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State Estimation for a Class of Nonlinear Differential Games using Differential Neural Networks

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***Abstract*—This paper deals with the problem of the state estimation for a certain class of nonlinear differential games, where the mathematical model of this class is completely unknown. Being thus, a Luenberger-like differential neural network observer is applied and a new learning law for its synaptic weights is suggested. Furthermore, by means of a Lyapunov stability analysis, the stability conditions for the state estimation error are established and the upper bound of this error is obtained. Finally, a numerical example illustrates the applicability of this approach.**

***Index Terms*—Differential games, dynamic neural networks, state observers.**

I. INTRODUCTION

NOWADAYS, an investigation field that has been developed widely is the design of controllers for certain systems that own conflicting interaction to each other, which can be modeled by means of the *game theory* (see [1]).

Nevertheless, the most of recent publications about the games and, particularly about the *differential games*, are based on the complete knowledge of the mathematical model that describes its dynamics (see [3] and [9]), which it is not always the case.

On the other hand, when the exact and complete knowledge on current states of a dynamic plant is impossible by different reasons, the use of a state estimator (observer) is compulsory to achieve a successful closed-loop control [7].

Being thus, the so-called *differential neural networks* (or continuous-time dynamic neural networks) have proved to be an excellent tool on the identification, state estimation and control of several systems (see [2], [6] and [8]) and, specifically, on a certain type of differential games (see [4] and [5]).

Therefore, the idea of designing a differential neural network that models a certain class of nonlinear differential games and estimates its state variables is a new approach that, as far as authors know, has not been treated.

So, the main goal of this paper is to show the modeling and the state observation of a class of nonlinear differential games through the designing of a *differential neural network observer*.

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More specifically, this differential neural network observer has a Luenberger-like structure and, by means of the Lyapunov's Second Method of Stability, a new *learning law* for its synaptic weights is obtained and a bounded stable *state estimation error* is inferred.

II. CLASS OF NONLINEAR DIFFERENTIAL GAMES

A. Definition

Consider a class of nonlinear differential games given by the following equations:

$$\dot{x}_t = f(t, x_t) + g^i(t, x_t)u_t{}^i \\ y_t = C x_t \tag{1}$$

where $i = 1,2,\dots,N$ denotes the number of players, $x_t \in \Re^n$ is the game state, $u_t{}^i \in \Re^{s_i}$ is the control action for the i player, $f:[0,\infty) \times \Re^n \to \Re^n$ and $g^i:[0,\infty) \times \Re^n \to \Re^{n\times s_i}$ are unknown functions, $y_t \in \Re^m$ is the game output and $C \in \Re^{m\times n}$ is a known constant matrix characterizing the state-output mapping.

B. Assumptions

In addition to the above, it is assumed that:

Assumption 1 *The control actions $u_t{}^i$ are bounded and measurable for all time t, that is:*

$$\|u_t{}^i\| \le \bar{u}^i < \infty \tag{2}$$

where $\bar{u}^i \in \Re$ is a known constant.

Assumption 2 *The functions $f(\cdot,\cdot)$ and $g^i(\cdot,\cdot)$ satisfy the Lipschitz condition, that is, there are constants $\varepsilon, \varepsilon^i > 0$ such that the equations:*

$$\|f(t,x_t) - f(t,\bar{x}_t)\| \le \varepsilon\|x_t - \bar{x}_t\| \\ \|g^i(t,x_t) - g^i(t,\bar{x}_t)\| \le \varepsilon^i\|x_t - \bar{x}_t\| \tag{3}$$

are fulfilled for all $t \in [0,\infty)$ and all $x_t, \bar{x}_t \in \Re^n$.

Assumption 3 *The class of nonlinear differential games given by (1) is stable in the sense of Lyapunov, that is, there exists a nonnegative function $E:[0,\infty) \times \Re^n \to \Re$ with continuous partial derivatives such that $\dot{E} \le 0$.*

Assumption 4 *The class of nonlinear differential games given by (1) is locally observable at a point x_ε, that is, the equation:*

$$\text{rank}\left(\frac{\partial}{\partial x_t}\begin{bmatrix} h(x_t) \\ L_f h(x_t) \\ L_f{}^2 h(x_t) \\ \vdots \\ L_f{}^{n-1} h(x_t) \end{bmatrix}\Bigg|_{x_s}\right) = n \tag{4a}$$

where n is the dimension of x_t, $h(x_t) = Cx_t$ and

$$\begin{aligned} L_f h(x_t) &= \left[\frac{\partial}{\partial x_t} h(x_t)\right] f(t, x_t) \\ L_f{}^2 h(x_t) &= \left[\frac{\partial}{\partial x_t} L_f h(x_t)\right] f(t, x_t) \\ &\vdots \\ L_f{}^{n-1} h(x_t) &= \left[\frac{\partial}{\partial x_t} L_f{}^{n-2} h(x_t)\right] f(t, x_t) \end{aligned} \tag{4b}$$

is satisfied.

