{s,n,1} = 0 \;\text{ and }\; (\mathcal{A} - \lambda_{sn})^{m_{sn}-1}\Phi_{s,n,1} \neq 0.$$

The other lower order linearly independent generalized eigenfunctions associated with $\lambda_{sn}$ can be found through $\Phi_{s,n,j} = (\mathcal{A} - \lambda_{sn})^{j-1}\Phi_{s,n,1}$, $j = 2,3,\cdots,m_{sn}$. Assume $\Phi_{s,n}$ is an eigenfunction of $\mathcal{A}$ corresponding to $\lambda_{sn}$ with $n \geq N$. Then

$$\left\{\left\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Phi_{s,n}\right\}_{n\geq N}\right\}_{s=1}^{2}$$
$$\bigcup\left\{\left\{\{\overline{\Phi_{2,n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\overline{\Phi_{2,n}}\right\}_{n\geq N}\right\}$$

are all linearly independent generalized eigenfunctions of $\mathcal{A}$. Let $\{\{\Psi_{s,n,j}\}_{j=1}^{m_{sn}}\}_{n<N} \cup \{\Psi_{s,n}\}_{n\geq N}$ be the bi-orthogonal sequence of $\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\}_{n<N}\} \cup \{\Phi_{s,n}\}_{n\geq N}$. Then

$$\left\{\left\{\{\Psi_{s,n,j}\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Psi_{s,n}\right\}_{n\geq N}\right\}_{s=1}^{2}$$
$$\bigcup\left\{\left\{\{\overline{\Psi_{2,n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\Psi_{2,n}\right\}_{n\geq N}\right\}$$

are all linearly independent generalized eigenfunctions of $\mathcal{A}^*$. It is well-known that these two sequences are minimal in $\mathcal{H}$ and from Theorem 5, they are also complete in $\mathcal{H}$.

Hence, in order to prove the Riesz basis of the system, it suffices to show that both eigenfunctions $\{\Phi_{s,n}\}_{n\geq N, s=1,2}$ and $\{\Psi_{s,n}\}_{n\geq N, s=1,2}$ of $\mathcal{A}$ and $\mathcal{A}^*$ respectively, are Bessel sequences in $\mathcal{H}$. Since $1 \leq \|\Phi_{s,n}\|\|\Psi_{s,n}\| \leq M$ for some constant $M$ independent of $n$ (see [20, p.19]), we may assume without loss of generality that $\Phi_{s,n} = (f_{sn}, \lambda_{sn}f_{sn}, h_{sn})$ and $\Psi_{s,n} = (f_{sn}, -\lambda_{sn}f_{sn}, h_{sn})$ given by (30), (32) and (35), (36) respectively, for all $s = 1,2, n \geq N$. It then follows from Lemma 5 and the expansions of (30), (32) and (35), (36) that all of $\{f''_{sn}\}_{n=N}^{\infty}, \{\pm\lambda_{sn}f_{sn}\}_{n=N}^{\infty}$ and $\{h_{sn}\}_{n=N}^{\infty}$, $s = 1,2$, are Bessel sequences in $L^2(0,1)$. Therefore both of

$$\{\Phi_{s,n}\}_{n\geq N, s=1,2}, \quad \{\Psi_{s,n}\}_{n\geq N, s=1,2}$$

are also Bessel sequences in $\mathcal{H}$ and the result follows. ∎

*Theorem 7:* Let $\mathcal{A}$ be defined by (10). Then the spectrum-determined growth condition $\omega(\mathcal{A}) = s(\mathcal{A})$ holds true for the $C_0$-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. Moreover, the system (11) is exponentially stable, that is, there exist two positive constants $M$ and $\omega$ such that the $C_0$-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ satisfies

$$\|e^{\mathcal{A}t}\| \leq Me^{-\omega t}. \tag{38}$$

*Proof:* The spectrum-determined growth condition follows from Theorem 6. By Lemma 1, for each $\lambda \in \sigma(\mathcal{A})$, we have $\mathrm{Re}\lambda < 0$. This, together with (26)-(29) and the spectrum-determined growth condition, shows that $e^{\mathcal{A}t}$ is exponentially stable. The proof is complete. ∎

## V. Gevrey Regularity

In what follows, we show that the $C_0$-semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ is of a Gevrey class $\delta$ with any $\delta > 2$. We recall the definition.

*Definition 1:* ([1], [17]) A $C_0$-semigroup $T(t)$ is of a Gevrey class $\delta > 1$ for $t > t_0$ if $T(t)$ is infinitely differentiable for $t > t_0$ and for every compact subset $K \subset (t_0, \infty)$ and each $\theta > 0$, there is a constant $C = C(K,\theta)$ such that

$$\|T^{(n)}(t)\| \leq C\theta^n(n!)^\delta, \quad \forall t \in K,\ n = 0,1,2,\ldots.$$

In order to get the Gevrey regularity of the system (11), we need the following theorem established by Taylor in [17, Theorem 4, Chapter 5].

*Theorem 8:* Let $e^{\mathcal{A}t}$ be a $C_0$-semigroup satisfying $\|e^{\mathcal{A}t}\| \leq Me^{\omega t}$. Suppose that for some $\mu \geq \omega$ and $\alpha$ satisfying $0 < \alpha \leq 1$,

$$\lim_{|\tau|\to\infty} \sup|\tau|^\alpha\|R(\mu + i\tau, \mathcal{A})\| = C < \infty, \quad \tau \in \mathbb{R}.$$

Then $e^{\mathcal{A}t}$ is of Gevrey class $\delta$ with $\delta > 1/\alpha$ for $t > 0$.

