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Enlarging Domain of Attraction of Switched Linear Systems in the Presence of Saturation Nonlinearity

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Abstract— **This work proposes an approach to compute domain of attraction of switched systems under arbitrary switching and in the presence of saturation nonlinearity. It is also shown that any estimation of domain of attraction using linear difference inclusion approaches is a subset of our result and hence our method is less conservative than the others.**

I. Introduction

This work considers the computation of the domain of attraction of discrete-time switched linear systems with saturation nonlinearity under static state feedback control:

$$\begin{cases} x^+ = A_i x + B_i sat(u) \quad , \quad i \in \mathcal{N} := \{1, 2, ..., N\} \\ u \;\; = K_i x \end{cases} \tag{1}$$

where, $x \in \mathbb{R}^n$ is the state, x^+ is the successor state and $u \in \mathbb{R}^m$ is the control input. Symbol $sat(\cdot)$ is the standard saturation function: when v is a scalar, $sat(v) = v$ if $|v| \leq 1$, 1 if $v > 1$ and -1 if $v < -1$ and $sat(u)$ is the vector of $sat(u_j)$ for each element j of u. The index "i" refers to the current mode of the system with i taking one of the possible indices in $\mathcal{N}$. The switching among the various modes is assumed to be arbitrary. Hence, i is not known a priori, but its instantaneous value is available at each sampling period. It is assumed hereafter that the problem data, satisfy the following assumptions: (**A1**) (A_i, B_i) is stabilizable for all $i \in \mathcal{N}$, (**A2**) $(A_i+B_iK_i, K_i)$ is observable for all $i \in \mathcal{N}$, (**A3**) $(A_i + B_iK_i)$ is discrete-time Hurwitz for all $i \in \mathcal{N}$, and (**A4**) the non-saturated switched system $x^+ = (A_i + B_iK_i), i \in \mathcal{N}$ is asymptotically stable. Of these assumptions, (A3) is not really restrictive since K_i can be designed to stabilize $A_i + B_iK_i$ for each $i \in \mathcal{N}$. However, stability of individual subsystems is only a sufficient condition for asymptotic stability of arbitrary switching systems [1]. Hence, for checking (A4), one can use one of the several techniques proposed in the literature, such as existence of a common Lyapunov function [1], pairwise switched Lyapunov functions [2], multiple Lyapunov functions [3], composite quadratic functions [4], [5] or polyhedral Lyapunov functions [6], [7].

While the literature on switched systems is rapidly growing, relatively few works are on saturated switched linear systems and their domain of attraction. We begin with a brief review on the domain of attraction of single saturated linear systems and extend it to the case where switching is present.

A non-switching saturated system can be described by

$$x^+ = Ax + Bsat(Kx) \tag{2}$$

where the subscripts to A, B and K are dropped. Given a non-empty set $\Omega \subset \mathbb{R}^n$, the well-known [8], [9] one-step set to Ω is the set of all states that can be driven to Ω in one step, or

$$\mathcal{Q}(\Omega) = \{x : Ax + Bsat(Kx) \in \Omega\} \tag{3}$$

In general, $\mathcal{Q}(\Omega)$ is not necessarily convex for a convex Ω. This non-convex nature of $\mathcal{Q}(\Omega)$ makes its use for computations of domain of attraction of saturated systems undesirable. While several approaches have been proposed to circumvent this problem, two of them appear promising. For their discussion, the following notations are used. Let $\mathcal{M} = \{1, ..., m\}$ be the set of integers and $S \subseteq \mathcal{M}$ be a subset of $\mathcal{M}$, $\mathcal{V}_{\mathcal{M}} = \{S : S \subseteq \mathcal{M}\}$ is the set of all subsets of $\mathcal{M}$ with cardinality 2^m and $S^c = \{i \in \mathcal{M} : i \notin S\}$ is the complement of S in $\mathcal{M}$. Also, let I_m be the $m \times m$ identity matrix. Given $x \in \mathbb{R}^n$ and $K \in \mathbb{R}^{m \times n}$, x_j is the j-th element of x; $K^{i\cdot} \in \mathbb{R}^n$, $K^{\cdot j} \in \mathbb{R}^m$ are the i-th row and the j-th column of matrix K, respectively.

A. LDI approach

The first approach [10], [11] uses a linear difference inclusion (LDI) representation of the saturation function. It uses an auxiliary feedback term and exploit their convex hull to represent the saturation function. Their work can be summarized by the following lemma:

Lemma 1: Suppose sets $\mathcal{M}$ and $S \in \mathcal{V}_{\mathcal{M}}$ are given. Let D_S be the $m \times m$ diagonal matrix with diagonal elements $D_S(j, j)$, whose value is 1 if $j \in S$ and 0 otherwise, and let $D_{S^c} = I_m - D_S$. Then, for all $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$ such that $|v_j| \leq 1$ for all $j \in \mathcal{M}$:

$$sat(u) \in co\{D_{S^c}u + D_S\, v : S \in \mathcal{V}_{\mathcal{M}}\}$$

This lemma shows that $sat(u)$ can be expressed as a convex hull of vectors formed by choosing some rows (those belonging to S) from v and the rest (those belonging to S^c) from u.