III. Differential Neural Network Observer

A. Definition

Consider a Luenberger-like differential neural network observer given by the following equations:

$$\begin{aligned} \dot{\hat{x}}_t &= W_{1,t}\sigma(V_{1,t}\hat{x}_t) + W_{2,t}{}^i\varphi^i(V_{2,t}{}^i\hat{x}_t)u_t{}^i \\ &\quad + K[y_t - \hat{y}_t] \\ \hat{y}_t &= C\hat{x}_t \end{aligned} \tag{5}$$

where $\hat{x}_t \in \Re^n$ is the observer state; $W_{1,t} \in \Re^{n\times k}$, $V_{1,t} \in \Re^{k\times n}$, $W_{2,t}{}^i \in \Re^{n\times r_i}$ and $V_{2,t}{}^i \in \Re^{r_i\times n}$ are synaptic weights; $\sigma: \Re^k \to \Re^k$ and $\varphi^i: \Re^{r_i} \to \Re^{r_i\times s_i}$ are activation functions; $K \in \Re^{n\times m}$ is the observer gain matrix; and $\hat{y}_t \in \Re^m$ is the observer output.

B. Design Conditions

The design conditions of the Luenberger-like differential neural network observer given by (5) are shown below:

Definition 1 *The modeling error is defined as:*

$$\begin{aligned} \Delta f_t &:= [f(t, x_t) + g^i(t, x_t)u_t{}^i] \\ &\quad - [W_{1,0}\sigma(V_{1,0}x_t) + W_{2,0}{}^i\varphi^i(V_{2,0}{}^i x_t)u_t{}^i] \end{aligned} \tag{6}$$

where $W_{1,0}$, $V_{1,0}$, $W_{2,0}{}^i$ and $V_{2,0}{}^i$ are the initial synaptic weights, that is to say, when $t = 0$.

Assumption 5 *The modeling error given by (6) is bounded and satisfies the following condition:*

$$\|\Delta f_t\| \le f_1 + f_2\bar{u}^i \tag{7}$$

where $f_1, f_2 \in \Re$ are positive constants.

Assumption 6 *There are values of Λ_1, Λ_2, $\Lambda_3{}^i$, $\Lambda_4{}^i$, Λ_5, Λ_6, $\Lambda_7{}^i$, $\Lambda_8{}^i$, Λ_σ, $\Lambda_\varphi{}^i$, Λ_f, A, Q, l_1, $l_2{}^i$ and δ, such that they provide a $P = P^T > 0$ solution to the following algebraic Riccati equation:*

$$P\bar{A} + \bar{A}^T P + PRP + \bar{Q} = 0 \tag{8}$$

where

$$\bar{A} = W_{1,0}A - KC \tag{9}$$

$$\begin{aligned} R &= W_{1,0}(\Lambda_1{}^{-1} + \Lambda_2{}^{-1})W_{1,0}{}^T + \Lambda_f{}^{-1} \\ &\quad + W_{2,0}{}^i\left(\Lambda_3{}^{i^{-1}} + \Lambda_4{}^{i^{-1}}\right)W_{2,0}{}^{i^T} \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{Q} &= \Lambda_\sigma + \delta(\Lambda_5 + \Lambda_6) + Q \\ &\quad + \Lambda_\varphi{}^i\bar{u}^i + \delta(\Lambda_7{}^i + \Lambda_8{}^i) \end{aligned} \tag{11}$$

the positive definite symmetric matrices Λ_1, Λ_2, $\Lambda_3{}^i$, $\Lambda_4{}^i$, Λ_5, Λ_6, $\Lambda_7{}^i$, $\Lambda_8{}^i$, Λ_σ, $\Lambda_\varphi{}^i$, Λ_f and Q are known constants of adequate dimensions, the scalars l_1, $l_2{}^i$ and δ are known positive constants, and A is a constant matrix such that $\bar{A} \in \Re^{n\times n}$ is Hurtwitz.