Now we establish the Gevrey regularity of the system (11).

*Theorem 9:* Let $\mathcal{A}$ be defined by (10). Then the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta > 2$ with $t_0 = 0$.

*Proof:* From Theorem 7, $\mathcal{A}$ generates a exponentially stable $C_0$-semigroup $e^{\mathcal{A}t}$ in $\mathcal{H}$. So, by Theorem 8, we only need to show

$$\lim_{|\tau|\to\infty} \|R(i\tau, \mathcal{A})\|^2 = \frac{C}{|\tau|} < \infty, \quad \tau \in \mathbb{R}. \tag{39}$$

By Theorem 6,

$$\left\{\left\{\{\Phi_{s,n,j}\}_{j=1}^{m_{sn}}\right\}_{n<N} \cup \left\{\Phi_{s,n}\right\}_{n\geq N}\right\}_{s=1}^{2}$$
$$\bigcup\left\{\left\{\{\overline{\Phi_{2,n,j}}\}_{j=1}^{m_{2n}}\right\}_{n<N} \cup \left\{\overline{\Phi_{2,n}}\right\}_{n\geq N}\right\}$$

forms a Riesz basis in $\mathcal{H}$. Then for each $Y \in \mathcal{H}$, we have

$$Y = \sum_{n=1}^{N-1} \sum_{s=1}^{2} \sum_{j=1}^{m_{sn}} a_{s,n,j} \Phi_{s,n,j} + \sum_{n=N}^{\infty} \sum_{s=1}^{2} a_{s,n} \Phi_{s,n} + \sum_{n=1}^{N-1} \sum_{j=1}^{m_{2n}} b_{2,n,j} \overline{\Phi_{2,n,j}} + \sum_{n=N}^{\infty} b_{2,n} \overline{\Phi_{2,n}}, \tag{40}$$

and

$$\|Y\|^2 \asymp \sum_{n=1}^{N-1} \sum_{s=1}^{2} \sum_{j=1}^{m_{sn}} |a_{s,n,j}|^2 + \sum_{n=N}^{\infty} \sum_{s=1}^{2} |a_{s,n}|^2 + \sum_{n=1}^{N-1} \sum_{j=1}^{m_{2n}} |b_{2,n,j}|^2 + \sum_{n=N}^{\infty} |b_{2,n}|^2. \tag{41}$$

Let $\tau \in \mathbb{R}$ and $\tau > 0$. Then we have $i\tau \in \rho(\mathcal{A})$, and, in addition,

$$\begin{aligned} &R(i\tau, \mathcal{A})Y \\ &= \sum_{n=1}^{N-1} \sum_{s=1}^{2} \sum_{j=1}^{m_{sn}} \frac{a_{s,n,j} \Phi_{s,n,j}}{i\tau - \lambda_{sn}} + \sum_{n=N}^{\infty} \sum_{s=1}^{2} \frac{a_{s,n} \Phi_{s,n}}{i\tau - \lambda_{sn}} \\ &+ \sum_{n=1}^{N-1} \sum_{j=1}^{m_{sn}} \frac{b_{2,n,j} \overline{\Phi_{2,n,j}}}{i\tau - \overline{\lambda_{2n}}} + \sum_{n=N}^{\infty} \frac{b_{2,n} \overline{\Phi_{2,n}}}{i\tau - \overline{\lambda_{2n}}} \\ &+ \sum_{n=1}^{N-1} \sum_{s=1}^{2} \mathcal{O}\left( \frac{1}{|i\tau - \lambda_{sn}|^2} \right) + \sum_{n=1}^{N-1} \mathcal{O}\left( \frac{1}{|i\tau - \overline{\lambda_{2n}}|^2} \right) \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\|R(i\tau, \mathcal{A})Y\|^2 \\ &\asymp \sum_{n=1}^{N-1} \sum_{s=1}^{2} \sum_{j=1}^{m_{sn}} \frac{|a_{s,n,j}|^2}{|i\tau - \lambda_{sn}|^2} + \sum_{n=N}^{\infty} \sum_{s=1}^{2} \frac{|a_{s,n}|^2}{|i\tau - \lambda_{sn}|^2} \\ &+ \sum_{n=1}^{N-1} \sum_{j=1}^{m_{2n}} \frac{|b_{2,n,j}|^2}{|i\tau - \overline{\lambda_{2n}}|^2} + \sum_{n=N}^{\infty} \frac{|b_{2,n}|^2}{|i\tau - \overline{\lambda_{2n}}|^2}, \end{aligned} \tag{43}$$

where $\{\lambda_{1n}, n \in \mathbb{N}\}$ and $\{\lambda_{2n}, \overline{\lambda_{2n}}, n \in \mathbb{N}\}$, given by (27), are eigenvalues of $\mathcal{A}$.

A direct computation yields that there is an $M > 0$ such that

$$\lim_{\tau \to \infty} \|R(i\tau, \mathcal{A})\|^2 = \frac{M}{|\tau|} < \infty. \tag{44}$$

On the other hand, when $\tau \in \mathbb{R}$ and $\tau < 0$, the same argument yields

$$\lim_{\tau \to -\infty} \|R(i\tau, \mathcal{A})\|^2 = \frac{M}{|\tau|} < \infty. \tag{45}$$

Therefore, this together with (44) yields (39), and by Theorem 8, the semigroup $e^{\mathcal{A}t}$, generated by $\mathcal{A}$, is of a Gevrey class $\delta > 2$ with $t_0 = 0$. The proof is complete. ■

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