Using this and assuming that some matrix, $H \in \mathbb{R}^{m \times n}$, is given, then

$$sat(Kx) \in co\{D_{S^c}Kx + D_SHx : S \in \mathcal{V}_{\mathcal{M}}\}$$

for all $x \in \mathcal{L}_H := \{x : ||Hx||_\infty \leq 1\}$. Let

$$\mathcal{G}_H(x, S) := \Big(A + \sum_{j \in S^c} B^{\cdot j}K^{j\cdot} + \sum_{j \in S} B^{\cdot j}H^{j\cdot}\Big)x \tag{4}$$

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be the successor state from x when some of the inputs are from the H matrix and the rest from K. Using this and lemma 1, it can be shown from (2) that for all $x \in \mathcal{L}_H$:

$$Ax + Bsat(Kx) \in co\{\mathcal{G}_H(x, S) : \text{ for all } S \in \mathcal{V}_{\mathcal{M}}\} \quad (5)$$

To compute the domain of attraction of (2) using such a representation, the H and P matrices must be chosen such that a common Lyapunov function, $V(x) = x^T P x$, exists for all linear systems of the form (4), obtained from all $S \in \mathcal{V}_{\mathcal{M}}$ for the choice of H. The largest $\mathcal{E}(P) := \{x : x^T P x \leq 1\}$ that is contained in $\mathcal{L}_H$ is then an estimate of domain of attraction of (2). The numerical determination of this maximal value of P and H can be computed using a Linear Matrix Inequality.

The LDI approach has been also extended to saturated switched systems [12], [13], [14], [15], [16]. The basic idea is to obtain auxiliary matrices (H_i) for the LDI representation of saturation function for each $i \in \mathcal{N}$ and the existence of the associated multiple Lyapunov functions that guarantee asymptotic stability of the resulting switched system. Again, the domain of attraction can be found by solving the following optimization problem with $P_i, H_i, i \in \mathcal{N}$ as variables:

$$\begin{cases} \max_{P_i, H_i} \ \det\left(P_i^{-1}\right) \\ s.t. \\ \left(A_c(i,S)\right)^T P_j \left(A_c(i,S)\right) - P_i < 0 \\ \qquad\qquad\qquad \forall (i,j) \in (\mathcal{N} \times \mathcal{N}),\ \forall S \in \mathcal{V}_{\mathcal{M}} \\ \mathcal{E}(P_i) \in \mathcal{L}_{H_i} \quad\ \forall i \in \mathcal{N} \end{cases} \quad (6)$$

where $A_c(i,S) = (A_i + B_i(D_{S^c} K_i + D_S H_i))$. The estimate of domain of attraction of switched system is then given by $\bigcap_{i \in \mathcal{N}} \mathcal{E}(P_i)$. In the above, the choice of i in the objective function that will result in the largest domain of attraction remains an open question.

B. SNS approach

The second approach to the estimation of domain of attraction of (2) is based on the concept of saturated and non-saturated (SNS)-invariance introduced in [17]. Given an index set $\mathcal{M}$ and a set $S \in \mathcal{V}_{\mathcal{M}}$, the response of system (2) can be written as

$$\mathcal{F}(x, S) := Ax + \sum_{j \in S^c} B^{\cdot j} K^{j \cdot} x + \sum_{j \in S} B^{\cdot j} sat(K^{j \cdot} x)$$

where $\mathcal{F}(x, S)$ is the successor state of x with inputs from $S \in \mathcal{V}_{\mathcal{M}}$ subjected to saturation nonlinearity but not the rest. Two associated definitions are now stated.

Definition 1: A set $\Omega \subset \mathbb{R}^n$ is said to be SNS-invariant w.r.t. (2), if $x \in \Omega$ implies that $\mathcal{F}(x, S) \in \Omega$ for all $S \in \mathcal{V}_{\mathcal{M}}$.