Assumption 7 *The activation functions $\sigma(\cdot)$ and $\varphi^i(\cdot)$ are sigmoid functions, that is, they are bounded and satisfy the following conditions:*

$$\begin{aligned} [\tilde{\sigma} - A(\hat{x}_t - x_t)]^T\Lambda_2[\tilde{\sigma} - A(\hat{x}_t - x_t)] &\le \\ [\hat{x}_t - x_t]^T\Lambda_\sigma[\hat{x}_t - x_t] & \\ [\tilde{\varphi}^i u_t{}^i]^T\Lambda_3{}^i[\tilde{\varphi}^i u_t{}^i] \le [\hat{x}_t - x_t]^T\Lambda_\varphi{}^i[\hat{x}_t - x_t]\bar{u}^i & \end{aligned} \tag{12}$$

$$\begin{aligned} \tilde{\sigma}' &= D_\sigma\tilde{V}_{1,t}\hat{x}_t + \eta_\sigma \\ \tilde{\varphi}^{i'}u_t{}^i &= \sum_{j=1}^{s}(D_\varphi{}^i)_j\tilde{V}_{2,t}{}^i\hat{x}_t(u_t{}^i)_j + \eta_\varphi{}^i u_t{}^i \end{aligned} \tag{13}$$

$$\begin{aligned} \eta_\sigma{}^T\Lambda_1\eta_\sigma &\le l_1[\tilde{V}_{1,t}\hat{x}_t]^T\Lambda_{\eta_\sigma}[\tilde{V}_{1,t}\hat{x}_t] \\ [\eta_\varphi{}^i u_t{}^i]^T\Lambda_4{}^i[\eta_\varphi{}^i u_t{}^i] &\le \\ l_2{}^i\left[\tilde{V}_{2,t}{}^i\hat{x}_t\right]^T\Lambda_{\eta_\varphi}{}^i\left[\tilde{V}_{2,t}{}^i\hat{x}_t\right]\bar{u}^i & \end{aligned} \tag{14}$$

where the positive definite symmetric matrices Λ_{η_σ} and $\Lambda_{\eta_\varphi}{}^i$ are known constants of adequate dimensions, and:

$$\begin{aligned} \tilde{\sigma} &:= \sigma(V_{1,0}\hat{x}_t) - \sigma(V_{1,0}x_t) \\ \tilde{\varphi}^i &:= \varphi^i(V_{2,0}{}^i\hat{x}_t) - \varphi^i(V_{2,0}{}^i x_t) \end{aligned} \tag{15}$$

$$\begin{aligned} \tilde{\sigma}' &:= \sigma(V_{1,t}\hat{x}_t) - \sigma(V_{1,0}\hat{x}_t) \\ \tilde{\varphi}^{i'} &:= \varphi^i(V_{2,t}{}^i\hat{x}_t) - \varphi^i(V_{2,0}{}^i\hat{x}_t) \end{aligned} \tag{16}$$

$$\begin{aligned} D_\sigma &:= \frac{\partial}{\partial z}\sigma(z)\Big|_{z=V_{1,t}\hat{x}_t} \in \Re^{k\times k} \\ (D_\varphi{}^i)_j &:= \frac{\partial}{\partial z}\left(\varphi^i(z)\right)_j\Big|_{z=V_{2,t}{}^i\hat{x}_t} \in \Re^{r\times r} \end{aligned} \tag{17}$$

$$
\begin{aligned}
\tilde{W}_{1,t} &:= W_{1,t} - W_{1,0} \\
\tilde{V}_{1,t} &:= V_{1,t} - V_{1,0} \\
\tilde{W}_{2,t}{}^{i} &:= W_{2,t}{}^{i} - W_{2,0}{}^{i} \\
\tilde{V}_{2,t}{}^{i} &:= V_{2,t}{}^{i} - V_{2,0}{}^{i}
\end{aligned} \tag{18}
$$

IV. Main Result on State Estimation for the Class of Nonlinear Differential Games

The main result on the state estimation for the class of nonlinear differential games (1), deals with both the development of an adaptive learning law for the synaptic weights of the differential neural network observer (5) and the inference of a maximum value of state estimation error, that is to say, an error between the real states and the estimated ones.

More formally, the main obtained result is described in the next theorem:

Theorem 1 *Let the differential neural network observer described at (5) be the one that makes the modeling and the state estimation of the class of nonlinear differential games given by (1). If the Assumptions 1-7 are fulfilled and the synaptic weights of the observer are adjusted with the following learning law:*

$$
\begin{aligned}
\dot{\tilde{W}}_{1,t} = -K_1 P N_\delta \Big[2C^T e_t + \delta\left[\Lambda_5{}^{-1}\right]^T N_\delta{}^T \\
P\tilde{W}_{1,t}\sigma\left(V_{1,t}\hat{x}_t\right)\Big]\left[\sigma\left(V_{1,t}\hat{x}_t\right)\right]^T
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\dot{\tilde{V}}_{1,t} = -K_2\big[l_1\Lambda_{\eta_\sigma}{}^T + D_\sigma{}^T W_{1,0}{}^T P N_\delta \\
\left[2C^T e_t + \delta\left[\Lambda_6{}^{-1}\right]^T N_\delta{}^T P W_{1,0} D_\sigma\right]\big]\tilde{V}_{1,t}\hat{x}_t\hat{x}_t{}^T
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\dot{\tilde{W}}_{2,t}{}^{i} = -K_3{}^{i} P N_\delta \Big[2C^T e_t + \delta\left[\Lambda_7{}^{i^{-1}}\right]^T N_\delta{}^T \\
P\tilde{W}_{2,t}{}^{i}\varphi^{i}\left(V_{2,t}{}^{i}\hat{x}_t\right)u_t{}^{i}\Big][u_t{}^{i}]^T\left[\varphi^{i}\left(V_{2,t}{}^{i}\hat{x}_t\right)\right]^T
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\dot{\tilde{V}}_{2,t}{}^{i} = -K_4{}^{i}\Big[l_2{}^{i}\bar{u}^{i}\left[\Lambda_{\eta_\varphi}{}^{i}\right]^T + \left[\left(D_\varphi{}^{i}u_t{}^{i}\right)_j\right]^T \\
\left[W_{2,0}{}^{i}\right]^T P N_\delta\Big[2C^T e_t + \delta\left[\Lambda_8{}^{i^{-1}}\right]^T N_\delta{}^T P W_{2,0}{}^{i} \\
\left(D_\varphi{}^{i}u_t{}^{i}\right)_j\Big]\Big]\tilde{V}_{2,t}{}^{i}\hat{x}_t\hat{x}_t{}^T
\end{aligned} \tag{22}
$$

where

$$
\begin{aligned}
e_t &:= \hat{y}_t - y_t \\
N_\delta &:= \left(C^T C + \delta I\right)^{-1}
\end{aligned} \tag{23}
$$

I is the identity matrix and K_1, K_2, $K_3{}^{i}$ and $K_4{}^{i}$ are known positive constants, then, it is possible to obtain the next maximum value of state estimation error "in average sense":

$$
\limsup_{t\to\infty}\frac{1}{t}\int_0^t \Delta_t{}^T Q\Delta_t\,dt \le f_1 + f_2\bar{u}^{i} \tag{24}
$$

where

$$
\Delta_t := \hat{x}_t - x_t \tag{25}
$$

Proof: By representing the nonlinear differential game given by (1) in terms of a differential neural network, the following is obtained:

$$
\dot{x}_t = W_{1,0}\sigma\left(V_{1,0}x_t\right) + W_{2,0}{}^{i}\varphi^{i}\left(V_{2,0}{}^{i}x_t\right)u_t{}^{i} + \Delta f_t \tag{26}
$$

Now, by substituting (5) and (26) in the derivative of Δ_t with respect to t, the following is obtained:

$$
\begin{aligned}
\dot{\Delta}_t = W_{1,t}\sigma\left(V_{1,t}\hat{x}_t\right) + W_{2,t}{}^{i}\varphi^{i}\left(V_{2,t}{}^{i}\hat{x}_t\right)u_t{}^{i} \\
+K\left[y_t - \hat{y}_t\right] - W_{1,0}\sigma\left(V_{1,0}x_t\right) \\
-W_{2,0}{}^{i}\varphi^{i}\left(V_{2,0}{}^{i}x_t\right)u_t{}^{i} - \Delta f_t
\end{aligned} \tag{27}
$$