Definition 2: Given a set Ω, the SNS-one-step set is defined by

$$\mathcal{Q}^{SNS}(\Omega) = \{x : \mathcal{F}(x, S) \in \Omega, \text{ for all } S \in \mathcal{V}_{\mathcal{M}}\} \quad (7)$$

Definition 1 shows that SNS-invariance is a stronger requirement than invariance. This follows since $\mathcal{M} \in \mathcal{V}_{\mathcal{M}}$, hence Ω is invariant w.r.t. (2) if it is SNS-invariant. Definition 2 shows an approach to compute the SNS-one-step set. Given Ω, the set of x such that $\mathcal{F}(x, S) \in \Omega$ can be computed for every $S \in \mathcal{V}_{\mathcal{M}}$. The intersection of these sets is $\mathcal{Q}^{SNS}(\Omega)$. More exactly, if Ω is a convex polyhedron given by $\Omega = \{x : Rx \leq \mathbf{1}\}$,

$$\mathcal{Q}^{SNS}(\Omega) := \bigcap_{S \in \mathcal{V}_{\mathcal{M}}} \left\{ x : R(A + \sum_{j \in S^c} B^{\cdot j} K^{j \cdot}) x \leq \mathbf{1} + \sum_{j \in S} \left| R B^{\cdot j} \right| \right\} \quad (8)$$

where $|\cdot|$ refers to the absolute operator applied element-wise. Using this operator, one can start from an initial domain of attraction, Ω^0, and enlarge it by computing $\Omega^k = \mathcal{Q}^{SNS}(\Omega^{k-1})$, at each step. The set sequence $\{\Omega^0, \Omega^1, \Omega^2, ...\}$ enlarges to an estimate of domain of attraction of (2), which is also SNS-invariant [17].

Subsequent content of the paper is arranged as follows. Section II extends the concept of SNS-invariance and SNS-one-step set for switched systems that preserve convexity. Then, a simple algorithm is proposed to enlarge domain of attraction of saturated switched systems. Section III presents a method for the computation of invariant sets of constrained switched systems in their linear region of operation. Section IV shows an important result - that the domain of attraction computed from our method, contains any domain of attraction obtained from the LDI approaches. Sections V and VI contain, respectively, examples and conclusions.

II. The Proposed Approach

This section shows an extension of SNS-invariance to switched saturated systems and estimation of their domain of attraction. Note that subscript s added to a variable/operator refers to that of the switched systems. An obvious extension of (3) to a switch system is the one-step set $\mathcal{Q}_s(\Omega)$ given by

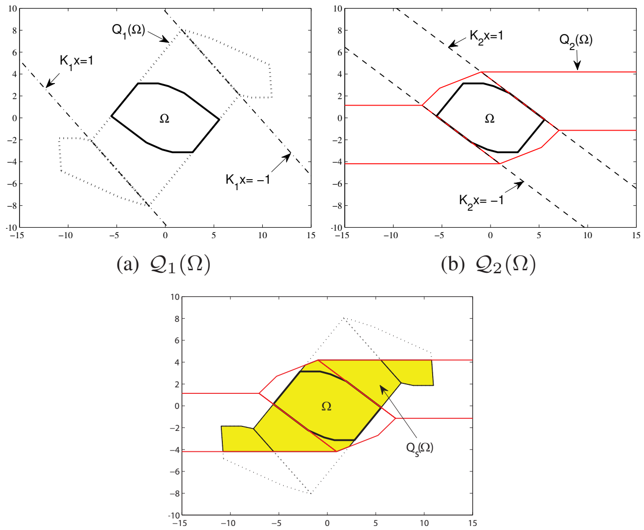
$$\begin{aligned} \mathcal{Q}_s(\Omega) &:= \{x : A_i x + B_i sat(K_i x) \in \Omega,\ \forall i \in \mathcal{N}\} \\ &= \bigcap_{i \in \mathcal{N}} \mathcal{Q}_i(\Omega) \end{aligned}$$

where $\mathcal{Q}_i(\Omega) = \{x : A_i x + B_i sat(K_i x) \in \Omega\}$. Like the single system case, $\mathcal{Q}_i(\Omega)$ and hence $\mathcal{Q}_s(\Omega)$ is not necessarily convex when Ω is. An example of this can be seen in $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B_1 = [10, 5]^T$, $K_1 = -0.1029 * [1, 1]$, $A_2 = \begin{bmatrix} 0 & -1 \\ 0.0001 & 1 \end{bmatrix}$, $B_2 = [0.5, -2]^T$ and $K_2 = 0.0938 * [2, 3]$. Figure 1 shows $\mathcal{Q}_1(\Omega)$ (dotted lines), $\mathcal{Q}_2(\Omega)$ (solid lines) and $\mathcal{Q}_s(\Omega)$ (in shade). The use of SNS-one-step set is useful for preserving convexity. For this purpose, several related definitions are given.

Definition 3: (**Switched-SNS-invariance:**) Let $\mathcal{F}_i(x, S) := A_i x + \sum_{j \in S^c} B_i^{\cdot j} K_i^{j \cdot} x + \sum_{j \in S} B_i^{\cdot j} sat(K_i^{j \cdot} x)$. Then, a set $\Omega \subset \mathbb{R}^n$ is said to be switched-SNS-invariant w.r.t. (1), if $x \in \Omega$ implies that $\mathcal{F}_i(x, S) \in \Omega$ for all $S \in \mathcal{V}_{\mathcal{M}}$ and for all $i \in \mathcal{N}$.