Then, by proposing the following Lyapunov candidate function:

$$
\begin{aligned}
V_t := \Delta_t{}^T P\Delta_t + \frac{1}{2}\operatorname{tr}\left\{\tilde{W}_{1,t}{}^T K_1{}^{-1}\tilde{W}_{1,t}\right\} \\
+\frac{1}{2}\operatorname{tr}\left\{\tilde{V}_{1,t}{}^T K_2{}^{-1}\tilde{V}_{1,t}\right\} \\
+\frac{1}{2}\operatorname{tr}\left\{\left[W_{2,t}{}^{i}\right]^T\left[K_3{}^{i}\right]^{-1}\tilde{W}_{2,t}{}^{i}\right\} \\
+\frac{1}{2}\operatorname{tr}\left\{\left[V_{2,t}{}^{i}\right]^T\left[K_4{}^{i}\right]^{-1}\tilde{V}_{2,t}{}^{i}\right\}
\end{aligned} \tag{28}
$$

and by calculating its derivative with respect to t, the following is obtained:

$$
\begin{aligned}
\dot{V}_t = 2\Delta_t{}^T P\dot{\Delta}_t + \operatorname{tr}\left\{\dot{\tilde{W}}_{1,t}{}^T K_1{}^{-1}\tilde{W}_{1,t}\right\} \\
+\operatorname{tr}\left\{\dot{\tilde{V}}_{1,t}{}^T K_2{}^{-1}\tilde{V}_{1,t}\right\} \\
+\operatorname{tr}\left\{\left[\dot{\tilde{W}}_{2,t}{}^{i}\right]^T\left[K_3{}^{i}\right]^{-1}\tilde{W}_{2,t}{}^{i}\right\} \\
+\operatorname{tr}\left\{\left[\dot{\tilde{V}}_{2,t}{}^{i}\right]^T\left[K_4{}^{i}\right]^{-1}\tilde{V}_{2,t}{}^{i}\right\}
\end{aligned} \tag{29}
$$

By substituting (27) in (29) and by adding and subtracting $\Delta_t{}^T Q\Delta_t$, the following is obtained:

$$
\begin{aligned}
\dot{V}_t = 2\Delta_t{}^T P\tilde{W}_{1,t}\sigma\left(V_{1,t}\hat{x}_t\right) + 2\Delta_t{}^T P W_{1,0}\tilde{\sigma}' \\
+2\Delta_t{}^T P W_{1,0}\left[\tilde{\sigma} - A\Delta_t\right] + 2\Delta_t{}^T P\left[W_{1,0}A - KC\right]\Delta_t \\
+2\Delta_t{}^T P\tilde{W}_{2,t}{}^{i}\varphi^{i}\left(V_{2,t}{}^{i}\hat{x}_t\right)u_t{}^{i} + 2\Delta_t{}^T P W_{2,0}{}^{i}\tilde{\varphi}^{i}u_t{}^{i}
\end{aligned} \tag{30}
$$

$$+2\Delta_t^T P W_{2,0}{}^i \tilde{\varphi}^{i'} u_t{}^i - 2\Delta_t^T P \Delta f_t \pm \Delta_t^T Q \Delta_t$$
$$+\mathrm{tr}\left\{\dot{\tilde{W}}_{1,t}{}^T K_1{}^{-1} \tilde{W}_{1,t}\right\} + \mathrm{tr}\left\{\dot{\tilde{V}}_{1,t}{}^T K_2{}^{-1} \tilde{V}_{1,t}\right\}$$
$$+\mathrm{tr}\left\{\left[\dot{\tilde{W}}_{2,t}{}^i\right]^T \left[K_3{}^i\right]^{-1} \tilde{W}_{2,t}{}^i\right\}$$
$$+\mathrm{tr}\left\{\left[\dot{\tilde{V}}_{2,t}{}^i\right]^T \left[K_4{}^i\right]^{-1} \tilde{V}_{2,t}{}^i\right\}$$

Now, by analyzing the first eight terms of (30) with the following inequality:

$$X^T Y + Y^T X \le X^T \Lambda^{-1} X + Y^T \Lambda Y \tag{31}$$

which is valid for any pair of matrices $X, Y \in \Re^{a\times b}$ and for any symmetric matrix $0 < \Lambda \in \Re^{a\times a}$ (a and b are positive integers), the following is obtained:

$$\dot{V}_t \le \Delta_t^T [P\bar{A} + \bar{A}^T P + PRP + \bar{Q}]\Delta_t + \mathrm{tr}\{L_{W_1}\} + \mathrm{tr}\{L_{V_1}\} + \mathrm{tr}\{L_{W_2}{}^i\} + \mathrm{tr}\{L_{V_2}{}^i\} - \Delta_t^T Q \Delta_t + f_1 + f_2 \bar{u}^i \tag{32}$$

where

$$L_{W_1} = \dot{\tilde{W}}_{1,t}{}^T K_1{}^{-1} \tilde{W}_{1,t} + \sigma(V_{1,t}\hat{x}_t)[2e_t^T C + \delta[\sigma(V_{1,t}\hat{x}_t)]^T \tilde{W}_{1,t}{}^T P N_\delta I \Lambda_5{}^{-1} I] N_\delta{}^T P \tilde{W}_{1,t} \tag{33}$$

$$L_{V_1} = \dot{\tilde{V}}_{1,t}{}^T K_2{}^{-1} \tilde{V}_{1,t} + \hat{x}_t\Big[l_1 \hat{x}_t^T \tilde{V}_{1,t}{}^T \Lambda_{\eta_\sigma} + \left[2C^T e_t + \delta \hat{x}_t^T \tilde{V}_{1,t}{}^T D_\sigma{}^T W_{1,0}{}^T P N_\delta I \Lambda_6{}^{-1} I\right] N_\delta{}^T P W_{1,0} D_\sigma\Big]\tilde{V}_{1,t} \tag{34}$$

$$L_{W_2}{}^i = \left[\dot{\tilde{W}}_{2,t}{}^i\right]^T \left[K_3{}^i\right]^{-1} \tilde{W}_{2,t}{}^i + \varphi^i(V_{2,t}{}^i\hat{x}_t)u_t{}^i \left[2e_t^T C + \delta[u_t{}^i]^T [\varphi^i(V_{2,t}{}^i\hat{x}_t)]^T \left[\tilde{W}_{2,t}{}^i\right]^T P N_\delta I \Lambda_7{}^{i^{-1}} I\right] N_\delta{}^T P \tilde{W}_{2,t}{}^i \tag{35}$$

$$L_{V_2}{}^i = \left[\dot{\tilde{V}}_{2,t}{}^i\right]^T \left[K_4{}^i\right]^{-1} \tilde{V}_{2,t}{}^i + \hat{x}_t\Big[l_2{}^i \bar{u}^i \hat{x}_t{}^T \left[\tilde{V}_{2,t}{}^i\right]^T \Lambda_{\eta_\varphi}{}^i + \left[2C^T e_t + \delta \hat{x}_t{}^T \left[\tilde{V}_{2,t}{}^i\right]^T \left[\left(D_\varphi{}^i u_t{}^i\right)_j\right]^T [W_{2,0}{}^i]^T P N_\delta I \Lambda_8{}^{i^{-1}} I\right] N_\delta{}^T P W_{2,0}{}^i \left(D_\varphi{}^i u_t{}^i\right)_j\Big] \tilde{V}_{2,t}{}^i \tag{36}$$

By equating (33)-(36) to zero, that is:

$$L_{W_1} = L_{V_1} = L_{W_2}{}^i = L_{V_2}{}^i = 0 \tag{37}$$

and by respectively solving for $\dot{W}_{1,t}$, $\dot{V}_{1,t}$, $\dot{W}_{2,t}{}^i$ and $\dot{V}_{2,t}{}^i$, the learning law described at (19)-(22) is obtained.

Thus, by solving the algebraic Riccati equation (8) in the first term of (32), the following is obtained:

$$\dot{V}_t \le -\Delta_t^T Q \Delta_t + f_1 + f_2 \bar{u}^i \tag{38}$$

By integrating both sides of (38) on the time interval $[0, t]$, the following is obtained:

$$\int_0^t \Delta_t^T Q \Delta_t \, dt \le V_0 - V_t + (f_1 + f_2 \bar{u}^i)t \le V_0 + (f_1 + f_2 \bar{u}^i)t \tag{39a}$$

Finally, by dividing (39) by t, that is:

$$\frac{1}{t}\int_0^t \Delta_t^T Q \Delta_t \, dt \le \frac{V_0}{t} + f_1 + f_2 \bar{u}^i \tag{39b}$$

and by calculating the upper limit as $t \to \infty$, the maximum value of state estimation error "in average sense" is the one described at (24).

Hence, the state estimation error is bounded between zero and this maximum value and, therefore, it is stable in the sense of Lyapunov. The theorem is proved.