Definition 4: (**Switched-SNS-one-step set:**) Given a set Ω, the switched-SNS-one-step set w.r.t. (1) is defined by

$$\mathcal{Q}_s^{SNS}(\Omega) = \{x : \mathcal{F}_i(x, S) \in \Omega,\ \forall S \in \mathcal{V}_{\mathcal{M}} \text{ and } \forall i \in \mathcal{N}\}$$



(a) $\mathcal{Q}_1(\Omega)$ (b) $\mathcal{Q}_2(\Omega)$

(c) $\mathcal{Q}_s(\Omega) = \mathcal{Q}_1(\Omega) \cap \mathcal{Q}_2(\Omega)$

Fig. 1. Illustration of non-convex one-step sets

Suppose that Ω is a convex polytope given by $\Omega = \{x : Rx \leq \mathbf{1}\}$, it follows from the previous discussion that

$$\mathcal{Q}_s^{SNS}(\Omega) = \bigcap_{i \in \mathcal{N}} \mathcal{Q}_i^{SNS}(\Omega) \tag{9}$$

Hence, the computation of $\mathcal{Q}_s^{SNS}(\Omega)$ follows directly from (8) by interseting over every $S \in \mathcal{V}_{\mathcal{M}}$ and then over all $i \in \mathcal{N}$. Figure 2 shows the convex set $\mathcal{Q}_s^{SNS}(\Omega)$ (dashed line) and compares it with $\mathcal{Q}_s(\Omega)$ (in shade) for the same example considered earlier. Note that $\mathcal{Q}_s^{SNS}(\Omega)$ is a convex set contained in $\mathcal{Q}_s(\Omega)$ and it is invariant w.r.t. $(A_1 + B_1K_1)x$, $(A_2 + B_2K_2)x$, $(A_1x + B_1 sat(K_1x))$ and $(A_2x + B_2 sat(K_2x))$.

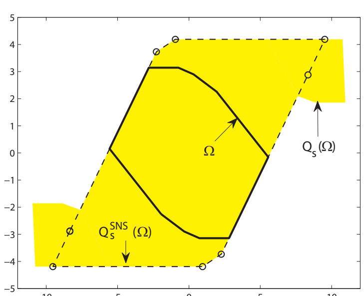


Fig. 2. Illustration of switched-SNS-one-step set, $\Omega \subseteq \mathcal{Q}_s^{SNS}(\Omega) \subseteq \mathcal{Q}_s(\Omega)$

Definition 5: (**SNS-domain of attraction:**) An initial condition x_0 belongs to the SNS-domain of attraction of (1), if the sequence $\{x_k : k \in \mathbb{Z}^+\}$ obtained from recursion

$$x_{k+1} = \mathcal{F}_i(x_k, S),$$

converges to the origin for every $S \in \mathcal{V}_{\mathcal{M}}$ and for every $i \in \mathcal{N}$ as $k \to \infty$.

Remark 1: Like the single system situation, the SNS-domain of attraction requires the satisfaction of all $S \in \mathcal{V}_{\mathcal{M}}$ and is, hence, a subset of true domain of attraction of (1).

The following theorem extends the domain of attraction of saturated switched systems beyond its region of linear behavior.

Theorem 1: Suppose that $\Phi \subseteq \mathbb{R}^n$ is a convex polyhedral domain of attraction of non-saturated system $x^+ = (A_i + B_iK_i)x$, $i \in \mathcal{N}$ and it contains the origin in its interior. Let $\Omega^0 := \Phi$ and consider the following recursion:

$$\Omega^{k+1} := \mathcal{Q}_s^{SNS}(\Omega^k) \tag{10}$$

Then, *(i)* Each Ω^k is a convex polyhedron. *(ii)* Each Ω^k is a switched-SNS-invariant set w.r.t. (1). *(iii)* The condition $\Omega^k \subseteq \Omega^{k+1}$ holds for all $k \geq 0$. *(iv)* Each Ω^k is a SNS-domain of attraction of switched saturated system (1). *(v)* A point $x \in \Omega^k$ is steered into Ω^0 in at most k steps. *(vi)* The set sequence $\{\Omega^0, \Omega^1, \Omega^2, ...\}$ converges to Ω^*, which is the maximal SNS-domain of attraction of (1).

Proof: See Appendix. ■

The recursion presented in Theorem 1 generates a sequence of domains of attraction of (1) and requires an initial set Ω^0 to start. The procedure to compute such a $\Omega^0 \subseteq \mathcal{L}_{\mathcal{K}} := \{x : ||K_ix||_\infty \leq 1, \text{ for all } i \in \mathcal{N}\}$ is described in the next section.

III. Computation of Domain of Attraction of Constrained Switched Systems

For the sake of simplicity, let $\tilde{A}_i := A_i + B_iK_i$ and consider the constrained switched system with arbitrary switching of the form

$$\begin{cases} x^+ = \tilde{A}_i x\,, \; i \in \mathcal{N} \\ x \in \mathcal{X} \end{cases} \tag{11}$$

where $\mathcal{X} \subset \mathbb{R}^n$ is a polytope with the origin in its interior.

Suppose that the constraint set has a characterization of $\mathcal{X} = \{x : F^0x \leq \mathbf{1}\}$. The procedure to compute the maximal invariant set w.r.t. (11), which is also a domain of attraction, uses the following iterative recursion starting with $\mathcal{X}$.

Algorithm 1 Computation of domain of attraction of Constrained Switched Systems

`Input:` $\tilde{A}_i$ and $\mathcal{X}$ (Constraint set).
`Output:` Φ (Polyhedral domain of attraction).

1) Set $k = 0$ and let
$$\phi^0 := \mathcal{X} = \{x : F^0 x \leq \mathbf{1}\}$$
2) For each $i \in \mathcal{N}$, compute:
$$\mathcal{X}_i^k = \left\{x : F^k \tilde{A}_i\, x \leq \mathbf{1}\right\}$$
3) Compute the polyhedron
$$\phi^{k+1} = \phi^k \bigcap_{\{i=1,\ldots,N\}} \mathcal{X}_i^k$$
and let it be represented by $\phi^{k+1} = \{x : F^{k+1}x \leq \mathbf{1}\}$ for some appropriate F^{k+1}.
4) If $\phi^{k+1} \equiv \phi^k$ set $\Phi = \phi^{k+1}$ and stop, else set $k = k+1$ and go to step (2).

Properties of the ϕ^k set are summarized in the following theorem.

Theorem 2: Suppose system (11) satisfies assumptions (A2)-(A4) and ϕ^k is generated based on Algorithm 1. Then, the following results are known: (i) $\phi^k \subseteq \mathcal{X}$ and $\phi^k \subseteq \phi^{k-1}$ for all k. (ii) $\phi^\infty := \lim_{k\to\infty} \phi^k$ exists and contains the origin in its interior (iii) ϕ^∞ is finitely determined, or equivalently, there exist a $k^* \in \mathbb{Z}^+$ such that $\phi^{k^*+1} = \phi^{k^*}$ and $\phi^{\bar{k}} = \phi^{k^*}$ for all $\bar{k} > k^*$. (iv) ϕ^∞ is the largest invariant set w.r.t. (11) and is the largest constrained-admissible domain of attraction of (1) contained in $\mathcal{X}$.

Proof: Part (i) of the theorem follows immediately from the definition of the set ϕ^k in step (3) of algorithm. The proofs of other results are non-trivial and are omitted due to space limitations. ■

IV. Comparison of SNS-domain of attraction with LDI approaches

In this section, first the definition of H-contractive sets obtained from the LDI representation of saturation function is presented. Then, it is proved that any H-contractive set obtained from LDI approach is included in SNS-domain of attraction obtained from Theorem 1 and hence our result is less conservative.

As described in Section I, the domain of attraction of saturated switched systems can be estimated by means of LDI representation of saturation function; Taking this LDI representation into account, it is clear that a given set $\Psi \subseteq \mathbb{R}^n$ is invariant w.r.t. (1) if it is invariant w.r.t. all linear systems of the form $\mathcal{G}_{H_i}(x, S)$ obtained from all $S \in \mathcal{V}_\mathcal{M}$ an all $i \in \mathcal{N}$. The notion of H-contractive set is now introduced:

Definition 6: Given H_i matrices, a set $\Psi \subseteq \mathcal{L}_\mathcal{H} := \{x : ||H_i x||_\infty \leq 1, \forall i \in \mathcal{N}\}$ is an H-contractive set if it is a convex set containing the origin and there is a $\lambda \in [0, 1)$ such that $x \in \Psi$ implies that $\mathcal{G}_{H_i}(x, S) \in \lambda\Psi$, for all $S \in \mathcal{V}_\mathcal{M}$ and for all $i \in \mathcal{N}$.

H-contractive sets can be obtained by methods described in Section I and it is clear that any H-contractive set constitutes an estimation of the domain of attraction of (1). The following theorem states that any H-contractive set is included in the SNS-domain of attraction of (1). That is, the estimation of the domain of attraction given by Theorem 1 is less conservative than the one obtained by means of H-contractive sets.