■

V. Numerical Example and Simulation

Consider a 2-player nonlinear differential game given by the following equations:

$$\dot{x}_t = f(t, x_t) + g^1(t, x_t)u_t{}^1 + g^2(t, x_t)u_t{}^2$$
$$y_t = C x_t \tag{40}$$

where

$$t \in [0, \infty), \quad x_0 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \quad C = [1 \quad 0],$$
$$g^1(t, x_t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g^2(t, x_t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$f(t, x_t) = \begin{bmatrix} -6x_{1,t} + 2x_{2,t} - x_{1,t}x_{2,t} \\ 2x_{1,t} - 6x_{2,t} - 2x_{2,t}{}^2 \end{bmatrix} \tag{41}$$

and the control actions are:

$$u_t{}^1 = u_t{}^2 = \sin(t) \tag{42}$$

Once verified that the Assumptions 1-4 hold, it is possible to model the nonlinear differential game (40) and to estimate its state variable $x_{2,t}$.

Being thus, consider now a Luenberger-like differential neural network observer given by the following equations:

$$\dot{\hat{x}}_t = W_{1,t}\sigma(V_{1,t}\hat{x}_t) + W_{2,t}{}^1\varphi^1(V_{2,t}{}^1\hat{x}_t)u_t{}^1 + W_{2,t}{}^2\varphi^2(V_{2,t}{}^2\hat{x}_t)u_t{}^2 + K[y_t - \hat{y}_t] \tag{43}$$

$$\hat{y}_t = C\hat{x}_t$$

where

$$t \in [0,\infty), \quad \hat{x}_0 = \begin{bmatrix}-0.1\\0.1\end{bmatrix}, \quad K = \begin{bmatrix}25\\25\end{bmatrix},$$
$$W_{1,0} = \begin{bmatrix}0.5 & 0.1 & 0 & 2\\0.2 & 0.5 & 0.6 & 0\end{bmatrix}, \quad V_{1,0} = \begin{bmatrix}1 & 0.5\\0.3 & 1.3\\0.5 & 1\\0.2 & 0.6\end{bmatrix}, \tag{44}$$
$$W_{2,0}{}^1 = \begin{bmatrix}0.5 & 0.1\\0 & 0.2\end{bmatrix}, \quad W_{2,0}{}^2 = \begin{bmatrix}1.3 & 0.8\\0.8 & 1.3\end{bmatrix}$$
$$V_{2,0}{}^1 = \begin{bmatrix}1 & 0.1\\0.1 & 2\end{bmatrix}, \quad V_{2,0}{}^2 = \begin{bmatrix}0.5 & 0.7\\0.9 & 0.1\end{bmatrix}$$

and the activation functions $\sigma: \Re^k \to \Re^k$, $\varphi^1: \Re^{r_1} \to \Re^{r_1 \times s_1}$ and $\varphi^2: \Re^{r_2} \to \Re^{r_2 \times s_2}$ are:

$$\sigma(z) = \frac{1}{1+e^{-0.8z}} - 0.5, \quad k = 4$$
$$\varphi^1(z) = \frac{1}{1+e^{-0.5z}}, \quad r_1 = 2, \quad s_1 = 1 \tag{45}$$
$$\varphi^2(z) = \frac{1}{1+e^{-0.5z}}, \quad r_2 = 2, \quad s_2 = 1$$

By proposing the values described at the Assumption 6 as:

$$\Lambda_1 = \Lambda_2 = 0.3I_{4\times4}$$
$$\Lambda_3{}^1 = \Lambda_3{}^2 = \Lambda_4{}^1 = \Lambda_4{}^2 = 0.3I_{2\times2}$$
$$\Lambda_5 = \Lambda_6 = \Lambda_7{}^1 = \Lambda_7{}^2 = \Lambda_8{}^1 = \Lambda_8{}^2 = 0.5I_{2\times2}$$
$$\Lambda_\sigma = 0.3I_{2\times2}$$
$$\Lambda_\varphi{}^1 = \Lambda_\varphi{}^2 = 0.3I_{2\times2}$$
$$\Lambda_{\eta_\sigma} = 0.8I_{4\times4}$$
$$\Lambda_{\eta_\varphi}{}^1 = \Lambda_{\eta_\varphi}{}^2 = 0.8I_{2\times2}$$
$$\Lambda_f = 0.1I_{2\times2}$$
$$l_1 = l_2{}^1 = l_2{}^2 = 0.3$$
$$\bar{u}^1 = \bar{u}^2 = 3 \tag{48}$$
$$\delta = 0.1$$
$$A = \begin{bmatrix}-1.3963 & -2.2632\\0.0473 & -6.1606\\0.426 & -7.4451\\-6.1533 & 0.8738\end{bmatrix}$$
$$Q = I_{2\times2}$$
$$K_1 = 1$$
$$K_2 = 1$$
$$K_3{}^1 = K_3{}^2 = 10$$
$$K_4{}^1 = K_4{}^2 = 1$$