Theorem 3: Let us suppose that $\Psi \subseteq \mathcal{L}_\mathcal{H}$ is an H-contractive set for given matrices H_i, $i \in \mathcal{N}$. Then Ψ is a switched-SNS-invariant set and there exist a $\hat{k} \in \mathbb{Z}^+$ such that $\Psi \subseteq \Omega^{\hat{k}} \subset \Omega^*$.

Proof: See Appendix. ■

V. Numerical Example

A. Computation of SNS-domain of attraction

In order to illustrate our results, consider a single input saturated switching discrete-time system taken from [13], with two subsystems:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B_1 = [10,\ 5]^T,\ K_1 = -0.1029 * [1,\ 1],$$

$$A_2 = \begin{bmatrix} 0 & -1 \\ 0.0001 & 1 \end{bmatrix},\ B_2 = [0.5,\ -2]^T,$$

$$K_2 = 0.0938 * [2,\ 3]$$

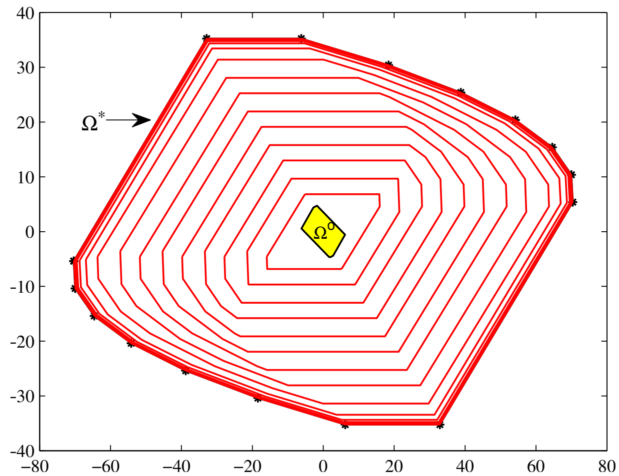


Fig. 3. Illustration of sequence of Ω^k in every 5 steps leading to SNS-domain of attraction

First an initial domain of attraction based on Algorithm I is computed. The algorithm starts from the initial polyhedron $\mathcal{L}_\mathcal{K}$ and converges in 4 iterations to Ω^0, which is shown with the shaded region in Figure 3. Using this set as the initial polyhedron in recursion (10), the sequence of Ω^k has been determined from Theorem 1. Given $\Omega^k = \{x : Rx \leq \mathbf{1}\}$, Ω^{k+1} is computed from:

$$\Omega^{k+1} = \mathcal{Q}_s^{SNS}(\Omega^k) = \mathcal{Q}_1^{SNS}(\Omega^k) \cap \mathcal{Q}_2^{SNS}(\Omega^k)$$

where,

$$\mathcal{Q}_1^{SNS}(\Omega^k) =$$
$$\{x : R(A_1 + B_1K_1)x \preceq \mathbf{1}\} \cap \{x : R(A_1)x \preceq \mathbf{1} + |RB_1|\}$$
$$\mathcal{Q}_2^{SNS}(\Omega^k) =$$
$$\{x : R(A_2 + B_2K_2)x \preceq \mathbf{1}\} \cap \{x : R(A_2)x \preceq \mathbf{1} + |RB_2|\}$$

Figure 3 depicts the sequence of Ω^k sets at every five steps (note that, as it is claimed in Theorem 1, $\Omega^k \subseteq \Omega^{k+1}, \forall k \geq 0$). The sequence converges to the maximal SNS-domain of attraction, Ω^*, at 81 steps. This set, which is characterized by 16 inequalities is also shown in Figure 3.

In the proposed recursion of Theorem 1, each Ω^k is obtained by computing the intersection of $N \times 2^m$ sets obtained from Ω^{k-1}. Most of the computational complexity is due to the solution of the linear programming problems required to eliminate the redundant linear constraints of the obtained sets. As claimed in part (*iv*) of Theorem 1, each one of the polyhedrons Ω^k is an estimation of the domain of attraction of the saturated switched system. This means that if a limited computational time is available, the recursion could be stopped at any desired step before converging to the maximal domain of attraction. In this paper, all the algorithms are implemented in Matlab 7 using multi-parametric programming toolbox solvers [18] and the computations are performed on a dual-core CPU with 3.2 GHz processor. The total computational time of SNS-domain of attraction is 0.978 seconds.

B. Comparison with H-contractive sets

Estimation of domain of attraction with an H-contractive is obtained from optimization problem (6), which results in:

$$P_1 = \begin{bmatrix} 0.001 & 0.0003 \\ 0.0003 & 0.002 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0004 & 0 \\ 0 & 0.0020 \end{bmatrix},$$

$H_1 = [-0.0174, -0.0421]$, $H_2 = [0, 0.0449]$.

The resulting H-contractive set, $\Psi = \mathcal{E}(P_1) \cap \mathcal{E}(P_2)$ is compared with SNS-domain of attraction in Figure 4. As claimed by Theorem 3, it is observed that $\Psi \subseteq \Omega^{32} \subset \Omega^*$. The relative sizes of Ω^* and $\Psi = \cap \mathcal{E}(P_i)$ is compared by the ratio $\frac{Area(\Omega^*)}{Area(\Psi)} \simeq 3.21$.

It is clear from Figure 4 that the SNS-domain of attraction has been significantly enlarged beyond the linear region of controllers and it contains any H-contractive set.

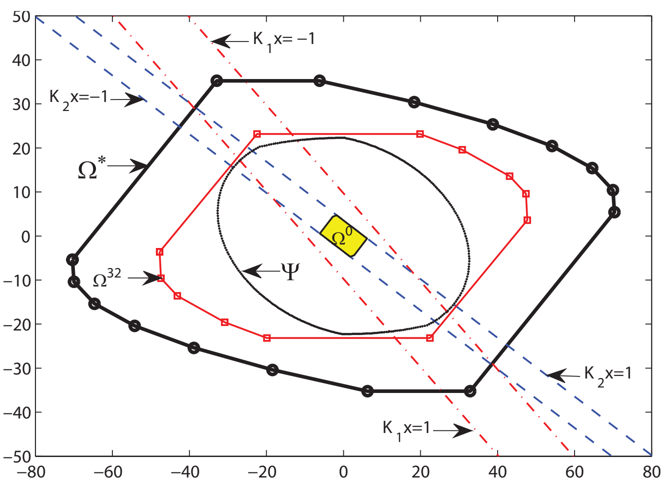


Fig. 4. Comparison of SNS-domain of attraction, Ω^*, with H-contractive set Ψ

VI. Conclusions

In this paper, a new method for computation of domain of attraction of saturated switched system was proposed based on the notion of switched-SNS-invariance. The proposed approach, starts from an initial domain of attraction inside the region of linear behavior of switched system and converges to the maximal SNS-domain of attraction. It was also shown that any estimate of domain of attraction using LDI approach is a subset of our result and hence our result is less conservative. Simulation results demonstrates the effectiveness of the proposed approach.

Appendix

Proof of Theorem 1:

(i) $\Omega^0 = \Phi$ is a convex domain of attraction of (1) in the region of linear behavior. $\mathcal{Q}_i^{SNS}(\Omega^0)$ is the intersection of 2^m polyhedrons, for each $i \in \mathcal{N}$, hence is a polyhedron. In each step, Ω^{k+1} is obtained by intersecting of N convex sets $\mathcal{Q}_i^{SNS}(\Omega^k)$ for $i = \{1, ..., N\}$, which is also convex. This proves that recursion (10) always yields a convex polyhedron.

(ii) Since Ω^0 belongs to $\mathcal{L}_\mathcal{K}$, it follows that for any $x \in \Omega^0$, $\mathcal{F}_i(x, S) = (A_i + B_i K_i)x$ for all $S \in \mathcal{V}_\mathcal{M}$ and for all $i \in \mathcal{N}$. From this and the fact that Ω^0 is a domain of attraction, it is concluded that if $x \in \Omega^0$, then $\mathcal{F}_i(x, S) \in \Omega^0$ for all $S \in \mathcal{V}_\mathcal{M}$ and for all $i \in \mathcal{N}$. This means that Ω^0 is a switched-SNS-invariant set. Now suppose Ω^k is switched-SNS-invariant, then $\Omega^k \subseteq \mathcal{Q}_i^{SNS}(\Omega^k), \forall i \in \mathcal{N}$, hence $\Omega^k \subseteq \bigcap_{i\in\mathcal{N}} \mathcal{Q}_i^{SNS}(\Omega^k) = \Omega^{k+1}$. Therefore, for any $x \in \Omega^{k+1}$, we have $\mathcal{F}_i(x, S) \in \Omega^k \subseteq \Omega^{k+1}, \forall S \in \mathcal{V}_\mathcal{M}, \forall i \in \mathcal{N}$, which means Ω^{k+1} is also switched-SNS-invariant.

(iii) From the definition of the set operation (8) and the property of SNS-invariant sets, it is immediate that for each $i \in \mathcal{N}$, $\Omega^k \subseteq \mathcal{Q}_i^{SNS}(\Omega^k)$. Therefore, $\Omega^k \subseteq \Omega^{k+1} = \bigcap_{i\in\mathcal{N}} \mathcal{Q}_i^{SNS}(\Omega^k)$.