the solution of the algebraic Riccati equation (8) results in:

$$P = \begin{bmatrix}0.0474 & -0.0293\\-0.0293 & 0.0922\end{bmatrix} \tag{46}$$

Finally, by applying the learning law (19)-(22) described at the Theorem 1, the value of the state estimation error "in average sense" on a time period of 25 seconds is:

$$\frac{1}{t}\int_0^t \Delta_t{}^T Q \Delta_t \, dt = 0.0076, \quad t \in [0,25] \tag{47}$$

The simulation of this example was made using the MATLAB-SIMULINK platform and its results are shown in the following figures.

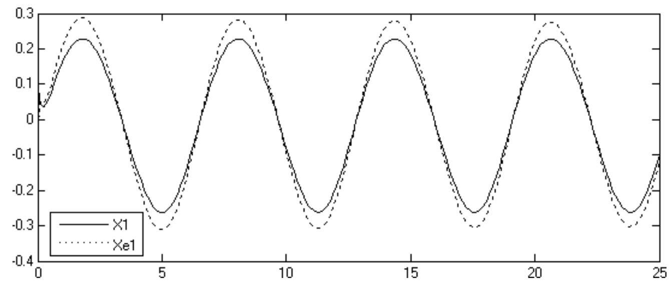

Fig. 1. Comparison between $x_{1,t}$ and $\hat{x}_{1,t}$.

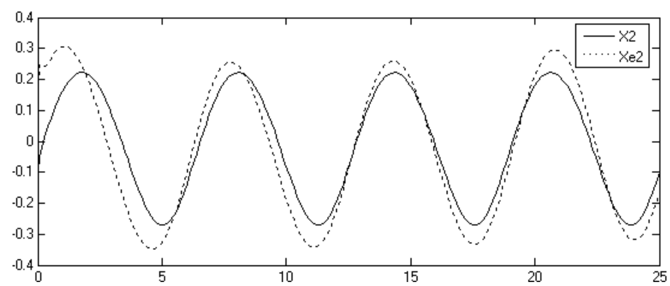

Fig. 2. Comparison between $x_{2,t}$ and $\hat{x}_{2,t}$.

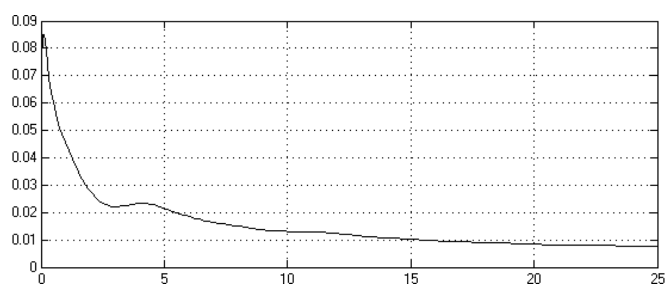
Fig. 3. Dynamics of the state estimation error $\frac{1}{t}\int_0^t \Delta_t{}^T Q \Delta_t \, dt$.

As is seen in the Figs. 1 and 2, the differential neural network observer (43) can perform both the modeling of the nonlinear differential game (40) and the observation of its state variable $x_{2,t}$.

It is important to mention that the performance of (43) depends on the number of neurons used and on the proposition of all the constant values described at the Assumption 6.

In this particular case, the design of the differential neural network observer (43) was made using only eight neurons: four for the output layer and two for the hidden layer of each player.

VI. CONCLUSIONS

According to the results of this paper, the differential neural network observer (5) solves the problem of the state estimation for the class of nonlinear differential games (1).

Being thus, the proposed learning law (19)-(22) obtains the maximum value of state estimation error (24) depending on the performance of the differential neural network observer (5), that is to say, depending on the number of neurons used and on the design conditions applied.

Finally, according to the simulation result of the numerical example, the effectiveness of differential neural network observer (5) is shown and the applicability of the Theorem 1 is verified.

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