(iv) First of all, $\Omega^0 = \Phi \subseteq \mathcal{L}_\mathcal{K}$ is a domain of attraction in the region of linear behavior of switched system and hence it belongs to SNS-domain of attraction of (1). If Ω^k belongs to SNS-domain of attraction, then $\Omega^{k+1} = \bigcap_{i\in\mathcal{N}} \mathcal{Q}_i^{SNS}(\Omega^k)$ also belongs to SNS-domain of attraction due to the fact that if $x \in \Omega^{k+1}$, $\mathcal{F}_i(x, S) \in \Omega^k$ for all $S \in \mathcal{V}_\mathcal{M}$ and for all $i \in \mathcal{N}$. Therefore, recursion (10) starting with $\Omega^0 = \Phi$ results in a sequence of SNS-invariant sets that each one belongs to SNS-domain of attraction of (1).

(v) Switched-SNS-invariance property of the sets resulting from recursion (10) inferred that if $x \in \Omega^{k+1}$, then $\mathcal{F}_i(x, S) \in \Omega^k$ for all $i \in \mathcal{N}$ and for all $S \in \mathcal{V}_\mathcal{M}$ including $S = \mathcal{M}$, hence, $x^+ = \mathcal{F}_i(x, \mathcal{M}) \in \Omega^k$ for all $i \in \mathcal{N}$. In other words, if $x \in \Omega^{k+1}$, then $x^+ \in \Omega^k$. This means that any $x \in \Omega^{k+1}$ is brought into Ω^k in one step. Consequently, Ω^k is the set of points that can be brought into Ω^0 at most in k steps.

(vi) The proof is by contradiction. Suppose x belongs to the SNS-domain of attraction of (1), but $x \notin \Omega^*$. Consider the recursion $x_{k+1} = \mathcal{F}_i(x_k, S)$ with $x_0 = x$. Since x belongs to the SNS-domain of attraction, $\lim_{k\to\infty} x_k = 0$. As Ω^0 is a domain of attraction with nonzero volume, it means that there exist a $q \in \mathbb{Z}^+$ such that $x_q \in \Omega^0$. This is equivalent to say that x is included in Ω^q, which is a contradiction.

Proof of Theorem 3:

First, we will show that Ψ is a switched-SNS-invariant set. For this purpose, we show that if $x \in \Psi$, then $\mathcal{F}_i(x, T) \in \lambda\Psi$ for all $T \in \mathcal{V}_\mathcal{M}$ and for all $i \in \mathcal{N}$. Since Ψ is H-contractive, from its definition it is inferred that

$$\mathcal{G}_{H_i}(x, S) \in \lambda\Psi, \ \forall S \in \mathcal{V}_\mathcal{M}, \ \forall i \in \mathcal{N} \qquad (12)$$

According to lemma 2 of [17], for given $H_i \in \mathbb{R}^{m\times n}$ matrices and any $T \in \mathcal{V}_\mathcal{M}$, if $x \in \mathcal{L}_\mathcal{H}$ then there exist an $S \in \mathcal{V}_\mathcal{M}$ such that

$$\mathcal{F}_i(x, T) \in co\{\mathcal{G}_{H_i}(x, S) : S \in \mathcal{V}_\mathcal{M}\}, \ \forall T \in \mathcal{V}_\mathcal{M}$$

From (12), it is concluded that any convex combination of $\mathcal{G}_{H_i}(x,S)$ is also inside $\lambda\Psi$, i.e.

$$\mathcal{F}_i(x,T) \in co\{\mathcal{G}_{H_i}(x,S) : S \in \mathcal{V}_{\mathcal{M}}\} \in \lambda\Psi, \ \forall T \in \mathcal{V}_{\mathcal{M}}$$

This proves that Ψ is a switched-SNS-invariant set. To show that Ψ belongs to SNS-domain of attraction, consider the following recursion:

$$x_{k+1} = \mathcal{F}_i(x_k, S) \quad, \forall S \in \mathcal{V}_{\mathcal{M}}, \forall i \in \mathcal{N} \tag{13}$$

where $x_0 \in \Psi$. Due to the λ-contractivity of Ψ, it is clear that $x_k \in \lambda^k\Psi$, for every $S \in \mathcal{V}_{\mathcal{M}}$ and for every $i \in \mathcal{N}$. Therefore, $\lim_{k\to\infty} x_k \to 0$. This and the fact that Ω^0 is a domain of attraction with nonzero volume, in turn, means that for any $x_0 \in \Psi$, there exist a $\bar{k} \in \mathbb{Z}^+$ such that $x_{\bar{k}} \in \Omega^0$. Now, let $\hat{k} := \max \bar{k}$ over all $x_0 \in \Psi$. Then, for any $x_0 \in \Psi$, $x_{\hat{k}}$ from recursion (13) is contained in Ω^0. This means that $\Psi \subseteq \Omega^{\hat{k}} \subset \Omega^*$, which proves the claim.

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